

FINITE-TIME ADAPTIVE MODIFIED FUNCTION PROJECTIVE MULTI-LAG GENERALIZED COMPOUND SYNCHRONIZATION FOR MULTIPLE UNCERTAIN CHAOTIC SYSTEMS

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In this paper, for multiple different chaotic systems with fully unknown parameters, a novel synchronization scheme called 'modified function projective multi-lag generalized compound synchronization' is put forward. As an advantage of the new method, not only the addition and subtraction, but also the multiplication of multiple chaotic systems are taken into consideration. This makes the signal hidden channels more abundant and the signal hidden methods more flexible. By virtue of finite-time stability theory and an adaptive control technique, a finite-time adaptive control scheme is established to realize the finite-time synchronization and to properly evaluate the unknown parameters. A detailed theoretical derivation and a specific numerical simulation demonstrate the feasibility and validity of the advanced scheme.

Keywords: finite-time adaptive control, modified function projective multiple-lag generalized compound synchronization, unknown parameter, chaotic systems.

1. Introduction

Due to its potential applications in secure communication, engineering science, biological systems and other fields, the chaotic synchronization problem has attracted increasing attention. The main idea of synchronization is to design an appropriate controller to make the state of the response system track the state of the drive system asymptotically (Song *et al.*, 2010; Lu *et al.*, 2010; Grzybowski *et al.*, 2009; Wang and Liu, 2007; Lee *et al.*, 2008; Liu *et al.*, 2012; Chen *et al.*, 2010; 2015; Mu and Chen, 2016).

Nevertheless, most of the previous research was primarily concerned with the asymptotic stability or exponential stability of the synchronization error system. In practice, especially in the field of engineering, optimizing the synchronization time is more important than achieving asymptotic synchronization. This implies optimality in settling time (Haimo, 1986). To realize fast synchronization, a finite-time control technique which

can drive the controlled system to its target within finite time has been formulated (Bhat and Bernstein, 2000). Moreover, compared with asymptotic control, the finite-time control method has many advanced properties, such as fast and finite-time convergence, robustness, the disturbance rejection ability and high tracking precision. Therefore, finite-time synchronization has attracted increasing attention in recent years (Liu *et al.*, 2018; Yu and Man, 2002; Wang *et al.*, 2009; Aghababa *et al.*, 2011).

So far, different synchronization schemes for chaotic systems have been proposed and abundant research and application results have been obtained, such as complete synchronization (CS) (Pecora and Carroll, 1990; Yu and Liu, 2003), anti-synchronization (AS) (Kim *et al.*, 2003), lag synchronization (Rosenblum *et al.*, 1997), intermittent lag synchronization (Boccaletti and Valladares, 2000), phase synchronization (Park *et al.*, 1999), generalized synchronization (Yang and Duan, 1998), projective synchronization (PS) (Mainieri and Rehacek, 1999; Wen and Xu, 2005), modified projective

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synchronization (MPS) (Cai *et al.*, 2010), function projective synchronization (FPS) (Du *et al.*, 2008), time scale synchronization (Hramov and Koronovskii, 2004), and so on. Recently, due to its higher security in secure communication, a more general FPS, called ‘modified function projective synchronization’ (MFPS) (Du *et al.*, 2009; Sudheer and Sabir, 2011), in which the drive system and the response system are synchronized up to a desired scaling function matrix, has attracted researchers attention. Therefore, research on MFPS is more valuable in practice. What is more, considering that time-delays exist widely in engineering, more recently, a general method called ‘modified function projective lag synchronization’ (MFPLS) for chaotic systems has been developed (Du *et al.*, 2010; Gao *et al.*, 2013; Wang and Wei, 2015; Wang *et al.*, 2016).

In practice, the level of communication security is dependent on the complexity level of the drive dynamical system and the signal loading scheme. If the driving system involves more chaotic subsystems and the way of combining subsystems is more complex, then the geometry topological manifold of the compound driving system becomes more complex. Further, the transmitted signal can be separated into several parts carried by different subsystems to improve the anti-attack ability and anti-translated capability.

The previous synchronization methods mentioned above involved only one drive system and one response system. To further improve the security of the synchronization technique in secure communication, combination synchronization containing two drive systems and one response system was proposed (Luo *et al.*, 2011; Luo and Wang, 2012). By adding another response system, Sun *et al.* (2014) extended combination synchronization to combination-combination synchronization among four chaotic systems. In order to further improve the anti-attack ability and anti-translated capability of the transmitted signals, Sun *et al.* (2015) introduced compound synchronization, in which the multiplication of two chaotic systems was considered to be the drive system. However, it is a pity that parameter uncertainty and the existence of synchronization-delay have not been taken into consideration in that article. Besides, it should be noted that chaotic systems and the controller studied in that paper are all specific. Thus the control method proposed is not general.

Moreover, as is well known, system parameters are probably partially unknown or even completely unknown, which may destroy the stability of the system. The adaptive control technique is considered to be an effective method to deal with uncertainties (Srinivasarengan *et al.*, 2018; Liu *et al.*, 2014; Li and Liu, 2017; Ben Brahim *et al.*, 2015; Kaczorek, 2016).

Therefore, how to combine finite time synchronization, and compound synchronization together with the adaptive control technique is an important issue. To the best of our knowledge, it is still open, which motivates our work.

In this paper, for a series of different chaotic systems with fully unknown parameters, finite-time adaptive modified function projective multi-lag generalized compound synchronization is investigated. Compared with the existing literature, there are two advantages which make the proposed scheme attractive. First, the modified function projective multi-lag generalized compound synchronization (MFPMGCS) is more general and more complex; it covers almost all of the existing synchronization methods. Second, the drive system is the compound of multiple chaotic systems by arithmetic operations of addition, subtraction and multiplication. In the process of secret communication, the transmitted signals can be split into several different parts and loaded in different sub-drive systems to improve the security of signal transmission. In addition, by adopting the multiplication of chaotic systems, the diameter of the compound chaotic topological manifold become much longer, which means that more types of signals can be transmitted.

The rest of this paper is organized as follows. Section 2 formulates the system model to be studied. In Section 3 some basic definitions and useful lemmas are introduced, and then the synchronization problem under investigation is proposed. In Section 4, based on Lyapunov stability theory, a controller is designed to realize finite-time compound synchronization, while an adaptive law is proposed to accurately estimate the unknown parameters. Section 5 presents a simulation example, and Section 6 concludes the paper and outlines and the future work.

2. System description

In our drive-response type compound synchronization scheme, $N_1 + N_2 + N_3$ different chaotic systems with unknown parameters are considered as the drive systems.

The l -th base drive system is given as

$$\begin{cases} \dot{x}_1^l(t) &= F_1^l(x^l(t))\theta^l + f_1^l(x^l(t)), \\ \dot{x}_2^l(t) &= F_2^l(x^l(t))\theta^l + f_2^l(x^l(t)), \\ &\vdots \\ \dot{x}_n^l(t) &= F_n^l(x^l(t))\theta^l + f_n^l(x^l(t)), \end{cases} \quad l = 1, 2, \dots, N_1. \quad (1)$$

The m -th scaling drive system is written as

$$\begin{cases} \dot{y}_1^m(t) = G_1^m(y^m(t))\phi^m + g_1^m(y^m(t)), \\ \dot{y}_2^m(t) = G_2^m(y^m(t))\phi^m + g_2^m(y^m(t)), \\ \vdots \\ \dot{y}_n^m(t) = G_n^m(y^m(t))\phi^m + g_n^m(y^m(t)), \end{cases} \quad m = 1, 2, \dots, N_2. \quad (2)$$

The j -th additive drive system is described as

$$\begin{cases} \dot{z}_1^j(t) = H_1^j(z^j(t))\eta^j + h_1^j(z^j(t)), \\ \dot{z}_2^j(t) = H_2^j(z^j(t))\eta^j + h_2^j(z^j(t)), \\ \vdots \\ \dot{z}_n^j(t) = H_n^j(z^j(t))\eta^j + h_n^j(z^j(t)), \end{cases} \quad j = 1, 2, \dots, N_3. \quad (3)$$

Meanwhile, the response system is described as

$$\begin{cases} \dot{w}_1(t) = R_1(w(t))\psi + r_1(w(t)) + u_1(t), \\ \dot{w}_2(t) = R_2(w(t))\psi + r_2(w(t)) + u_2(t), \\ \vdots \\ \dot{w}_n(t) = R_n(w(t))\psi + r_n(w(t)) + u_n(t), \end{cases} \quad (4)$$

where $x^l = [x_1^l, x_2^l, \dots, x_n^l]^T \in \mathbb{R}^n$ represents the state vector of the l -th base drive system, $y^m = [y_1^m, y_2^m, \dots, y_n^m]^T \in \mathbb{R}^n$ denotes the state vector of the m -th scaling system, $z^j = [z_1^j, z_2^j, \dots, z_n^j]^T \in \mathbb{R}^n$ refers to the state vector of the j -th additive drive system, $w = [w_1, w_2, \dots, w_n]^T \in \mathbb{R}^n$ stands for the state vector of the response system, $f_i^l(x^l(t))$, $g_i^m(y^m(t))$, $h_i^j(z^j(t))$, and $r_i(w(t))$ are continuous nonlinear functions, $F_i^l(x^l(t))$, $G_i^m(y^m(t))$, $H_i^j(z^j(t))$ and $R_i(w(t))$ are the i -th rows of the continuous linear function matrices $F^l(x^l(t))$, $G^m(y^m(t))$, $H^j(z^j(t))$ and $R(w(t))$, respectively, $\theta^l = [\theta_1^l, \theta_2^l, \dots, \theta_n^l]^T$, $\phi^m = [\phi_1^m, \phi_2^m, \dots, \phi_n^m]^T$, $\eta = [\eta_1^j, \eta_2^j, \dots, \eta_n^j]^T$ and $\psi = [\psi_1, \psi_2, \dots, \psi_n]^T$ are unknown parameter vectors, $u = [u_1, u_2, \dots, u_n]^T$ is the control input.

3. Preliminary definition and lemmas

In this section, we introduce several precise definitions and important lemmas which are necessary for our further study.

Assumption 1. The unknown parameters θ^l , ϕ^m , η^j and ψ are all bounded, that is to say, there exist known positive constants $\bar{\theta}^l$, $\bar{\phi}^m$, $\bar{\eta}^j$ and $\bar{\psi}$ such that

$$\begin{aligned} \|\theta^l\| &\leq \bar{\theta}^l, & \|\phi^m\| &\leq \bar{\phi}^m, \\ \|\eta^j\| &\leq \bar{\eta}^j, & \|\psi\| &\leq \bar{\psi}, \end{aligned} \quad (5)$$

where $l = 1, 2, \dots, N_1$, $m = 1, 2, \dots, N_2$, $j = 1, 2, \dots, N_3$, and $\|\cdot\|$ refers to the 2-norm.

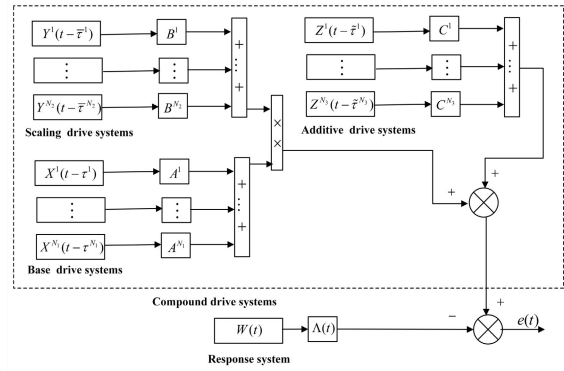


Fig. 1. Framework of MFPMLGCS.

Lemma 1. (Xu *et al.*, 2016) Assume that a continuous and positive definite function $V(t)$ is a Lyapunov candidate of a system. If $V(t)$ satisfies

$$\dot{V}(t) \leq -\alpha_1 V(t) - \alpha_2 V^\rho(t), \quad t \geq 0, \quad V(0) \geq 0, \quad (6)$$

where $\alpha_1 > 0$, $\alpha_2 > 0$ and $0 < \rho < 1$ are constants. Then the system is exponentially stable in a finite time T which is given by

$$T = \frac{1}{\alpha_1(1-\rho)} \ln \left(1 + \frac{\alpha_1 V^{1-\rho}(0)}{\alpha_2} \right). \quad (7)$$

Lemma 2. (Wang *et al.*, 2009) Suppose that a_1, a_2, \dots, a_n, v are all real numbers and $0 < v < 2$. Then

$$\begin{aligned} |a_1|^v + |a_2|^v + \dots + |a_n|^v \\ \geq (a_1^2 + a_2^2 + \dots + a_n^2)^{\frac{v}{2}}. \end{aligned}$$

Before presenting the definitions, let us introduce the following notation:

$$\begin{aligned} X^l(t) &= \text{diag}\{x_1^l(t), x_2^l(t), \dots, x_n^l(t)\}, \\ Y^m(t) &= \text{diag}\{y_1^m(t), y_2^m(t), \dots, y_n^m(t)\}, \\ Z^j(t) &= \text{diag}\{z_1^j(t), z_2^j(t), \dots, z_n^j(t)\}, \\ W(t) &= \text{diag}\{w_1(t), w_2(t), \dots, w_n(t)\}. \end{aligned}$$

Definition 1. We say that the three groups of drive systems (1)–(3) and the response system (4) are a modified function projective multiple-lag generalized compound synchronization (MFPMLGCS) if there exist $N_1 + N_2 + N_3$ different delay times τ^l ($l = 1, 2, \dots, N_1$), $\bar{\tau}^m$ ($m = 1, 2, \dots, N_2$), $\tilde{\tau}^j$ ($j = 1, 2, \dots, N_3$), $N_1 + N_2 + N_3$ constant scaling diagonal matrices

$$\begin{aligned} A^l &= \text{diag}\{a_1^l, a_2^l, \dots, a_n^l\}, \\ B^m &= \text{diag}\{b_1^m, b_2^m, \dots, b_n^m\}, \\ C^j &= \text{diag}\{c_1^j, c_2^j, \dots, c_n^j\} \end{aligned}$$

and a function scaling diagonal matrix

$$\Lambda(t) = \text{diag}\{\lambda_1(t), \lambda_2(t), \dots, \lambda_n(t)\},$$

such that

$$\lim_{t \rightarrow \infty} \left\| \sum_{m=1}^{N_2} B^m Y^m(t - \bar{\tau}^m) \sum_{l=1}^{N_1} A^l X^l(t - \tau^l) + \sum_{j=1}^{N_3} C^j Z^j(t - \tilde{\tau}^j) - \Lambda(t)W(t) \right\| = 0, \quad (8)$$

or

$$\lim_{t \rightarrow \infty} \left| \sum_{m=1}^{N_2} \sum_{l=1}^{N_1} b_i^m a_i^l x_i^l(t - \tau^l) y_i^m(t - \bar{\tau}^m) + \sum_{j=1}^{N_3} c^j z^j(t - \tilde{\tau}^j) - \lambda_i(t)w_i(t) \right| = 0, \quad i = 1, 2, \dots, n, \quad (9)$$

where $\Lambda(t)$ is a reversible function matrix and each element of $\Lambda(t)$ is a continuously differentiable nonzero function with a bound. This novel synchronization scheme is shown in Fig.1.

Definition 2. We say that $N_1 + N_2 + N_3$ systems (1)–(3) realize a finite-time modified function projective multi-lag generalized compound synchronization with the response system (4) if there exists a real constant $T > 0$ such that

$$\lim_{t \rightarrow T_-} \left\| \sum_{m=1}^{N_2} B^m Y^m(t - \bar{\tau}^m) \sum_{l=1}^{N_1} A^l X^l(t - \tau^l) + \sum_{j=1}^{N_3} C^j Z^j(t - \tilde{\tau}^j) - \Lambda(t)W(t) \right\| = 0, \quad (10)$$

and for any $t \geq T$

$$\left\| \sum_{m=1}^{N_2} B^m Y^m(t - \bar{\tau}^m) \sum_{l=1}^{N_1} A^l X^l(t - \tau^l) + \sum_{j=1}^{N_3} C^j Z^j(t - \tilde{\tau}^j) - \Lambda(t)W(t) \right\| = 0. \quad (11)$$

It is easy to see that (10) is equivalent to

$$\lim_{t \rightarrow T_-} \left| \sum_{m=1}^{N_2} \sum_{l=1}^{N_1} b_i^m a_i^l x_i^l(t - \tau^l) y_i^m(t - \bar{\tau}^m) + \sum_{j=1}^{N_3} c^j z^j(t - \tilde{\tau}^j) - \lambda_i(t)w_i(t) \right| = 0, \quad i = 1, 2, \dots, n. \quad (12)$$

Remark 1. As shown in Table 1, the proposed MFPMLGCS is more general, and it covers most of the previous synchronization schemes. Selecting a specific scaling matrix and specific delay times, MFPMLGCS will be reduced to different specific ones. Here $\Lambda = \text{diag}\{\lambda_1, \dots, \lambda_n\}$ and I denotes an $n \times n$ unit matrix.

Remark 2. In practice, the level of communication security is dependent on the complexity level of the drive dynamical system and the signal loading scheme. The drive system in MFPMLGCS is a compound of multiple chaotic systems. Not only the addition and subtraction, but also the multiplication of multiple chaotic systems is involved. This means the signal hidden channels are more diversified and the signal hidden methods are more flexible. Therefore, the complexity effectively improves the anti-attack ability and the anti-translated capability of the communication scheme.

To deal with the finite-time MFPMLGCS problem, the error vector of synchronization is defined as follows:

$$e(t) = \left[\sum_{m=1}^{N_2} B^m Y^m(t - \bar{\tau}^m) \sum_{l=1}^{N_1} A^l X^l(t - \tau^l) + \sum_{j=1}^{N_3} C^j Z^j(t - \tilde{\tau}^j) - \Lambda(t)W(t) \right] \Gamma, \quad (13)$$

where $\Gamma = [1, 1, \dots, 1]^T \in \mathbb{R}^n$. Furthermore, the element $e_i(t)$ of the vector $e(t)$ can be obtained as

$$e_i(t) = \sum_{m=1}^{N_2} \sum_{l=1}^{N_1} b_i^m a_i^l x_i^l(t - \tau^l) y_i^m(t - \bar{\tau}^m) + \sum_{j=1}^{N_3} c^j z^j(t - \tilde{\tau}^j) - \lambda_i(t)w_i(t), \quad (14)$$

where $i = 1, 2, \dots, n$.

Taking the time derivative of $e_i(t)$ and applying (1)–(4), the error dynamic system can be obtained as follows:

$$\begin{aligned} \dot{e}_i(t) &= \sum_{m=1}^{N_2} \sum_{l=1}^{N_1} b_i^m a_i^l [x_i^l(t - \tau^l) g_i^m(y^m(t - \bar{\tau}^m)) \\ &\quad + y_i^m(t - \bar{\tau}^m) f_i^l(x^l(t - \tau^l))] \\ &\quad + \sum_{m=1}^{N_2} \sum_{l=1}^{N_1} b_i^m a_i^l [x_i^l(t - \tau^l) G_i^m(y^m(t - \bar{\tau}^m)) \phi^m \\ &\quad + y_i^m(t - \bar{\tau}^m) F_i^l(x^l(t - \tau^l)) \theta^l] \\ &\quad + \sum_{j=1}^{N_3} c_i^j h_i^j(z^j(t - \tilde{\tau}^j)) + \sum_{j=1}^{N_3} c_i^j H_i^j(z^j(t - \tilde{\tau}^j)) \eta^j \\ &\quad - \dot{\lambda}_i(t)w_i(t) - \lambda_i(t)r_i(w(t)) - \lambda_i(t)R_i(w(t))\psi \\ &\quad - \lambda_i(t)u_i(t). \end{aligned} \quad (15)$$

Table 1. Comparison of the proposed synchronization scheme with other existing ones.

Parameter selection	Synchronous name	Mathematical model
	MFPMLGCS	$e(t) = [\sum_{m=1}^{N_2} B^m Y^m(t - \bar{\tau}^m) \sum_{l=1}^{N_1} A^l X^l(t - \tau^l) + \sum_{j=1}^{N_3} C^j Z^j(t - \tilde{\tau}^j) - \Lambda(t)W(t)]\Gamma$
Case 1 $N_1 = N_2 = N_3 = 2$	a simple MFPMLGCS	$e(t) = \{[B^1 Y^1(t - \bar{\tau}^1) + B^2 Y^2(t - \bar{\tau}^2)][A^1 X^1(t - \tau^1) + A^2 X^2(t - \tau^2)] + [C^1 Z^1(t - \tilde{\tau}^1) + C^2 Z^2(t - \tilde{\tau}^2)] - \Lambda(t)W(t)\}\Gamma$
Case 2 $N_2 = 1, N_1 = 2, N_3 = 0, \bar{\tau}^1 = \tau^l = 0, \Lambda(t) = 0$	compound synchronization	$e(t) = \{BY(t)[A^1 X^1(t) + A^2 X^2(t)] - \Lambda(t)W(t)\}\Gamma$
Case 3 $N_1 = N_2 = 0, N_3 = 2, \tilde{\tau}^j = 0, \Lambda(t) = 0$	combination synchronization	$e(t) = C^1 z^1(t) + C^2 z^2(t) - \Lambda w(t)$
Case 4 $N_1 = N_2 = 0, N_3 = 1, C^1 = I$	MFPLS	$e(t) = z(t - \tilde{\tau}) - \Lambda(t)w(t)$
Case 5 $N_1 = N_2 = 0, N_3 = 1, C^1 = I, \tilde{\tau}^1 = 0$	MFPS	$e(t) = z(t) - \Lambda(t)w(t)$
Case 6 $N_1 = N_2 = 0, N_3 = 1, C^1 = I, \tilde{\tau}^1 = 0, \Lambda(t) = \Lambda$	PS	$e(t) = z(t) - \Lambda w(t)$
Case 7 $N_1 = N_2 = 0, N_3 = 1, C^1 = \Lambda(t) = I, \tilde{\tau}^1 = 0$	complete synchronization	$e(t) = z(t) - w(t)$
Case 8 $N_1 = N_2 = 0, N_3 = 1, C^1 = -\Lambda(t) = I, \tilde{\tau}^1 = 0$	anti-synchronization	$e(t) = z(t) + w(t)$

For convenience, write

$$\begin{aligned}
 &S_i(x^l, y^m, z^j, w) \\
 &= \sum_{m=1}^{N_2} \sum_{l=1}^{N_1} b_i^m a_i^l [x_i^l(t - \tau^l) g_i^m(y^m(t - \bar{\tau}^m)) \\
 &\quad + y_i^m(t - \bar{\tau}^m) f_i^l(x^l(t - \tau^l))] \\
 &\quad + \sum_{j=1}^{N_3} c_i^j h_i^j (z^j(t - \tilde{\tau}^j)) - \lambda_i(t) r_i(w(t)) \\
 &\quad - \dot{\lambda}_i(t) w_i(t), \tag{16}
 \end{aligned}$$

and

$$\bar{u}_i(t) = \lambda_i(t) u_i(t). \tag{17}$$

Then, the error dynamic system (15) is reduced as follows:

$$\begin{aligned}
 \dot{e}_i(t) &= \sum_{m=1}^{N_2} \sum_{l=1}^{N_1} b_i^m a_i^l [x_i^l(t - \tau^l) G_i^m(y^m(t - \bar{\tau}^m)) \phi^m \\
 &\quad + y_i^m(t - \bar{\tau}^m) F_i^l(x^l(t - \tau^l)) \theta^l] \\
 &\quad + \sum_{j=1}^{N_3} c_i^j H_i^j (z^j(t - \tilde{\tau}^j)) \eta^j - \lambda_i(t) R_i(w(t)) \psi \\
 &\quad + S_i(x^l, y^m, z^j, w) - \bar{u}_i(t). \tag{18}
 \end{aligned}$$

4. Design of a finite-time adaptive control scheme

In fact, the finite-time MFPMLGCS problem is directly equivalent to finite-time stabilization of the error system (18). In this section, we will devote our efforts to design

an adaptive control scheme to ensure the trajectory of each error element $e_i(t)$ converges to zero within a finite time.

In order to achieve finite-time synchronization, the controller is designed as follows:

$$\begin{aligned}
 u_i(t) &= \frac{1}{\lambda_i(t)} \left\{ \sum_{m=1}^{N_2} \sum_{l=1}^{N_1} b_i^m a_i^l [x_i^l(t - \tau^l) G_i^m(y^m(t - \bar{\tau}^m)) \right. \\
 &\quad \cdot \hat{\phi}^m + y_i^m(t - \bar{\tau}^m) F_i^l(x^l(t - \tau^l)) \hat{\theta}^l] \\
 &\quad + \sum_{j=1}^{N_3} [c_i^j H_i^j (z^j(t - \tilde{\tau}^j)) \hat{\eta}^j] + S_i(x^l, y^m, z^j, w) \\
 &\quad - \lambda_i(t) R_i(w(t)) \hat{\psi} + k_i e_i + \rho_i e_i^{\frac{q}{p}} \\
 &\quad \left. + \Omega \cdot \Delta_i \right\}, \tag{19}
 \end{aligned}$$

where the positive constants k_i and ρ_i are the control gains which will be designed according to the requirements of the designer. $\hat{\theta}^l, \hat{\phi}^m, \hat{\eta}^j$ and $\hat{\psi}$ represent the estimations of the unknown parameters θ^l, ϕ^m, η^j and ψ , respectively, $k^* = \min\{k_1, \dots, k_n\}, \rho^* = \min\{\rho_1, \dots, \rho_n\}, i = 1, 2, \dots, n$, and

$$\begin{aligned}
 \Omega &= \sigma_1 \left[\sum_{l=1}^{N_1} (\|\hat{\theta}^l\| + \bar{\theta}^l)^2 + \sum_{m=1}^{N_2} (\|\hat{\phi}^m\| + \bar{\phi}^m)^2 \right. \\
 &\quad \left. + \sum_{j=1}^{N_3} (\|\hat{\eta}^j\| + \bar{\eta}^j)^2 + (\|\hat{\psi}\| + \bar{\psi})^2 \right] + \sigma_2
 \end{aligned}$$

$$\begin{aligned} & \cdot \left[\sum_{l=1}^{N_1} (\|\hat{\theta}^l\| + \bar{\theta}^l)^{\frac{p+q}{p}} + \sum_{m=1}^{N_2} (\|\hat{\phi}^m\| + \bar{\phi}^m)^{\frac{p+q}{p}} \right. \\ & \left. + \sum_{j=1}^{N_3} (\|\hat{\eta}^j\| + \bar{\eta}^j)^{\frac{p+q}{p}} + (\|\hat{\psi}\| + \bar{\psi})^{\frac{p+q}{p}} \right], \\ \Delta_i &= \begin{cases} \frac{e_i}{\|e\|^2} & \text{if } \|e\| \neq 0, \\ 0 & \text{if } \|e\| = 0. \end{cases} \end{aligned} \quad (20)$$

Meanwhile, the following adaptive law is given to estimate the unknown parameters:

$$\begin{aligned} \dot{\hat{\theta}}^l &= \left\{ A^l \left[\sum_{m=1}^{N_2} B^m Y^m(t - \bar{\tau}^m) \right] F^l(x^l(t - \tau^l)) \right\}^T e, \\ \hat{\theta}^l(0) &= \hat{\theta}_0^l, \quad l = 1, 2, \dots, N_1, \\ \dot{\hat{\phi}}^m &= \left\{ B^m \left[\sum_{l=1}^{N_1} A^l X^l(t - \tau^l) \right] G^m(y^m(t - \bar{\tau}^m)) \right\}^T e, \\ \hat{\phi}^m(0) &= \hat{\phi}_0^m, \quad m = 1, 2, \dots, N_2, \\ \dot{\hat{\eta}}^j &= [C^j H^j(z^j(t - \bar{\tau}^j))]^T e, \quad \hat{\eta}^j(0) = \hat{\eta}_0^j, \\ j &= 1, 2, \dots, s, \\ \dot{\hat{\psi}} &= -[\Lambda(t)R(w(t))]^T e, \quad \hat{\psi}(0) = \hat{\psi}_0. \end{aligned} \quad (21)$$

Theorem 1. Under the actions of the controller (19) and the adaptive law (21), $N_1 + N_2 + N_3 + 1$ systems (1)–(4) can achieve MFPLGCS in a finite time T given by

$$T = \frac{p}{\mu(p-q)} \ln \left(1 + \frac{2\mu V^{\frac{p-q}{2p}}(0)}{2^{\frac{p+q}{2p}} \gamma} \right), \quad (22)$$

where $\mu = \min\{k^*, \sigma_1\}$, $\gamma = \min\{\rho^*, \sigma_2\}$, p and q are two positive odd integers satisfying $p > q$, and $p + q$ is an even number.

Proof. Choosing the following Lyapunov function:

$$V(t) = V_1(t) + V_2(t), \quad (23)$$

in which

$$\begin{aligned} V_1(t) &= \frac{1}{2} \|e(t)\|^2, \\ V_2(t) &= \frac{1}{2} \left(\sum_{l=1}^{N_1} \|\hat{\theta}^l - \theta^l\|^2 + \sum_{m=1}^{N_2} \|\hat{\phi}^m - \phi^m\|^2 \right. \\ & \left. + \sum_{j=1}^{N_3} \|\hat{\eta}^j - \eta^j\|^2 + \|\hat{\psi} - \psi\|^2 \right), \end{aligned} \quad (24)$$

when $\|e(t)\| \neq 0$, and substituting the controller (19) into (18), we get

$$\begin{aligned} \dot{e}_i(t) &= \sum_{l=1}^{N_1} \sum_{m=1}^{N_2} b_i^m a_i^l [x_i^l(t - \tau^l) G_i^m(y^m(t - \bar{\tau}^m)) \\ & \cdot (\phi^m - \hat{\phi}^m) + y_i^m(t - \bar{\tau}^m) F_i^l(x^l(t - \tau^l)) \\ & \cdot (\theta^l - \hat{\theta}^l)] + \sum_{j=1}^{N_3} c_i^j H_i^j(z^j(t - \bar{\tau}^j)) (\eta^j - \hat{\eta}^j) \\ & - \lambda_i(t) R_i(w(t)) (\psi - \hat{\psi}) - k_i e_i - \rho_i e_i^{\frac{q}{p}} \\ & - \Omega \cdot \frac{e_i}{\|e(t)\|^2}. \end{aligned} \quad (25)$$

Taking the time derivative of $V_1(t)$ along the error system and using the fact that

$$\sum_{i=1}^n e_i \cdot \frac{e_i}{\|e(t)\|^2} = 1,$$

we obtain

$$\begin{aligned} \dot{V}_1(t) &= e^T \dot{e} = \sum_{i=1}^n e_i \dot{e}_i \\ &= \sum_{l=1}^{N_1} \sum_{m=1}^{N_2} \sum_{i=1}^n e_i b_i^m a_i^l x_i^l(t - \tau^l) \\ & \cdot G_i^m(y^m(t - \bar{\tau}^m)) (\phi^m - \hat{\phi}^m) \\ & + \sum_{l=1}^{N_1} \sum_{m=1}^{N_2} \sum_{i=1}^n e_i b_i^m a_i^l y_i^m(t - \bar{\tau}^m) \\ & \cdot F_i^l(x^l(t - \tau^l)) (\theta^l - \hat{\theta}^l) \\ & + \sum_{j=1}^{N_3} \sum_{i=1}^n e_i c_i^j H_i^j(z^j(t - \bar{\tau}^j)) (\eta^j - \hat{\eta}^j) \\ & - \sum_{i=1}^n e_i \lambda_i(t) R_i(w(t)) (\psi - \hat{\psi}) \\ & - \sum_{i=1}^n k_i e_i^2 - \sum_{i=1}^n \rho_i e_i^{\frac{p+q}{p}} - \Omega. \end{aligned} \quad (26)$$

Meanwhile, the time derivative of $V_2(t)$ is calculated as

$$\begin{aligned} \dot{V}_2(t) &= \sum_{l=1}^{N_1} (\hat{\theta}^l - \theta^l)^T \dot{\hat{\theta}}^l + \sum_{m=1}^{N_2} (\hat{\phi}^m - \phi^m)^T \dot{\hat{\phi}}^m \\ & + \sum_{j=1}^{N_3} (\hat{\eta}^j - \eta^j)^T \dot{\hat{\eta}}^j + (\hat{\psi} - \psi)^T \dot{\hat{\psi}} \\ & = \sum_{l=1}^{N_1} \{ (\hat{\theta}^l - \theta^l)^T \sum_{m=1}^{N_2} [B^m A^l Y^m(t - \bar{\tau}^m) \end{aligned} \quad (27)$$

$$\begin{aligned}
 & \cdot F^l(x^l(t - \tau^l))]^T e\} + \sum_{m=1}^{N_2} \{(\hat{\phi}^m - \phi^m)^T \\
 & \cdot \sum_{l=1}^{N_1} [B^m A^l X^l(t - \tau^l) G^m(y^m(t - \bar{\tau}^m))]^T e\} \\
 & + \sum_{j=1}^{N_3} (\eta^j - \hat{\eta}^j)^T [C^j H^j(z^j(t - \tilde{\tau}^j))]^T e \\
 & - (\hat{\psi} - \psi)^T [\Lambda(t) R(w(t))]^T e. \tag{28}
 \end{aligned}$$

Combining (26) with (27), and using the facts that

$$\begin{aligned}
 & \sum_{l=1}^{N_1} \sum_{i=1}^n e_i b_i^m a_i^l x_i^l(t - t^l) G_i^m(y^m(t - \bar{t}^m)) \\
 & \cdot (\phi^m - \hat{\phi}^m) \\
 & = (\phi^m - \hat{\phi}^m)^T \left\{ B^m \left[\sum_{l=1}^{N_1} A^l X^l(t - t^l) \right] G^m(y^m(t - \bar{t}^m)) \right\}^T e, \\
 & \sum_{m=1}^{N_2} \sum_{i=1}^n e_i b_i^m a_i^l y_i^m(t - \bar{t}^m) F_i^l(x^l(t - t^l)) (\theta^l - \hat{\theta}^l) \\
 & = (\theta^l - \hat{\theta}^l)^T \left\{ A^l \left[\sum_{m=1}^{N_2} B^m Y^m(t - \bar{t}^m) \right] \right. \\
 & \quad \left. \cdot F^l(x^l(t - t^l)) \right\}^T e, \\
 & \sum_{j=1}^{N_3} \sum_{i=1}^n e_i c_i^j H_i^j(z^j(t - \tilde{t}^j)) (\eta^j - \hat{\eta}^j) \\
 & = \sum_{j=1}^{N_3} (\eta^j - \hat{\eta}^j)^T [C^j Z^j(t - \tilde{t}^j) H^j(z^j(t - \tilde{t}^j))]^T e,
 \end{aligned}$$

and

$$\begin{aligned}
 & - \sum_{i=1}^n e_i \lambda_i(t) R_i(w(t)) (\psi - \hat{\psi}) \\
 & = -(\psi - \hat{\psi})^T [\Lambda(t) R(w(t))]^T e,
 \end{aligned}$$

we obtain

$$\begin{aligned}
 \dot{V}(t) & = \dot{V}_1(t) + \dot{V}_2(t) \\
 & = - \sum_{i=1}^n \left[k_i (e_i)^2 + \rho_i (e_i)^{\frac{p+q}{p}} \right] - \Omega \\
 & \leq -k^* \sum_{i=1}^n (e_i)^2 - \rho^* \sum_{i=1}^n (e_i)^{\frac{p+q}{p}} - \Omega.
 \end{aligned}$$

Since

$$\begin{aligned}
 -\Omega & \leq -\sigma_1 \left[\sum_{l=1}^{N_1} \|\hat{\theta}^l - \theta^l\|^2 + \sum_{m=1}^{N_2} \|\hat{\phi}^m - \phi^m\|^2 \right. \\
 & \quad \left. + \sum_{j=1}^{N_3} \|\hat{\eta}^j - \eta^j\|^2 + \|\hat{\psi} - \psi\|^2 \right] \\
 & - \sigma_2 \left[\sum_{l=1}^{N_1} \|\hat{\theta}^l - \theta^l\|^{\frac{p+q}{p}} + \sum_{m=1}^{N_2} \|\hat{\phi}^m - \phi^m\|^{\frac{p+q}{p}} \right. \\
 & \quad \left. + \sum_{j=1}^s \|\hat{\eta}^j - \eta^j\|^{\frac{p+q}{p}} + \|\hat{\psi} - \psi\|^{\frac{p+q}{p}} \right],
 \end{aligned}$$

we have

$$\begin{aligned}
 \dot{V}(t) & \leq -\mu \left[\sum_{i=1}^n (e_i)^2 + \sum_{l=1}^{N_1} \|\hat{\theta}^l - \theta^l\|^2 \right. \\
 & \quad \left. + \sum_{m=1}^{N_2} \|\hat{\phi}^m - \phi^m\|^2 + \sum_{j=1}^{N_3} \|\hat{\eta}^j - \eta^j\|^2 \right. \\
 & \quad \left. + \|\hat{\psi} - \psi\|^2 \right] - \gamma \left[\sum_{i=1}^n (e_i)^{\frac{p+q}{p}} + \sum_{l=1}^{N_1} \|\hat{\theta}^l - \theta^l\|^{\frac{p+q}{p}} \right. \\
 & \quad \left. + \sum_{m=1}^{N_2} \|\hat{\phi}^m - \phi^m\|^{\frac{p+q}{p}} + \sum_{j=1}^{N_3} \|\hat{\eta}^j - \eta^j\|^{\frac{p+q}{p}} \right. \\
 & \quad \left. + \|\hat{\psi} - \psi\|^{\frac{p+q}{p}} \right] \\
 & = -2\mu V(t) - \gamma \left[\sum_{i=1}^n (e_i)^{\frac{p+q}{p}} + \sum_{l=1}^{N_1} \|\hat{\theta}^l - \theta^l\|^{\frac{p+q}{p}} \right. \\
 & \quad \left. + \sum_{m=1}^{N_2} \|\hat{\phi}^m - \phi^m\|^{\frac{p+q}{p}} + \sum_{j=1}^{N_3} \|\hat{\eta}^j - \eta^j\|^{\frac{p+q}{p}} \right. \\
 & \quad \left. + \|\hat{\psi} - \psi\|^{\frac{p+q}{p}} \right]. \tag{29}
 \end{aligned}$$

According to Lemma 2, we get

$$\begin{aligned}
 \dot{V}(t) & \leq -2\mu V(t) - \gamma \left[\sum_{i=1}^n (e_i)^2 + \sum_{l=1}^{N_1} \|\hat{\theta}^l - \theta^l\|^2 \right. \\
 & \quad \left. + \sum_{m=1}^{N_2} \|\hat{\phi}^m - \phi^m\|^2 + \sum_{j=1}^{N_3} \|\hat{\eta}^j - \eta^j\|^2 \right. \\
 & \quad \left. + \|\hat{\psi} - \psi\|^2 \right]^{\frac{p+q}{2p}} \\
 & = -2\mu V(t) - 2^{\frac{p+q}{2p}} \gamma V^{\frac{p+q}{2p}}(t). \tag{30}
 \end{aligned}$$

From Lemma 1 it follows that the error trajectory $e_i(t)$ converges to zero within a finite time T given by (22), i.e., the finite time MFPLGCS of (1)–(4) is achieved. This completes the proof. \blacksquare

Remark 3. In the adaptive control laws (19)–(20), the control gain vectors $k = (k_1, \dots, k_n)$ and $\rho = (\rho_1, \dots, \rho_n)$ determine the speed at which the error approaches zero, while σ_1 and σ_2 determine the speed at which the unknown parameters are tracked. If the main purpose of controller design is to implement fast synchronization, rather than fast tracking of parameters, one can choose sufficiently small σ_1 and σ_2 to reduce the cost.

Remark 4. In practice, the term Δ_i in Eqn. (20) is often improved as $e_i/(\|e\|^2 + \varepsilon)$ or

$$\Delta_i = \begin{cases} \frac{e_i}{\|e\|^2} & \text{if } \|e\| \geq \delta, \\ 0 & \text{if } \|e\| < \delta, \end{cases} \quad (31)$$

where the switching gain ε or δ is a sufficiently small positive constant which can be preset according to the accuracy requirement in a practical problem. This method is commonly used in dealing with the problem of finite-time control or sliding mode control (Liu et al., 2013a; Liu et al., 2013b; Cheng et al., 2018; Fedele et al., 2018; Xia et al., 2018).

5. Numerical simulation

In this section, a Rössler system and a Lü system with fully unknown parameters are chosen as the base drive systems, a Chen system is considered as the scaling drive systems while a Liu system is used as the response system.

The above four famous chaotic systems can be described as follows:

- Rössler system,

$$\begin{pmatrix} \dot{x}_1^1 \\ \dot{x}_2^1 \\ \dot{x}_3^1 \end{pmatrix} = \underbrace{\begin{pmatrix} -x_2^1 - x_3^1 & 0 & 0 \\ x_1^1 & x_2^1 & 0 \\ 0 & 0 & -x_3^1 \end{pmatrix}}_{F^1(x^1(t))} \underbrace{\begin{pmatrix} 1 \\ 0.2 \\ 5.7 \end{pmatrix}}_{\theta^1} + \underbrace{\begin{pmatrix} 0 \\ 0 \\ x_1^1 x_3^1 + 0.2 \end{pmatrix}}_{f^1(x^1(t))}.$$

- Lü system,

$$\begin{pmatrix} \dot{x}_1^2 \\ \dot{x}_2^2 \\ \dot{x}_3^2 \end{pmatrix} = \underbrace{\begin{pmatrix} x_2^2 - x_1^2 & 0 & 0 \\ 0 & x_2^2 & 0 \\ 0 & 0 & -x_3^2 \end{pmatrix}}_{F^2(x^2(t))} \underbrace{\begin{pmatrix} 36 \\ 20 \\ 3 \end{pmatrix}}_{\theta^2} + \underbrace{\begin{pmatrix} 0 \\ -x_1^2 x_3^2 \\ x_1^2 x_2^2 \end{pmatrix}}_{f^2(x^2(t))}.$$

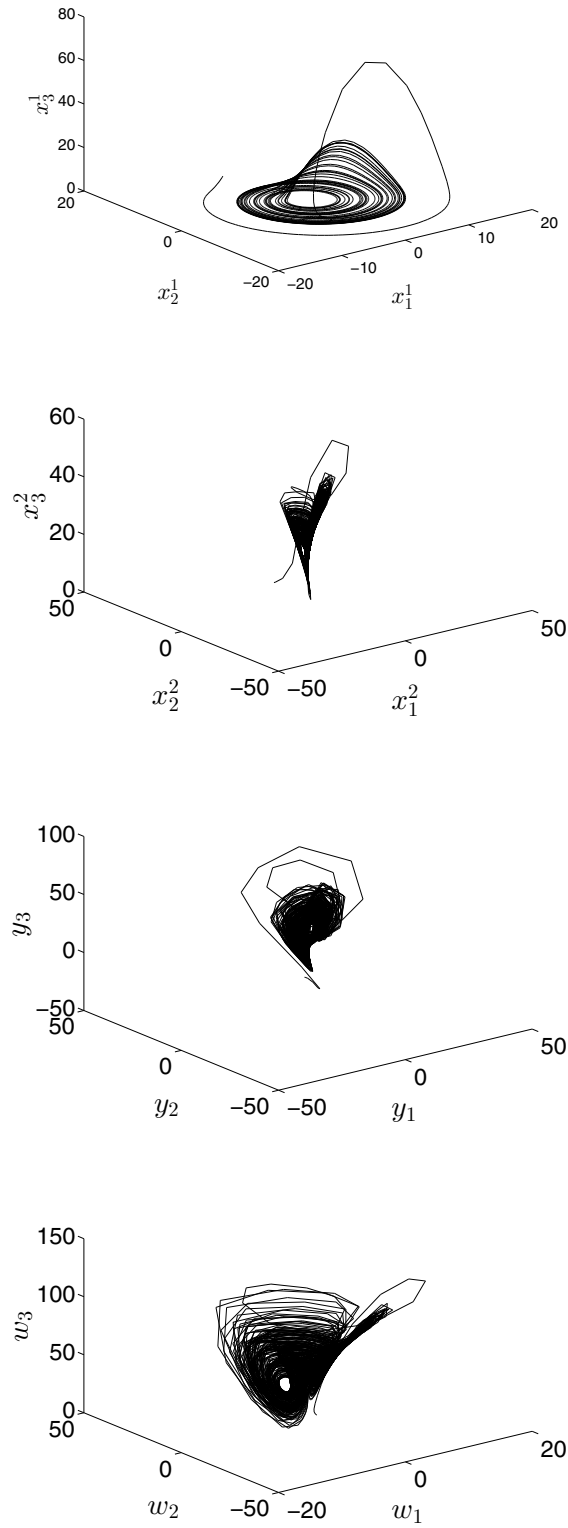


Fig. 2. Phase portraits of the chaotic systems involved.

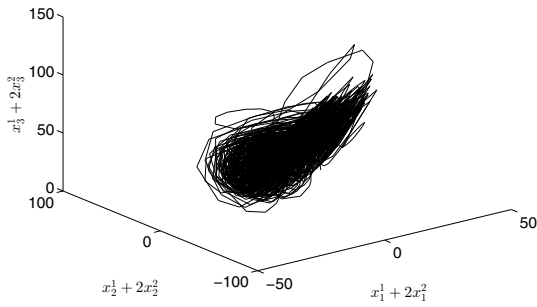


Fig. 3. Phase portrait of the combined drive system $x^1 + 2x^2$.

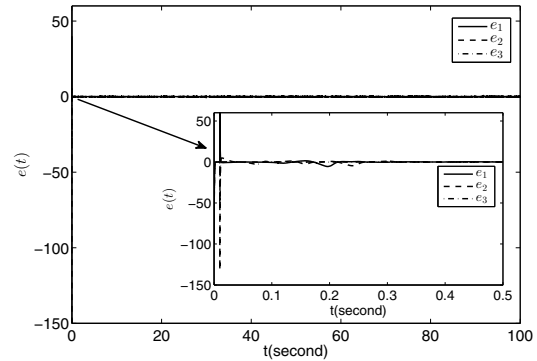


Fig. 5. Time response of the MFPMLGCS error e .

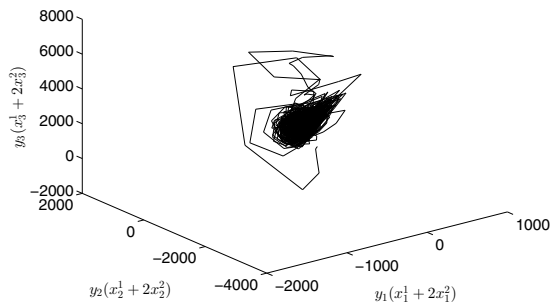


Fig. 4. Phase portrait of the compound drive system $y(x^1 + 2x^2)$.

In the simulation, the drive systems are initialized with $x^1(0) = x^2(0) = (-5, 11, 4.5)$ and $y(0) = (-6, -6, -6)$, while the response system is started from $w(0) = (2, 2, 2)$; the control gains are selected as $k = (1500, 1500, 1500)$, $\rho = (60, 60, 60)$, $p = 5, q = 1$ and $\sigma_1 = \sigma_2 = 0.01$, while the bound vectors are given as $\hat{\theta}^l = 10, \hat{\theta}^2 = \hat{\phi} = \hat{\psi} = 50$. Choose $\tau^0 = \tau^1 = \tau^2 = 0.1$ and take the following scaling matrices:

$$A^1 = A^2 = B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

and $\Lambda(t) = \text{diag}\{10 + 0.1 \sin t, 10 - 0.1 \sin t, 10 + 0.1 \cos t\}$. The simulation results are shown in Figs. 2–6.

The comparison of Figs. 2 and 3 shows that the compound drive system becomes more complex without loss of its chaotic characteristic. In addition, Figs. 3 and 4 show that, compared with the combined drive system, by replacing the constant proportional coefficient with a scaling-driven system, the diameter of the chaotic manifold under the compound scheme becomes much longer, which means more types of signals can be transmitted.

As revealed in Fig. 5, under the actions of the controller (19) and the modified term Δ_i given by (30), in which $\delta = 0.01$, each MFPMLGCS error $e_i(t)$ converges to zero within a very short time. This is revealed in Fig. 5. Meanwhile, from Fig. 6 one can see that $\hat{\theta}^l, \hat{\phi}$ and $\hat{\psi}$ converge to the values θ^l, ϕ and ψ , respectively. The simulation results illustrate the effectiveness of the proposed synchronization scheme.

- Chen system,

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \end{pmatrix} = \underbrace{\begin{pmatrix} y_2 - y_1 & 0 & 0 \\ -y_1 & y_1 + y_2 & 0 \\ 0 & 0 & -y_3 \end{pmatrix}}_{G(y(t))} \underbrace{\begin{pmatrix} 35 \\ 28 \\ 3 \end{pmatrix}}_{\phi} + \underbrace{\begin{pmatrix} 0 \\ -y_1 y_3 \\ y_1 y_2 \end{pmatrix}}_{g(y(t))}.$$

- Liu system,

$$\begin{pmatrix} \dot{w}_1 \\ \dot{w}_2 \\ \dot{w}_3 \end{pmatrix} = \underbrace{\begin{pmatrix} w_2 - w_1 & 0 & 0 \\ 0 & w_1 & 0 \\ 0 & 0 & -w_3 \end{pmatrix}}_{R(w(t))} \underbrace{\begin{pmatrix} 10 \\ 40 \\ 2.5 \end{pmatrix}}_{\psi} + \underbrace{\begin{pmatrix} 0 \\ -w_1 w_3 \\ 4(w_1)^2 \end{pmatrix}}_{r(w(t))} + \underbrace{\begin{pmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{pmatrix}}_{u(t)}.$$

6. Conclusion

In this paper, the problem of finite-time MFPMLGCS for multiple different chaotic systems with fully unknown parameters was studied. Based on the adaptive control technique and Lyapunov stability theory, an adaptive control scheme was designed to realize

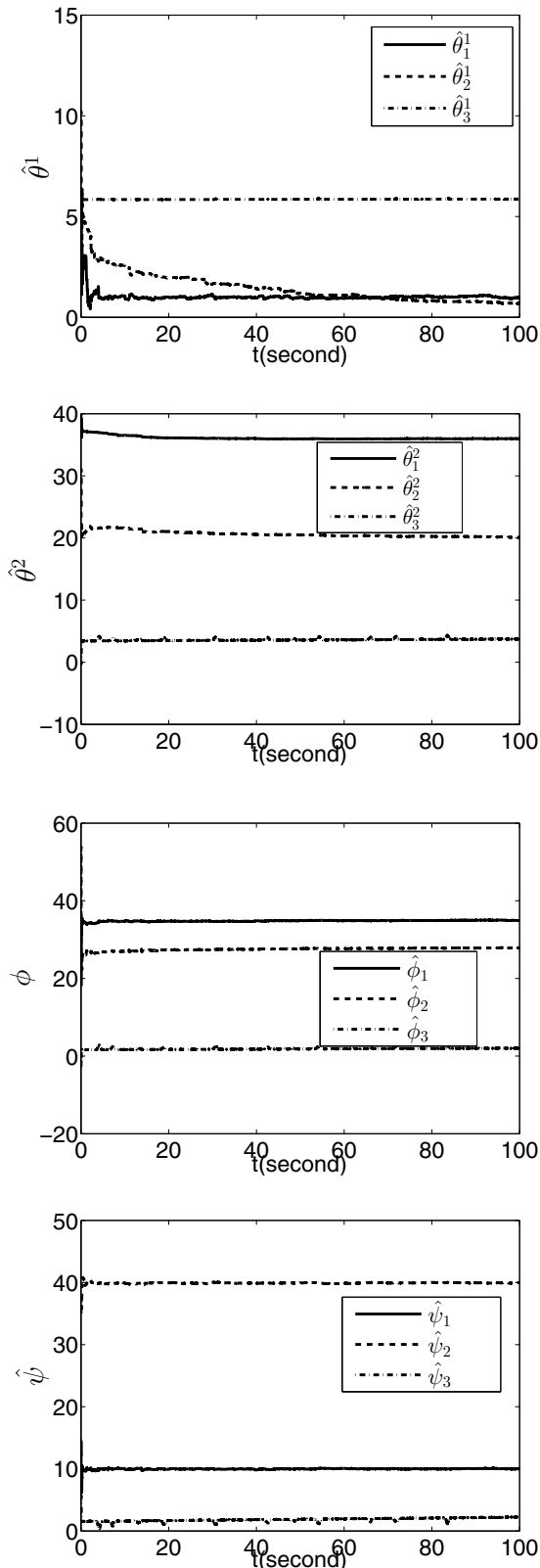


Fig. 6. Estimated values of the unknown parameters.

finite-time synchronization and parameter tracking. A theoretical proof and numerical simulation demonstrated the correctness and effectiveness of the advanced scheme. The future work covers application of this novel synchronization scheme and improvement of the control technology.

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