DE GRUYTER OPEN

Int. J. Appl. Math. Comput. Sci., *2016*, *Vol. 26*, *No. 4*, *803–813*
DOI: 10.1515/amcs-2016-0057

amcs

SET–MEMBERSHIP IDENTIFIABILITY OF NONLINEAR MODELS AND RELATED PARAMETER ESTIMATION PROPERTIES

CARINE JAUBERTHIE[a], LOUISE TRAVÉ-MASSUYÈS[a,*],
NATHALIE VERDIÈRE[b]

[a] LAAS-CNRS
University of Toulouse, UPS, 7 avenue du Colonel Roche, 31400 Toulouse, France
email: {cjaubert,louise}@laas.fr

[b] UNIHAVRE, LMAH
Normandy University, FR-CNRS-3335, ISCN, 76600 Le Havre, France
email: verdiern@univ-lehavre.fr

Identifiability guarantees that the mathematical model of a dynamic system is well defined in the sense that it maps unambiguously its parameters to the output trajectories. This paper casts identifiability in a set-membership (SM) framework and relates recently introduced properties, namely, SM-identifiability, μ-SM-identifiability, and ε-SM-identifiability, to the properties of parameter estimation problems. Soundness and ε-consistency are proposed to characterize these problems and the solution returned by the algorithm used to solve them. This paper also contributes by carefully motivating and comparing SM-identifiability, μ-SM-identifiability and ε-SM-identifiability with related properties found in the literature, and by providing a method based on differential algebra to check these properties.

Keywords: identifiability, bounded uncertainty, set-membership estimation, nonlinear dynamic models.

1. Introduction

Identifiability is an important concept that decides to what extent the parameter values of a mathematical model can be *uniquely* inferred from input-output measurements, assuming that the model has the same structure as the system (Nelles, 2002). Mathematically, this means that there exists an unambiguous mapping between the model parameters and the output trajectories. Identifiability is hence a pre-condition for safely running a parameter estimation algorithm and obtaining reliable results.

In the last years, there has been quite a lot of emphasis on bounded-error models, as opposed to stochastic ones, for achieving several tasks, e.g., fault diagnosis and fault tolerant control (Puig, 2010; Seybold *et al.*, 2015), robust robot localization (Kieffer *et al.*, 2000), reachability analysis (Auer *et al.*, 2013; Maiga *et al.*, 2016). This has been stressed by the success of operational estimation methods aiming at computing sets guaranteed to contain the *feasible parameter/state set*, i.e., the set of all the parameter/state vectors consistent with the specified bounds. This is why bounded, or also called *set-membership*, estimation is qualified as *guaranteed* (Kieffer *et al.*, 2002). In this paper, we use the term *set-membership*, abbreviated as SM, having in mind that the type of sets can be of different kinds, such as ellipsoids (Kurzhanski and Valyi, 1997), boxes (Kieffer and Walter, 2011), parallelotopes (Chiscii *et al.*, 1996), zonotopes (Alamo *et al.*, 2005), or other polytopes.

Interval analysis has brought a set of tools that indifferently apply to linear and nonlinear systems (Jaulin *et al.*, 2001) as opposed to ellipsoidal and zonotope-based estimation methods. Furthermore, its efficiency has been considerably enhanced by recent constraint propagation techniques (Chabert and Jaulin, 2009; Kieffer and Walter, 2011; Maiga *et al.*, 2013), resulting in the most appropriate paradigm to deal with nonlinearities.

Identifiability of SM nonlinear models has been shown to give rise to three concepts: SM-identifiability, μ-SM-identifiability, and ε-SM-identifiability that we introduced earlier (Jauberthie *et al.*, 2011; 2013). In this paper, we are interested in the way these properties impact the SM parameter estimation (SM-PE) problem. This

*Corresponding author.
The authors are listed in alphabetical order.

issue is characterized by two new properties. *Soundness* guarantees that the feasible parameter set (FPS) is reduced to one single bounded connected set. On the other hand, ε-*consistency* is a numerical property that guarantees that the FPS and the solution set returned by a parameter estimation algorithm with precision ε are composed of an equal number of mutually disjoint connected sets. While abundant literature exists on SM-PE (Jaulin *et al.*, 2001; Raïssi *et al.*, 2004; Kieffer and Walter, 2011; Milanese *et al.*, 2013; Herrero *et al.*, 2016), these problems have never been discussed in relation to SM-identifiability.

The paper is organized as follows. Section 2 recalls the definitions of SM-identifiability, μ-SM-identifiability, and ε-SM-identifiability, and a method for checking these properties. Section 3 brings a first contribution with a thorough analysis of the links between these ones and related properties existing in the literature. Section 4 introduces the properties of SM-PE problems, namely, soundness and ε-consistency, as Sections 5 and 6 derive the conditions that guarantee these properties. Finally, Section 7 concludes the paper and discusses perspectives of the work.

2. Set-membership identifiability

This section resumes the framework proposed by Jauberthie *et al.* (2013) for SM-identifiability for the class of systems formalized below.

2.1. Class of systems.

The models considered in this paper are bounded-error uncertain nonlinear models, controlled or uncontrolled, of the following form:

$$\Gamma = \begin{cases} \dot{x}(t,p) = f(x(t,p), u(t), p), \\ y(t,p) = h(x(t,p), p), \\ x(t_0, p) = x_0 \in X_0, \\ p \in P \subset \mathcal{U}_{\mathcal{P}},\ t_0 \leq t \leq T, \end{cases} \tag{1}$$

where

- $x(t,p) \in \mathbb{R}^n$ and $y(t,p) \in \mathbb{R}^m$ denote the state variables and the outputs at time, t respectively;
- $u(t) \in \mathbb{R}^r$ is the input vector at time t; in the case of uncontrolled models, $u(t)$ is equal to 0;
- the initial conditions x_0, if any, are assumed to belong to a bounded set X_0, and one assumes that X_0 does not contain equilibrium points of the system;
- the parameter vector p belongs to a connected set P assumed to be included in $\mathcal{U}_{\mathcal{P}}$, where $\mathcal{U}_{\mathcal{P}} \subseteq \mathbb{R}^p$ is an *a priori* known set of admissible parameters; the components of p are denoted by p_i;
- the functions f and h are real and analytic[1] on M, where M is an open set of $\mathbb{R}^n$ such that $x(t,p) \in M$ for every $t \in [t_0,\ T]$ and $p \in P$, T is a finite or infinite time bound.

In the following, $Y(P)$ denotes the set of output trajectories, the solution of Γ for any $p \in P$, and is also called the output of Γ arising from P. P^c denotes the complement of P in $\mathcal{U}_{\mathcal{P}}$.

2.2. Useful concepts.

Let us consider a nonempty connected set Π of $\mathbb{R}^p$, $\|\cdot\|$ a usual norm on $\mathbb{R}^p$ and d its associated distance.

The *distance*[2] between two sets Π_1 and Π_2 of $\mathbb{R}^p$ is defined by

$$d(\Pi_1, \Pi_2) = \min_{\pi_1 \in \Pi_1, \pi_2 \in \Pi_2} d(\pi_1, \pi_2).$$

Let us define $\delta(\Pi)$ as the *diameter* of Π. $\delta(\Pi)$ is given by the least upper bound of $\{d(\pi_1,\ \pi_2),\ \ \pi_1, \pi_2 \in \Pi\}$. If Π is not bounded, we define $\delta(\Pi) = +\infty$ (Bourbaki, 1989). On the metric space (Π, d), let μ be a continuous map from Π to Π. As an extension of the definition of contraction by Munkres (1975), we define μ as a *set contraction* if there is a nonnegative number $k < 1$ such that for all $\Pi_1,\ \Pi_2 \subseteq \Pi$, $d(\mu(\Pi_1), \mu(\Pi_2)) < kd(\Pi_1, \Pi_2)$. In the following, $\|\cdot\|$ denotes the Euclidean norm, $\|\cdot\|_\infty$ the maximum norm, and $\|\cdot\|_1$ the norm 1. These may be defined on $\mathbb{R}^\alpha$, where $\alpha \in \{n, p, m\}$, depending on the case.

2.3. Definitions.

The proposed definitions are given for controlled systems but they can be formulated in a similar manner for uncontrolled ones assuming that $u(t) = 0$.

Definition 1. Given the model Γ expressed by (1), consider a nonempty connected set $P^* \subseteq \mathcal{U}_{\mathcal{P}}$ and another set $\bar{P} \subseteq \mathcal{U}_{\mathcal{P}}$. Then P^* is *globally SM-identifiable* if there exists an input u such that $Y(P^*) \neq \emptyset$ and $Y(P^*) \cap Y(\bar{P}) \neq \emptyset \Longrightarrow P^* \cap \bar{P} \neq \emptyset$.

Definition 1 states that a connected set P^* is globally SM-identifiable if the output of Γ arising from P^* does not share any trajectory with the output of Γ arising from any set $\bar{P} \subseteq P^{*c}$. As an example, consider the following nonlinear system of the form (1):

$$\dot{x} = x + t\cos(p), \quad x(t_0) = x_0, \tag{2}$$

[1] The assumption that f and h are analytic on M, and hence infinitely differentiable, is needed in Section 3.3 for the use of differential algebra. In particular, proving sufficiency in Theorem 1 requires $y(t,p)$ to be expressed as a Taylor series. This proof is provided by Jauberthie *et al.* (2013).

[2] To keep the concept intuitive, it is a deliberate abuse of language to call $d(\Pi_1, \Pi_2)$ a *distance* between the two sets Π_1 and Π_2 of $\mathbb{R}^p$, even though it does not satisfy all the assumptions of a distance, in particular the triangular inequality.

where p is a bounded-error parameter for which the admissible set is $\mathcal{U}_\mathcal{P} = [0, 2\pi]$. The solution of (2) is $x(t) = x_0e^t + (-1 - t + e^t)\cos(p)$. It is clear that this system is not globally identifiable. It is enough to notice that any pair $(p_1 = \pi - \alpha, p_2 = \pi + \alpha)$, with $\alpha \in [0, \pi]$, results in the same trajectory, since $\cos(\pi - \alpha) = \cos(\pi + \alpha)$ for $\alpha \in [0, \pi]$. However, the trajectories arising from any set $P^* = [\pi - \alpha, \pi + \alpha]$ with $\alpha \in [0, \pi]$ are different from any trajectory arising from other regions of the parameter space. P^* is then said to be *globally SM-identifiable*.

The definition of μ-SM-identifiability has been proposed to ensure that the set P^* may be contracted as small as desired while still retaining the SM-identifiability property. For this purpose, a contraction μ is applied to P^* and, by the Banach fixed-point theorem, it implies that the diameter of $\mu(P^*)$ tends to zero (Munkres, 1975).

Definition 2. A nonempty connected set $P^* \subseteq \mathcal{U}_\mathcal{P}$ is *globally μ-SM-identifiable* if $\mu(P^*)$ is globally SM-identifiable for any contraction μ from P^* to P^*.

This implies the following proposition.

Proposition 1. *If the nonempty connected set $P^* \subseteq \mathcal{U}_\mathcal{P}$ is globally μ-SM-identifiable, then it is globally SM-identifiable. The converse is not true.*

Proof. For the converse, consider the system (2) and the set $P^* = [\pi - \alpha, \pi + \alpha]$ with $\alpha \in [0, \pi]$ as before. P^* has been shown to be globally SM-identifiable but it is not μ-SM-identifiable since, assuming $\alpha_1, \alpha_2 \in]0, \pi[\,, \alpha_1 \geq \alpha_2$, any set $P_1^* = [\pi - \alpha_1, \pi - \alpha_2] \subseteq P^*$ shares trajectories with its complementary set P_1^{*c} that contains $[\pi + \alpha_2, \pi + \alpha_1]$. ∎

If the diameter of $\mu(P^*)$, $\delta(\mu(P^*))$, cannot be lower than ε without loosing SM-identifiability, we refer to *ε-SM-identifiability* (Jauberthie *et al.*, 2013).

Definition 3. Consider an SM-identifiable nonempty connected set $P^* \subseteq \mathcal{U}_\mathcal{P}$. Then P^* is *globally ε-SM-identifiable* if there exists a set contraction μ from P^* to P^* such that $\delta(\mu(P^*)) = \varepsilon$ and $\mu(P^*))$ is globally SM-identifiable, and for all $\tilde{\mu}$ such that $\tilde{\mu}(P^*) \subset \mu(P^*)$, $\tilde{\mu}(P^*)$ is not globally SM-identifiable.

To summarize, interpreting identifiability in the SM framework leads to two definitions, depending on whether one considers a set as a whole (SM-identifiability) or also cares about the properties of its proper subsets (μ-SM-identifiability). μ-SM-identifiability can be seen as subsuming classical identifiability in the sense that if P^* is μ-SM-identifiable, it implies that any $p \in P^*$ is identifiable in the classical sense (Ljung and Glad, 1994). ε-SM-identifiability is a kind of structural μ-SM-identifiability since subsets of a delimited diameter ε that are SM-identifiable although not μ-SM-identifiable are accepted. The reader is referred to the work of Jauberthie *et al.* (2011) for the extension to structural and local counterparts of these properties.

3. SM-identifiability and related concepts

The links between (μ)-SM-identifiability and classical and interval identifiability were provided by Jauberthie *et al.* (2011; 2013). In this section, we are interested in the links with ε-global identifiability (Braems *et al.*, 2001) and partial injectivity (Lagrange *et al.*, 2008). These links allow us to propose a method for checking (μ)-SM-identifiability.

3.1. Links with ε-global identifiability.

Global identifiability in $P^* \subset \mathcal{U}_\mathcal{P}$ (g.i.i.P^*) was proposed by Braems *et al.* (2001) as a means to provide a stronger conclusion than structural identifiability, guaranteeing that atypical regions of nonidentifiability do not exist in the parameter space.

Definition 4. Given $(u, x_0) \in \mathbb{R}^r \times X_0$, the parameter p_i is *globally identifiable in* P^* (g.i.i.P^*) if

$$\forall (p, \bar{p}) \in P^*,\ y(\cdot, p) \equiv y(\cdot, \bar{p}) \Rightarrow p_i = \bar{p}_i, \qquad (3)$$

and the parameter vector p is g.i.i.P^* if all its components are g.i.i.P^*.

The originality of Braems *et al.* (2001) is to propose a practical way to formulate the condition of Definition 4, which consists in checking the condition

$$\nexists (p, \bar{p}) \in P^* \times P^* \\ \text{such that } y(\cdot, p) \equiv y(\cdot, \bar{p}),\ \|\bar{p} - p\|_\infty > 0. \qquad (4)$$

This is a *constraint satisfaction problem* (CSP) that can be solved in a guaranteed way by *interval constraint propagation* (ICP). In practice, Braems *et al.* (2001) state that checking the condition (4) comes back to checking

$$\nexists (p, \bar{p}) \in P^* \times P^* \\ \text{such that } y(\cdot, p) \equiv y(\cdot, \bar{p}),\ \|\bar{p} - p\|_\infty > \varepsilon, \qquad (5)$$

which is defined as ε-g.i.i.P^*. We have the following results.

Proposition 2. *P^* is globally μ-SM-identifiable with respect to P^* (in the sense that $\mathcal{U}_\mathcal{P}$ is reduced to P^*) if and only if (4) is satisfied.*

Proof. Jauberthie *et al.* (2013) provided the proof that if P^* is globally μ-SM-identifiable, then equivalently any p in P^* is globally identifiable with respect to P^*, hence satisfying the conditions (3) and (4). ∎

Proposition 3. *If P^* is globally ε-SM-identifiable with respect to P^*, then the condition (5) is satisfied.*

Proof. P^* is globally ε-SM-identifiable (cf. Definition 3) with respect to P^* if and only if there exists some subset $\tilde{P} \subset P^*$ such that $\delta(\tilde{P}) = \varepsilon$ and the interior of $\tilde{P}$, denoted by int($\tilde{P}$), as well as any $\tilde{P}' \subseteq \tilde{P}$ is not globally SM-identifiable, hence not globally identifiable. In such a case, for all p, $\bar{p} \in P^* \backslash \mathrm{int}(\tilde{P})$ satisfies the condition (5). The converse is not true because, when the condition (5) is satisfied, it does not provide any information about subsets $\tilde{P}' \subseteq \tilde{P}$ such that $\delta(\tilde{P}) \leq \varepsilon$. ∎

From the above propositions, the condition (5) does not allow one to decide between μ-SM-identifiability and ε-SM-identifiability. It can be considered to check μ-SM-identifiability accepting a numerical precision of ε.

3.2. Links with partial injectivity.

The definition of partial injectivity of a function was introduced in Lagrange *et al.* (2008). This notion perfectly characterizes μ-SM-identifiability. A second definition named *restricted-partial injectivity* is proposed in this paper in order to characterize global SM-identifiability.

Definition 5. Consider a function $f : \mathcal{A} \to \mathcal{B}$ and any set $\mathcal{A}_1 \subseteq \mathcal{A}$. The function f is said to be a *partial injection* of $\mathcal{A}_1$ over $\mathcal{A}$, or $(\mathcal{A}_1, \mathcal{A})$-injective, if $\forall a_1 \in \mathcal{A}_1$, $\forall a \in \mathcal{A}$,

$$a_1 \neq a \Rightarrow f(a_1) \neq f(a).$$

f is said to be $\mathcal{A}$-injective if it is $(\mathcal{A}, \mathcal{A})$-injective.

In Lagrange *et al.* (2008), an algorithm based on interval analysis for testing the injectivity of a given differentiable function is presented and a solver called IAVIA (injectivity analysis using interval analysis) implemented in C++ is mentioned.[3] For a given function, the solver partitions a given box in two domains: a domain on which the function is partially injective and an indeterminate domain on which the function may or may not be injective.

In order to characterize global SM-identifiability, the notion of *restricted-partial injectivity* is introduced.

Definition 6. Consider a function $f : \mathcal{A} \to \mathcal{B}$ and any set $\mathcal{A}_1 \subseteq \mathcal{A}$. The function f is said to be a *restricted-partial injection* of $\mathcal{A}_1$ over $\mathcal{A}$, or $(\mathcal{A}_1, \mathcal{A})$-R-injective, if

$$\forall a_1 \in \mathcal{A}_1,\ \forall a \in \mathcal{A}_1^c,\ f(a_1) \neq f(a).$$

In the following proposition, partial injectivity and restricted partial injectivity are interpreted in terms of trajectories, and this formulation makes the direct link with the definition of SM-identifiability and μ-SM-identifiability possible.

Consider the set of outputs S_u arising from $\mathcal{U}_\mathcal{P}$ for a given input u.

[3] Let us notice that the solver IAVIA has been implemented for functions $f : \mathbb{R} \to \mathbb{R}^2$ and $f : \mathbb{R}^2 \to \mathbb{R}^2$.

Proposition 4. *Given the model Γ, P^* is globally SM-identifiable (resp. μ-SM-identifiable) for an input u if and only if the function $\varphi : \mathcal{U}_\mathcal{P} \to S_u : p \to y(\cdot, p)$ is $(P^*, \mathcal{U}_\mathcal{P})$-R-injective (resp. $(P^*, \mathcal{U}_\mathcal{P})$-injective).*

Proof.
(Necessity) From the definition of global SM-identifiability, P^* and its complementary do not share trajectories; hence, there do not exist common trajectories arising from these two sets, which implies that φ is $(P^*, \mathcal{U}_\mathcal{P})$-R-injective.

If P^* is μ-SM-identifiable, then the property of global SM-identifiability is satisfied for any $\mu(P^*)$, μ being a contraction from P^* to P^*, which implies that for any $\bar{P}$ included in the complement of $\mu(P^*)$, $Y(\mu(P^*))$ and $Y(\bar{P})$ have no common trajectories. In other words, from the Banach fixed-point theorem, the trajectory arising from $p \in P^*$ is different from any trajectory arising from $\mathcal{U}_\mathcal{P} \setminus \{p\}$ and hence φ is $(P^*, \mathcal{U}_\mathcal{P})$-injective.

(Sufficiency) If $\bar{P}$ is such that $P^* \cap \bar{P} = \emptyset$, $\bar{P}$ is included in the complementary of P^*, and since φ is $(P^*, \mathcal{U}_\mathcal{P})$-R-injective, there exist no common trajectories arising from these two sets; hence P^* is globally SM-identifiable.

Assume now that φ is $(P^*, \mathcal{U}_\mathcal{P})$-injective and that, for a contraction μ, $Y(\mu(P^*))$ and $Y(\bar{P})$ have common trajectories. Then these trajectories arise from the same parameter. This implies that $\mu(P^*)$ and $\bar{P}$ have a nonempty intersection and that P^* is globally μ-SM-identifiable. ∎

Corollary 1. *The following properties are equivalent:*

- *P^* is globally μ-SM-identifiable;*
- *the function $\varphi : \mathcal{U}_\mathcal{P} \to S_u : p \to y(\cdot, p)$ is $(P^*, \mathcal{U}_\mathcal{P})$-injective;*
- *the condition (4) is satisfied.*

Proof. The proof directly comes from Propositions 2 and 4. ∎

Corollary 2. *P^* is globally ε-SM-identifiable implies that φ is $(\tilde{P}, \mathcal{U}_\mathcal{P})$-R-injective, with $\tilde{P} \subseteq P^*$ and $\delta(\tilde{P}) \geq \varepsilon$. The converse is not true.*

Proof. The necessity part of the proof of Proposition 4 applies and the inverse is not true for the same reasons as in the proof of Proposition 1. ∎

Testing $(P^*, \mathcal{U}_\mathcal{P})$-injectivity or $(P^*, \mathcal{U}_\mathcal{P})$-R-injectivity numerically can be done with an adaptation of IAVIA (Lagrange *et al.*, 2008) but it does not allow us to decide between μ-SM-identifiability and ε-SM-identifiability or SM-identifiability and ε-SM-identifiability.

3.3. Differential algebra method to perform SM-identifiability analysis. Proposition 4 points at an operational method to check SM and μ-SM-identifiability provided that the function $\varphi : \mathcal{U}_{\mathcal{P}} \to S_u : p \to y(\cdot, p)$ that maps parameters and trajectories is known. Differential algebra (Kolchin, 1973) was shown to provide a way to derive an implicit form of this function (Jauberthie *et al.*, 2011) [4].

This method, whose main result is given by Theorem 1 below, is based on the use of relations linking outputs, inputs and parameters of the model. These relations are more precisely differential polynomials whose indeterminates are the variables y and u, and coefficients are rational expressions in p.

For obtaining such polynomials, the Rosenfeld–Groebner algorithm, which is an elimination algorithm (Boulier, 1994) implemented in the package DifferentialAlgebra of Maple, is an efficient tool. The Rosenfeld–Groebner algorithm is used to eliminate state variables—to obtain the relations linking only outputs, inputs and parameters. With the elimination order $\{p\} < \{y, u\} < \{x\}$ (Kolchin, 1973; Denis-Vidal *et al.*, 2001b), several solutions are delivered by the algorithm. One is called the *characteristic presentation* because it corresponds to the general solution, the others being particular solutions.

The characteristic presentation contains differential polynomials linking outputs, inputs and parameters of the form

$$R_i(y, u, p) = m_0^i(y, u) + \sum_{k=1}^{n_i} \theta_k^i(p) m_k^i(y, u), \quad i = 1, \ldots, m, \tag{6}$$

where $(\theta_k^i(p))_{1 \le k \le n_i}$ are rational in p, $\theta_u^i \neq \theta_v^i$ $(u \neq v)$, $m_k^i(y, u))_{0 \le k \le n_i}$ are differential polynomials with respect to y, u and $m_0^i(y, u) \neq 0$. $\{\theta_k^i(p)\}_{1 \le k \le n_i}$ is called the exhaustive summary of R_i.

The size of the system is the number of outputs. For the time being, we assume that $i = 1$; that is, there is one output and $n_1 = n$, $R_1 = R$, $m_k^1(y, u) = m_k(y, u)$. The case of more outputs is considered at the end of this section.

Consider t_0^+ the right limit of t_0 [5] and l the higher order derivative of y in (6). $\Delta R(y, u)$ denotes the *functional determinant* formed from $\{m_k(y, u)\}_{1 \le k \le n}$ and given by the Wronskian (Denis-Vidal *et al.*, 2001a)

$$\Delta R(y, u) = \left| \begin{pmatrix} m_1(y, u) & \ldots & m_n(y, u) \\ m_1(y, u)^{(1)} & \ldots & m_n(y, u)^{(1)} \\ & \ddots & \\ m_1(y, u)^{(n-1)} & \ldots & m_n(y, u)^{(n-1)} \end{pmatrix} \right|. \tag{7}$$

Theorem 1. (Jauberthie *et al.*, 2011) *Assume that the functional determinant $\Delta R(y, u)$ is not identically equal to zero.*[6] *Consider P^* a connected subset of $\mathcal{U}_{\mathcal{P}}$. If the function defined by $\phi : p = (p_1, \ldots, p_p) \mapsto (\theta_1(p), \ldots, \theta_n(p), y(t_0^+, p), \ldots, y^{(l-1)}(t_0^+, p))$ is $(P^*, \mathcal{U}_{\mathcal{P}})$-R-injective (resp. $(P^*, \mathcal{U}_{\mathcal{P}})$-injective) then P^* is globally SM-identifiable (resp. μ-SM-identifiable). Furthermore, if for a contraction μ, $\mu(P^*)$ has a diameter equal to ε and ϕ is $(\mu(P^*), \mathcal{U}_{\mathcal{P}})$-R-injective but not $(\mu(P^*), \mathcal{U}_{\mathcal{P}})$-injective then P^* is ε-SM-identifiable. In the two cases, if the coefficient of $y^{(l)}$ in (6) is not equal to 0 at t_0, then the converse is valid.* [7]

Remark 1. If $m \ge 1$, then for each of the obtained m differential polynomials $R_i(y, u, p)$, the functional determinant is evaluated. If it is not identically equal to zero, the associated exhaustive summary is added to the image of the function ϕ for which (partial) injectivity has to be studied.

Theorem 1 has been used by Ravanbod *et al.* (2014) to provide an operational method for analyzing identifiability in an SM framework. First, the μ-SM-identifiable parameter subsets are determined with IAVIA. Then, determining the maxima and minima of the function ϕ allows one to assess SM-identifiable subsets and subsets that are neither SM nor μ-SM-identifiable.

4. SM parameter estimation and properties

In this section, the SM-PE problem is presented and two important properties are introduced, namely, *soundness* and *ε-consistency*. SM-identifiability is shown to play a key role in relation to this problem.

Classical parameter estimation considers a time series of noisy measured output data $y_m(t_i), i = 0, \ldots, h$, where $y_m(\cdot) \in \mathbb{R}^m$, generated by the real system on the interval $[0, T]$. The problem is formulated as finding the parameter vector p^* for which the outputs produced by the

[4] Another technique based on the power series expansion method inspired by Pohjanpalo (1978) was also proposed by Jauberthie *et al.* (2011).

[5] t_0^+ is considered to ensure the existence of derivatives.

[6] This assumption consists in verifying the linear independence of the $m_k(y, u)$, $k = 1, \ldots, n$. To this end, it is sufficient to find a time point at which the Wronskian is nonzero. In the framework of differential algebra, this condition consists in verifying that this functional determinant is not ideally obtained after eliminating state variables. In practice, it can be checked with the function Belong_To of the package Differential Algebra of Maple 16.

[7] When initial conditions are not considered, the function ϕ becomes $\phi : p = (p_1, \ldots, p_p) \mapsto (\theta_1(p), \ldots, \theta_n(p))$ and the converse of the theorem is not valid.

model best match the measured data according to some criterion. Minimal least squares is a common method, which is formulated as

$$p^* = \arg\min_{p\in\mathcal{U}_\mathcal{P}} \sum_{t=t_0}^{t_h} \|y_m(t) - y(t,p)\|^2.$$

The SM-PE problem assumes that the measured outputs are corrupted by bounded-error terms that may originate from the system parameters varying within specified bounds, bounded noise, or sensor precision such that $y_m(t_i) \in Y_m(t_i), i = 0,\dots,h$, where the $Y_m(t_i)$'s are connected sets of $\mathbb{R}^m$. The SM-PE problem is formulated as finding the set of parameter vectors $\mathcal{P} \subseteq \mathbb{R}^p$ such that the arising trajectories hit all the output data sets, i.e.,

$$p^* \in \mathcal{P} \Leftrightarrow y(t_i, p^*) \in Y_m(t_i), \quad \forall i = 0,\dots,h.$$

$\mathcal{P}$ is called the *feasible parameter set* (FPS). SM-PE problems are generally solved with a branch and bound algorithm that enumerates candidate solutions thanks to a rooted tree and assumes the full parameter space as the root set. At every node, the set of trajectories arising from the parameter set considered is checked for consistency against the measurements and labelled feasible, unfeasible or undetermined. Unfeasible sets are rejected while undetermined sets are split and checked in turn until the diameter of the candidate solution set is smaller than or equal to a given threshold ε provided by the user. Here ε is the *precision threshold* (or the *precision* for short) of the SM-PE algorithm. The SIVIA (set inversion via interval analysis) algorithm (Jaulin and Walter, 1993) can be cited to exemplify the above principles (branch and bound (bisection) and interval analysis). The number of bisections to be performed is generally prohibitive. Hence, recent algorithms take advantage of constraint propagation techniques to reduce the width of the boxes to be checked. In this context, the model is interpreted as the set of constraints of a constraint satisfaction problem (CSP). For solving such a CSP, various types of the so-called *contractors* can be used (Chabert and Jaulin, 2009).

It should be noted that such algorithms are anytime by nature, i.e., they provide a guaranteed solution independently of the stopping time, which redeems in some way their exponential complexity. The returned solution is an overestimation of the FPS given by the convex union of the candidates that have been labelled feasible and undetermined. Interestingly, the convex union may consist of one set or more (cf. the work of Jaulin *et al.* (2001) for several variants). In the following, we refer to the SM-PE algorithm as to a generic SM-PE algorithm based on these principles.

When considering an SM-PE problem, one would like to know beforehand whether or not $\mathcal{P}$ is reduced to one single connected set. In much the same may as for classical parameter estimation, this property indicates whether the problem is mathematically well-posed.

Definition 7. An SM-PE problem is said to be *sound* if $\mathcal{P} \subseteq \mathcal{U}_\mathcal{P}$ is reduced to one single connected set. In this case, $\mathcal{P}$ is also said to be sound.

Given an SM-PE algorithm with precision threshold ε, we denote by $\mathcal{P}_\varepsilon$ the solution set. Then it is important to know the properties of $\mathcal{P}_\varepsilon$ in relation to $\mathcal{P}$.

Definition 8. Assume that $\mathcal{P}$ is equal to the union of $\kappa \geq 1$ mutually disjoint connected sets. Then the solution set $\mathcal{P}_\varepsilon$ is said to be *ε-consistent* if $\mathcal{P}_\varepsilon$ is equal to the union of κ_ε mutually disjoint connected sets and $\kappa_\varepsilon = \kappa$.

Overdetermination and algorithm precision result in $\mathcal{P}_\varepsilon$ overestimating $\mathcal{P}$, which may imply $\kappa_\varepsilon < \kappa$. In this case, at least one of the sets composing $\mathcal{P}_\varepsilon$ includes several sets composing $\mathcal{P}$. ε-consistency is analyzed in Section 6.

5. Soundness

5.1. Conditions for soundness.

Proposition 5. *Consider the system Γ and assume that the set $\mathcal{P} \subseteq \mathcal{U}_\mathcal{P}$ is the FPS of an SM-PE problem for Γ. Then $\mathcal{P}$ is* sound *if and only if $\mathcal{P}$ is globally SM-identifiable.*

Proof. By definition, if $\mathcal{P}$ is a globally SM-identifiable set, those of Γ arising from $\mathcal{P}$ are different from the trajectories arising from the complement $\mathcal{P}^c = \mathcal{U}_\mathcal{P} \setminus \mathcal{P}$. In addition, $\mathcal{P}$ is connected, hence $\mathcal{P}$ is sound. Conversely, if $\mathcal{P}$ is sound, by definition it is globally SM-identifiable. ∎

In addition to being SM-identifiable, assume that $\mathcal{P}$ is μ-SM-identifiable for Γ. In this case, it is interesting to note that $\mathcal{P}$ preserves soundness when the bounded error corrupting the output data is getting smaller and smaller. In this case, $\mathcal{P}$ is said to be *μ-sound*. This is stated by the following result.

Proposition 6. *Given the output data sets $Y_m(t_i), i = 1,\dots,h$, assume that $\mathcal{P}$ is sound. Then, if $\mathcal{P}$ is μ-SM-identifiable, for Γ, the FPS of the same problem with contracted output data sets $\mu_i(Y_m(t_i)), i = 1,\dots,h$, where the μ_i's are contractions, is also sound.*

Proof. This proof uses Proposition 5. The result simply comes from the fact that if $\mathcal{P}$ is μ-SM-identifiable for Γ, then $\mathcal{P}$ is obviously SM-identifiable, and for all $P \subset \mathcal{P}$, P is also SM-identifiable. ∎

5.2. Example. Consider the model

$$\begin{cases} \dot{x}_1 = (p_1 + 2(1-p_2)\cos(p_1))x_1^2 + (1-p_2)x_2, \\ \dot{x}_2 = \sin(p_1)x_1, \\ y = x_1, \end{cases} \tag{8}$$

where $(p_1, p_2) \in [-1, 4] \times [0, 1/10] = \mathcal{U}_{\mathcal{P}}$.

By setting $c_1 = \sin(p_1)$, with the elimination order $\{c_1, p_2\} < \{y\} < \{x_1, x_2\}$, the Rosenfeld–Groebner algorithm gives the following differential polynomial:

$$\begin{aligned} &R(y, u) \\ &\quad = \ddot{y} - 2(p_1 + 2(1-p_2)\cos(p_1))\dot{y}y \\ &\qquad - (1-p_2)\sin(p_1)y. \end{aligned} \tag{9}$$

In that case, the functional determinant is reduced to $\triangle R(y) = \det(\dot{y}y, y) = -y^2\ddot{y}$ and is not identically equal to 0.

In order to consider the initial condition, the function $\phi : (p_1, p_2) \to ((p_1 + 2(1-p_2)\cos(p_1)), (1-p_2)\sin(p_1))$ has to be studied. By using the algorithm proposed in Ravanbod *et al.* (2014), Fig. 1 (right) is obtained. $\mathcal{U}_{\mathcal{P}} = [-1, 4] \times [0, 1/10]$ has been partitioned in two domains: a domain on which the function ϕ is partially injective and hence corresponding to μ-SM-identifiable subsets in grey color in the figure and two subsets in white color, each of them producing the same image.[8] If a parameter estimation problem is formulated such that the FPS $\mathcal{P}$ is in $\mathcal{U}_{\mathcal{P}}$, we can now decide whether or not $\mathcal{P}$ is sound. Indeed, if the inverse image of the trajectories hitting the output data sets entirely lies in a μ-SM-identifiable subset, then $\mathcal{P}$ is sound. On the contrary, $\mathcal{P}$ is unsound.

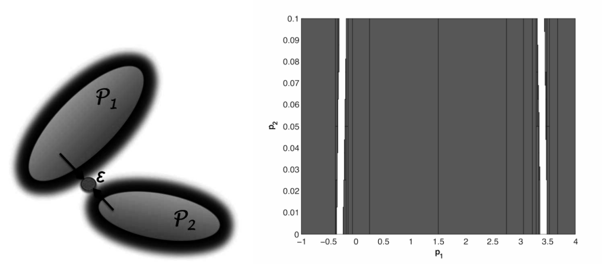

Fig. 1. Set of diameter ε interposed between $\mathcal{P}_1$, $\mathcal{P}_2$ and their undistinguishability neighborhoods (left), and the partition of the parameter domain: the grey color subsets are μ-SM-identifiable (right).

6. ε-consistency

6.1. Conditions for ε-consistency. As expressed in Definition 8, ε-consistency is a property of the solution set $\mathcal{P}_\varepsilon$ returned by the SM-PE algorithm with a specified precision threshold ε. Among the problems that impact ε-consistency, two are analyzed in this paper:

- the SM-PE algorithm may not be able to separate the mutually disjoint connected sets forming $\mathcal{P}$ by testing topologically relevant candidate solution sets;
- trajectories arising from solution parameters may not be distinguishable from trajectories arising from nonsolution parameters, given the precision of the sensors.

Proposition 7. *If the FPS $\mathcal{P}$ is sound, then the solution set $\mathcal{P}_\varepsilon$ is ε-consistent for any ε.*

Proof. If $\mathcal{P}$ is sound, it is reduced to one single connected set. Then, from the principle of branch and bound algorithms, the solution set $\mathcal{P}_\varepsilon$ is also reduced to one single connected set although it may be an overestimation of $\mathcal{P}$. ■

Let us now assume that $\mathcal{P}$ is unsound and consists of κ mutually disjoint connected sets, say $\mathcal{P}_i, i = 1, \ldots, \kappa$. The fact that the SM-PE algorithm is able to separate the $\mathcal{P}_i$'s is a topological problem involving the distance between the $\mathcal{P}_i$'s and the diameter of the smallest candidate solution sets considered by the branch and bound SM-PE algorithm.

Proposition 8. *If $\mathcal{P}$ is unsound and consists of κ mutually disjoint connected sets $\mathcal{P}_i, i = 1, \ldots, \kappa$, then a necessary condition for the solution set $\mathcal{P}_\varepsilon$ returned by the SM-PE algorithm with precision threshold ε to be ε-consistent is that $d(\mathcal{P}_i, \mathcal{P}_j) > \varepsilon$, $\forall i, j = 1, \ldots, \kappa, i \neq j$.*

Proof. With no loss of generality, consider two mutually disjoint connected sets $\mathcal{P}_1$ and $\mathcal{P}_2$. The successive partitions of the parameter space arising from the branch and bound procedure provide candidate solution sets whose diameter is greater than or equal to ε. $\mathcal{P}_1$ and $\mathcal{P}_2$ are separable if any candidate solution set of diameter ε can be *interposed* anywhere between the two sets, in particular just where the two sets are closest. [9] Only in such a case, i.e., if $d(\mathcal{P}_1, \mathcal{P}_2) > \varepsilon$, can the interposed candidate solution be labelled unfeasible, hence rejected by the algorithm, and $\mathcal{P}_\varepsilon$ composed of two mutually disjoint sets. ■

Let us now consider the second problem related to the fact that the output data sets $Y_m(t_i), i = 0, \ldots, h$, rely on sensors with a given precision λ, i.e., $v = v_{\text{mes}} \pm \lambda$, where v is the true value and v_{mes} the measured one. In this case, two trajectories $y(\cdot, p)$ and $y(\cdot, \bar{p})$ must be

[8] Notice that IAVIA gives no information about these two domains: they are labelled undetermined.

[9] *Interposed just where the two sets are closest* means that the candidate set can be aligned with the segment $[p_1, p_2]$ that connects the two points $p_1 \in \mathrm{Fr}(\mathcal{P}_1)$ and $p_2 \in \mathrm{Fr}(\mathcal{P}_2)$ which are at a minimum distance, and that its intersection with either $\mathcal{P}_1$ or $\mathcal{P}_2$ is empty.

distant by λ, i.e., be such that there exists $t \in [t_0, T]$, $\|y(t,p) - y(t,\bar{p})\|_\infty > \lambda$, to be distinguishable. If the trajectories arising from nonsolution parameters are not distinguishable from those arising from parameters of the the solution sets $\mathcal{P}_i$, then $\mathcal{P}_\varepsilon$ may not be ε-consistent.

The following proposition, whose proof is based on the Gronwall lemma, proves that, under some conditions, y is Lipschitz continuous with respect to the parameter vector. It provides the Lipschitz constant $K_{y,p}$ explicitly so that the conditions about parameters under which the output trajectories are distant by a given λ can be determined.

Recall first that, if a function g is real and analytic on M, an open set of $\mathbb{R}^n$, for every compact set $\mathcal{K} \subset M$, there exists a constant $K > 0$ such that, for every x in $\mathcal{K}$, the following bound holds:

$$\left\| \frac{\mathrm{d}g}{\mathrm{d}x}(x) \right\|_\infty \leq K,$$

Since f and h defining Γ are assumed to be real and analytic on M, assuming $x \in \mathcal{K}$, $\mathcal{K}$ to be a compact, they are Lipschitz continuous with respect to x. Their Lipschitz constants are respectively denoted $K_{f,x}$ and $K_{h,x}$.

Consider the following assumptions:

(i) f and h defined on $[t_0, T] \times \mathcal{U}_\mathcal{P}$ are Lipschitz continuous according to p, and their Lipschitz constants are respectively denoted $K_{f,p}$ and $K_{h,p}$;

(ii) the solution $x(t,p)$ of Γ is in the compact $\mathcal{K}$;

(iii) if the initial conditions depend on p, the function $p \mapsto x(t_0, p)$ is assumed to be Lipschitz continuous according to p and its Lipschitz constant is denoted by $K_{x_0,p}$.

Proposition 9. *Assume that the assumptions (i)–(iii) are satisfied. Then y is Lipschitz continuous with respect to p and its Lipschitz constant $K_{y,p}$ is given by*

$$K_{y,p} = K_{h,x}(K_{x_0,p} + K_{f,p}(T - t_0))e^{K_{f,x}(T-t_0)} + K_{h,p}.$$

If the initial conditions do not depend on p, $K_{y,p}$ is given by

$$K_{y,p} = K_{h,x}K_{f,p}(T - t_0)e^{K_{f,x}(T-t_0)} + K_{h,p}.$$

Proof. First, integrating both the sides of the equation $\dot{x}(t,p) = f(x(t,p), u(t), p)$ on $[0, t]$, we get

$$\begin{aligned}
&\|x(t,p) - x(t,\bar{p})\| \\
&\leq \|x(t_0,p) - x(t_0,\bar{p})\| \\
&\quad + \int_{t_0}^{t} \|f(x(s,p), u(s), p) - f(x(s,\bar{p}), u(s), \bar{p})\| \,\mathrm{d}s \\
&\leq \|x(t_0,p) - x(t_0,\bar{p})\| \\
&\quad + \int_{t_0}^{t} \|f(x(s,p), u(s), p) - f(x(s,p), u(s), \bar{p})\| \,\mathrm{d}s \\
&\quad + \int_{t_0}^{t} \|f(x(s,p), u(s), \bar{p}) - f(x(s,\bar{p}), u(s), \bar{p})\| \,\mathrm{d}s.
\end{aligned} \tag{10}$$

Using the assumption (i) about the Lipschitz continuity of f, we deduce that

$$\begin{aligned}
&\|x(t,p) - x(t,\bar{p})\| \\
&\quad \leq \|x(t_0,p) - x(t_0,\bar{p})\| \\
&\qquad + K_{f,p}(T - t_0)\|p - \bar{p}\| \\
&\qquad + K_{f,x} \int_{t_0}^{t} \|x(s,p) - x(s,\bar{p})\| \,\mathrm{d}s.
\end{aligned} \tag{11}$$

Then the application of the Gronwall lemma and the assumption (iii) gives

$$\begin{aligned}
&\|x(t,p) - x(t,\bar{p})\| \\
&\leq (K_{x_0,p} + K_{f,p}(T - t_0))\, \|p - \bar{p}\| e^{K_{f,x}(T-t_0)}.
\end{aligned} \tag{12}$$

Finally, using the hypothesis on h, the following inequalities are obtained:

$$\begin{aligned}
&\|y(t,p) - y(t,\bar{p})\| \\
&\leq \|h(x(t,p), p) - h(x(t,\bar{p}), p)\| \\
&\quad + \|h(x(t,\bar{p}), p) - h(x(t,\bar{p}), \bar{p})\| \\
&\leq K_{h,x}\|x(t,p) - x(t,\bar{p})\| + K_{h,p}\|p - \bar{p}\| \\
&\leq K_{h,x}\,(K_{x_0,p} + K_{f,p}(T - t_0))\, \|p - \bar{p}\| e^{K_{f,x}(T-t_0)} \\
&\quad + K_{h,p}\|p - \bar{p}\| \\
&\leq (K_{h,x}(K_{x_0,p} + K_{f,p}(T - t_0))e^{K_{f,x}(T-t_0)} \\
&\quad + K_{h,p})\|p - \bar{p}\|.
\end{aligned} \tag{13}$$

■

If the initial conditions do not depend on the parameters, then it is sufficient to set $K_{x_0,p} = 0$ in the previous proof.

From this result, one can determine the minimal distance between two parameter vectors p and $\bar{p}$ for which the trajectories $y(t,p)$ and $y(t,\bar{p})$ are ensured to be distant by λ. Applying the Lipschitz result of $y(t,p)$ with respect to p, the following corollary is deduced.

Corollary 3. *Let us consider two trajectories $y(\cdot, p)$ and $y(\cdot, \bar{p})$ arising from Γ. Then, under conditions (i)–(iii), if*

$$\exists t \in [t_0, T], \|y(t,p) - y(t,\bar{p})\|_\infty > \lambda$$

then

$$\|p - \bar{p}\| > \frac{\lambda}{K_{y,p}}.$$

Proof. Since y is Lipschitz continuous according to the parameter vector p, we get

$$\lambda < \|y(t,p) - y(t,\bar{p})\|_\infty < K_{y,p}\|p - \bar{p}\|, \tag{14}$$

which implies

$$\|p - \bar{p}\| > \frac{\lambda}{K_{y,p}}.$$

■

This result means that the $\mathcal{P}_i$'s forming $\mathcal{P}$ are surrounded by a neighborhood that may generate trajectories that are not distinguishable from those arising from their inside parameters.

Putting together the results of Proposition 8 and Corollary 3, we obtain the following condition for ε-consistency.

Proposition 10. *Consider the system Γ along with the assumptions (i)–(iii) and assume that $\mathcal{P}$ is unsound and consists of κ mutually disjoint connected sets $\mathcal{P}_i, i = 1, \dots, \kappa$. If the solution set $\mathcal{P}_\varepsilon$ returned by the SM-PE algorithm with precision threshold ε is ε-consistent, then*

$$d(\mathcal{P}_i, \mathcal{P}_j) > \varepsilon + \frac{2\lambda}{K_{y,p}}, \quad \forall i, j = 1, \dots, \kappa,$$

where λ is the precision of the sensors.

Proof. With no loss of generality, assume that $\mathcal{P}$ consists of two mutually disjoint connected sets $\mathcal{P}_1$ and $\mathcal{P}_2$. Denote by $\mathcal{N}_{\mathcal{P}_1}$ and $\mathcal{N}_{\mathcal{P}_2}$ the undistinguishability neighborhoods of $\mathcal{P}_1$ and $\mathcal{P}_2$, respectively. The sets to be separated by the SM-PE algorithm are hence $\mathcal{P}_1 \cup \mathcal{N}_{\mathcal{P}_1}$ and $\mathcal{P}_2 \cup \mathcal{N}_{\mathcal{P}_2}$. Then Proposition 8 applied to these sets implies $d(\mathcal{P}_1 \cup \mathcal{N}_{\mathcal{P}_1}, \mathcal{P}_2 \cup \mathcal{N}_{\mathcal{P}_2}) > \varepsilon$. The characterization of $\mathcal{N}_{\mathcal{P}_1}$ and $\mathcal{N}_{\mathcal{P}_2}$ provided by Corollary 3 then implies $d(\mathcal{P}_i, \mathcal{P}_j) > \varepsilon + 2\lambda/K_{y,p}$ as illustrated in Fig. 1 (left).

■

Remark 2. The converse is also true if the inclusion function $P \to [Y](P)$ used by the SM-PE algorithm to predict the set of trajectories arising from a given candidate parameter set P is such that $[Y](P) = Y(P)$, which is rarely the case.

6.2. Example.

Consider the following example defined on $[0, T]$:

$$\begin{cases} \dot{x}_1(t,p) = \cos(p)x_2(t,p),\ x_1(0) = (\frac{1}{\sqrt{2}} + 1)/2, \\ \dot{x}_2(t,p) = -x_1(t,p)\cos(p) + \big(1 - x_1(t,p)^2 \\ \qquad\qquad -2x_2(t,p)^2\big)\, x_2(t,p), \quad x_2(0) = 0, \\ y(t,p) = x_1(t,p). \end{cases} \tag{15}$$

The functions f and h are defined by

$$f(x,p) = (\cos(p)x_2, -x_1\cos(p) + (1 - x_1^2 - 2x_2^2)x_2)^T,$$

and $h(x,p) = x_1$ where $x = (x_1, x_2)^T$ and ^{T} denotes the transpose of the considered vector. The solution $(x_1(t), x_2(t))^T$ remains in the ring R defined by the two circles centered at $(0,0)$ with radii $1/\sqrt{2}$ and 1. Indeed, we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{x_1^2 + x_2^2}{2}\right) &= x_1\frac{\mathrm{d}x_1}{\mathrm{d}t} + x_2\frac{\mathrm{d}x_2}{\mathrm{d}t} \\ &= (1 - x_1^2 - 2x_2^2)x_2^2. \end{aligned}$$

Since $1 - x_1^2 - 2x_2^2$ is positive for $x_1^2 + x_2^2 < 1/2$ and negative for $x_1^2 + x_2^2 > 1$, $x_1^2 + x_2^2$ increases when $x_1^2 + x_2^2 < 1/2$ and decreases when $x_1^2 + x_2^2 > 1$. One can conclude that, according to the initial condition, the solution remains in the ring R.

The following step consists in finding the Lipschitz constants. Let us consider $z = (z_1, z_2)^T \in R$. Clearly, $K_{h,p} = 0$, $K_{h,x} = 1$ and $K_{f,p} = 1$ since $\|x\| = \|(x_1, x_2)^T\| < 1$. For $K_{f,x}$, by reordering the terms and adding $x_1^2 z_2 - x_1^2 z_2$ in line 3, we get

$$\begin{aligned} &\|f(x,p) - f(z,p)\|_1 \\ &\le |\cos(p)|(|x_2 - z_2| + |x_1 - z_1|) \\ &\quad + |x_2 - z_2| + |(x_1^2 - 2x_2^2)x_2 - (z_1^2 - 2z_2^2)z_2| \\ &\le (|x_1 - z_1| + 2|x_2 - z_2|) \\ &\quad + |x_1^2 x_2 - x_1^2 z_2 + x_1^2 z_2 \\ &\quad - z_1^2 z_2 - 2(x_2^3 - z_2^3)| \\ &\le (|x_1 - z_1| + 2|x_2 - z_2|) + x_1^2|x_2 - z_2| \\ &\quad + |z_2||x_1^2 - z_1^2| + 2|x_2^3 - z_2^3|. \end{aligned} \tag{16}$$

Since $|x_1| < 1$, $|z_1| < 1$, we deduce, on the one hand, that $|x_1^2 - z_1^2| \le |x_1 - z_1|(|x_1| + |z_1|) \le 2|x_1 - z_1|$, and on the other that

$$\begin{aligned} |z_2^3 - x_2^3| &= |x_2 - z_2||z_2^2 + 2x_2 z_2 + x_2^2| \\ &\le 4|x_2 - z_2|. \end{aligned}$$

Hence

$$\begin{aligned} &\|f(x,p) - f(z,p)\|_1 \\ &\le (|x_1 - z_1| + 2|x_2 - z_2|) \\ &\quad + |x_2 - z_2| + 2|x_1 - z_1| \\ &\quad + 8|x_2 - z_2| \le 11\|x - z\|_1. \end{aligned} \tag{17}$$

Using the equivalence between norms 1 and maximum norm, we get

$$\|f(x,p) - f(z,p)\|_\infty \le 22\|x - z\|.$$

Hence $K_{f,x} = 22$ and from Proposition 9 the Lipschitz constant $K_{y,p}$ is equal to Te^{22T}. Taking $[0,T] = [0,1]$, we get sensor precision $\lambda = 0.01$, and SM-PE algorithm precision threshold $\varepsilon = 0.001$. By Proposition 10 the solution set is ε-consistent, which implies that

$$\begin{aligned} d(\mathcal{P}_i, \mathcal{P}_j) &> \varepsilon + \frac{2\lambda}{K_{y,p}} \\ &\simeq 0.001 + 5.579 \cdot 10^{-12} \\ &\simeq 0.001, \quad \forall i,j = 1,\dots,\kappa. \end{aligned}$$

In this example, the SM-PE algorithm precision is dominant over the sensor precision with respect to ε-consistency.

7. Discussion and conclusions

This paper casts identifiability in an SM framework and relates the properties introduced by Jauberthie *et al.* (2011; 2013), namely, SM-/μ-SM-/ε-SM-identifiability, to the properties of SM-PE problems. Soundness and ε-consistency are proposed to characterize an SM-PE problem. Soundness is a theoretical property that assesses that the SM-PE problem is well-posed. Note that ε-consistency guarantees that the structure of the FPS is well reflected in the solution returned by the SM-PE algorithm.

SM-/μ-SM-/ε-SM-identifiability is compared with related properties existing in the literature, in particular partial injectivity. The differential algebra based method proposed to check these properties leads to checking partial-injectivity and a newly introduced property named partial-R-injectivity. The algorithm proposed for this (Ravanbod *et al.*, 2014) remains of exponential complexity like many interval-based algorithms, but it is still useful for medium-size problems.

Accordingly, ε-consistency is a complex property for which only necessary conditions are provided. It is impacted by several features of the SM-PE problem, including sensor precision and the overestimation involved in the computation of the image of a parameter set. Evaluating this overestimation and how it impacts ε-consistency remains an open problem.

Acknowledgment

This work was supported by the French National Research Agency (ANR) in the framework of the project ANR-11-INSE-006 (MAGIC-SPS).

References

Alamo, T., Bravo, J.M. and Camacho, E.F. (2005). Guaranteed state estimation by zonotopes, *Automatica* **41**(6): 1035–1043.

Auer, E., Kiel, S. and Rauh, A. (2013). A verified method for solving piecewise smooth initial value problems, *International Journal of Applied Mathematics and Computer Science* **23**(4): 731–747, DOI: 10.2478/amcs-2013-0055.

Boulier, F. (1994). *Study and Implementation of Some Algorithms in Differential Algebra*, Ph.D. thesis, Université des Sciences et Technologie de Lille, Lille.

Bourbaki, N. (1989). *Elements of Mathematics*, Springer-Verlag, Berlin/Heidelberg.

Braems, I., Jaulin, L., Kieffer, M. and Walter, E. (2001). Guaranteed numerical alternatives to structural identifiability testing, *Proceedings of the 40th IEEE Conference on Decision and Control, Orlando, FL, USA*, pp. 3122–3127.

Chabert, G. and Jaulin, L. (2009). Contractor programming, *Artificial Intelligence* **173**(11): 1079–1100.

Chiscii, L., Garulli, A. and Zappa, G. (1996). Recursive state bounding by parallelotopes, *Automatica* **32**(7): 1049–1055.

Denis-Vidal, L., Joly-Blanchard, G. and Noiret, C. (2001a). Some effective approaches to check identifiability of uncontrolled nonlinear systems, *Mathematics and Computers in Simulation* **57**(1–2): 35–44.

Denis-Vidal, L., Joly-Blanchard, G., Noiret, C. and Petitot, M. (2001b). An algorithm to test identifiability of non-linear systems, *Proceedings of the 5th IFAC Symposium on Nonlinear Control Systems, St. Petersburg, Russia*, Vol. 7, pp. 174–178.

Herrero, P., Delaunay, B., Jaulin, L., Georgiou, P., Oliver, N. and Toumazou, C. (2016). Robust set-membership parameter estimation of the glucose minimal model, *International Journal of Adaptive Control and Signal Processing* **30**(2): 173–185.

Jauberthie, C., Verdière, N. and Travé-Massuyès, L. (2011). Set-membership identifiability: Definitions and analysis, *Proceedings of the 18th IFAC World Congress, Milan, Italy*, pp. 12024–12029.

Jauberthie, C., Verdière, N. and Travé-Massuyès, L. (2013). Fault detection and identification relying on set-membership identifiability, *Annual Reviews in Control* **37**(1): 129–136.

Jaulin, L., Kieffer, M., Didrit, O. and Walter, E. (2001). *Applied Interval Analysis: With Examples in Parameter and State Estimation, Robust Control and Robotics*, Springer, Londres.

Jaulin, L. and Walter, E. (1993). Set inversion via interval analysis for nonlinear bounded-error estimation, *Automatica* **29**(4): 1053–1064.

Kieffer, M., Jaulin, L. and Walter, E. (2002). Guaranteed recursive nonlinear state bounding using interval analysis, *International Journal of Adaptive Control and Signal Processing* **6**(3): 193–218.

Kieffer, M., Jaulin, L., Walter, É. and Meizel, D. (2000). Robust autonomous robot localization using interval analysis, *Reliable Computing* **6**(3): 337–362.

Kieffer, M. and Walter, E. (2011). Guaranteed estimation of the parameters of nonlinear continuous-time models: Contributions of interval analysis, *International Journal of Adaptive Control and Signal Processing* **25**(3): 191–207.

Kolchin, E. (1973). *Differential Algebra and Algebraic Groups*, Academic Press, New York, NY.

Kurzhanski, A.B. and Valyi, I. (1997). *Ellipsoidal Calculus for Estimation and Control*, Nelson Thornes, Birkhäuser.

Lagrange, S., Delanoue, N. and Jaulin, L. (2008). Injectivity analysis using interval analysis: Application to structural identifiability, *Automatica* **44**(11): 2959–2962.

Ljung, L. and Glad, T. (1994). On global identifiability for arbitrary model parametrizations, *Automatica* **30**(2): 265–276.

Maiga, M., Ramdani, N. and Travé-Massuyès, L. (2013). A fast method for solving guard set intersection in nonlinear hybrid reachability, *Proceedings of the 52nd IEEE Conference on Decision and Control, CDC 2013, Firenze, Italy*, pp. 508–513.

Maiga, M., Ramdani, N., Travé-Massuyès, L. and Combastel, C. (2016). A comprehensive method for reachability analysis of uncertain nonlinear hybrid systems, *IEEE Transactions on Automatic Control* **61**(9): 2341–2356, DOI:10.1109/TAC.2015.2491740.

Milanese, M., Norton, J., Piet-Lahanier, H. and Walter, É. (2013). *Bounding Approaches to System Identification*, Springer Science & Business Media, New York, NY.

Munkres, J.R. (1975). *Topology—A First Course*, Prentice Hall, Upper Saddle River, NJ.

Nelles, O. (2002). *Nonlinear System Identification*, Springer-Verlag, Berlin/Heidelberg.

Pohjanpalo, H. (1978). System identifiability based on the power series expansion of the solution, *Mathematical Biosciences* **41**(1): 21–33.

Puig, V. (2010). Fault diagnosis and fault tolerant control using set-membership approaches: Application to real case studies, *International Journal of Applied Mathematics and Computer Science* **20**(4): 619–635, DOI: 10.2478/v10006-010-0046-y.

Raïssi, T., Ramdani, N. and Candau, Y. (2004). Set-membership state and parameter estimation for systems described by nonlinear differential equations, *Automatica* **40**(10): 1771–1777.

Ravanbod, L., Verdière, N. and Jauberthie, C. (2014). Determination of set-membership identifiability sets, *Mathematics in Computer Science* **8**(3–4): 391–406.

Seybold, L., Witczak, M., Majdzik, P. and Stetter, R. (2015). Towards robust predictive fault-tolerant control for a battery assembly system, *International Journal of Applied Mathematics and Computer Science* **25**(4): 849–862, DOI: 10.1515/amcs-2015-0061.

Carine Jauberthie has been an associate professor at the University Paul Sabatier of Toulouse (France) since 2005. She is a researcher at CNRS Laboratoire d'Analyse et d'Architecture des Systèmes (LAAS) in the Diagnosis and Supervisory Control (DISCO) Research Team. She obtained a Ph.D. degree in 2002 in applied mathematics, specializing in system control, from the University of Technology of Compiègne, France, at the ONERA Center of Lille in collaboration with the Laboratory of Applied Mathematics. Her research interests concern fault detection and diagnosis as well as the analysis of related properties, including identifiability and diagnosability of nonlinear dynamical systems with mixed uncertainties.

Louise Travé-Massuyès holds the position of the research director at Laboratoire d'Analyse et d'Architecture des Systémes, Centre National de la Recherche Scientifique (LAAS-CNRS), Toulouse, France, and was the head of the Diagnosis and Supervisory Control (DISCO) team from 1994 to 2015. She graduated in control engineering from Institut National des Sciences Appliques (INSA), Toulouse, France, in 1982 and received the Ph.D. degree from INSA in 1984. Her research interests are in dynamic systems supervision, with special focus on qualitative and model-based methods, as well as data mining. She has been particularly active in bridging the AI and control engineering diagnosis fields as the leader of the BRIDGE Task Group of the MONET European Network of Excellence. She is the coordinator of the Maintenance & Diagnosis Strategic Field within the Aerospace Valley World Competitiveness Cluster, and serves as the contact evaluator for the French Research Funding Agency. She serves on editorial boards of artificial intelligence journals. She is also a member of the IFAC Safeprocess Technical Committee.

Nathalie Verdière has been an associate professor at the University of Le Havre (France) since 2006. She is a researcher at the Laboratory of Applied Mathematics of Le Havre in the Dynamical Systems Research Team. She obtained a Ph.D. degree in 2005 in applied mathematics (specialization in control systems) from the University of Technology of Compiègne, France. From 2005 to 2006, she was a temporary associate professor at the University of Technology of Compiègne. Her research interests concern bifurcations, identifiability analysis, parameter estimation, fault detection and diagnosis for models described by ordinary or partial differential equations.

Received: 10 March 2016
Revised: 7 July 2016
Accepted: 28 August 2016