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A DISCRETE–TIME QUEUEING SYSTEM WITH CHANGES IN THE VACATION TIMES

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This paper considers a discrete-time queueing system in which an arriving customer can decide to follow a last come first served (LCFS) service discipline or to become a negative customer that eliminates the one at service, if any. After service completion, the server can opt for a vacation time or it can remain on duty. Changes in the vacation times as well as their associated distribution are thoroughly studied. An extensive analysis of the system is carried out and, using a probability generating function approach, steady-state performance measures such as the first moments of the busy period of the queue content and of customers delay are obtained. Finally, some numerical examples to show the influence of the parameters on several performance characteristics are given.

Keywords: discrete-time queueing theory, changes in the remaining vacation times, busy period, sojourn times.

1. Introduction

The development of queueing theory dates back more than a century to the work of A.K. Erlang from the early 1900s, who derived important formulae for teletraffic engineering. Nowadays the range of applications has grown and queueing theory provides models, structural insights, problem solutions and algorithms to many areas, and due to its practical applicability to production, manufacturing, communications technology, etc., more and more complex systems require more elaborated models, techniques, algorithms, etc.

Focusing on computer communications, a communication channel can be viewed as a server and the messages as customers; the random times at which messages request the use of the channel would be the arrival process, and the random lengths of service time that the messages hold the channel while being transmitted would constitute the service process. Referred to biological cells, the queueing analogy for a single ClpXP (protease) molecule is then a server (catalytic core) that selects customers (protein substrate) at random from a finite capacity waiting line (SspB-substrate complex associated with ClpX binding sites) and processes (degrades) the customers (substrate).

For a good review on queueing theory the reader is referred to the books by Kleinrock (1976), Cooper (1981), Takagi (1993), Bocharov *et al*. (2004) or Lakatos *et al*. (2013), and for applications on biological systems to the works of Hochendoner *et al*. (2014) as well as Wieczorek (2010).

Discrete-time models are very suitable in many situations. A feature that makes the analysis of discrete time systems technically more involved than its continuous time counterparts is the fact that the basic unit is a binary code and the occurrence of simultaneous events does not happen in continuous time. The regulation of these simultaneous events in queueing systems (arrivals and departures) was already studied in depth with respect to steady-state behavior by Hunter (1983). A detailed discussion and applications of discrete-time queues can be found in the books by Hunter (1983), Bruneel and Kim (1993), Woodward (1994), Alfa (2010) and the references therein.

In classical queueing models, the server is always available but this assumption is practically unrealistic. In queueing parlance, the period when the server is not attending a certain task is called vacation. In this paper, possible changes in the vacation times are contemplated, which is a quite realistic consideration. The vacation period can be, in some cases, shortened or enlarged, according to the necessities of the company. A staff shortage, due to illnesses or any other causes, may make it advisable to call back to the firm employees on vacation and, on the contrary, a decrease in the activity of the firm may make it advisable to extend the vacation period. To the author's knowledge, the changes in the vacation times have never been considered before. In the literature, there has been a number of contributions with respect to vacation models (e.g., Tian and Zhang, 2002; Xeung *et al*., 2007; Oliver and Olubukola, 2014); the book of Tian and Zhang (2006) offers a great survey on vacation models with examples.

Another feature that is usually found when a message is being processed in computers, in communications switching queues, etc., is that sometimes the information coming to the server is more actual than the one on service. In that case, the message is moved to another place if the contained information can be used later on, or if the information is not any more valuable it is deleted: in both cases the server is interrupted and upgraded. For a general survey on queues with service interruptions we refer to the papers by Fiems *et al*. (2002), Walraevens *et al*. (2006), Krishnamoorthy *et al.* (2012; 2013) or Atencia (2015), and for a detailed review on queues with service interruptions to the work of Krishnamoorthy *et al*. (2014).

The mechanism of moving messages by the arrival of one of them is called synchronized or triggered motion. There are several mechanisms on how and where the messages are moved; for a survey on those we refer to Artalejo (2000), Gelenbe and Labed (1998) and Atencia (2014). In the remainder of this paper the term "customer" will be used, which can be understood as messages, proteins, etc, depending on the context in which the model is applied.

A certain type of movement can be also considered when customers are deleted. The process of deleting customers or killing them is defined in queueing theory as negative customers. This type of movement can be related to the arrival of viruses at the system. The pioneering work on discrete-time considering negative arrivals without retrials was done by Atencia and Moreno (2004; 2005), who considered several killing strategies for negative customers. For a survey on this topics we refer to Gelenbe and Labed (1998) or Artalejo (2000), for applications in engineering to Chao *et al*. (1999), and for application in communication networks and packet transmission systems to Harrison *et al*. (2000), Park *et al*. (2009), or Piórkowski and Werewka (2010).

The strategy used in this paper for moving customers is the one that displaces them from the server to the first place of the queue, and it seems a realistic one because the displaced customer can begin its service after service completion for the customer that has caused its displacement. The arrival of negative customers has the effect, in this model, of eliminating the job that is currently being served, and has no influence on the system if the server is idle.

The question on how the triggered customers will resume their service after being interrupted depends on a pre-agreed policy. In this paper a preemptive repeat policy is considered, that is, it does not take into account the duration of the last service. For other pre-agreed policies, refer to Fiems *et al*. (2002), Moreno (2006), or Fiems and Bruneel (2013), and for an inverse order discipline to Pechinkin and Svischeva (2004), Cascone *et al*. (2011), or Milovanova and Pechinkin (2013) as well as Meykhanadzhyan *et al*. (2014).

Smart homes with sensing, actuation, and networked devices have been anticipated for a long time, although the term "home automation" has caught the attention of the media and researchers not long ago (see Lucero and Burden, 2010). For exploring on how households are configured and manage access control, etc., one can refer to Kim *et al*. (2010), Brush *et al*. (2011), or Guzmán-Navarro and Merino-Córdoba (2015).

The system proposed in this paper has applications to home automation, where the priority customers can be regarded as telegrams that have information of a technical alert. Those types of telegrams are distinguished from the rest of them by the information in their priority bits, that is, 00-system functions, 01-alert functions, 10-high priority commands.

It is usual that in a domotic system collisions occur between telegrams, for example, in decentralized systems that connect and manage dispositive and remote control processes. In order to avoid these collisions the term "negative telegram" has been introduced, which is used in systems without collisions such as CSMA/CA of the international KNX or in the MODBUS standard.

In any wireless technology there always exists the problem of energy saving, and therefore it would be interesting to have a certain control saving mechanism. This control is given in the paper through the concept of vacation service control, which can be found in the Zigbee technology, that leaves the server in standby, which is precise when the server is not operative.

Therefore, the system introduced in the paper can be considered a management system in home automation under the standards of MODBUS or Zigbee.

The rest of this paper is organized as follows. The next section gives a description of the queueing model. In Section 3 the Markov chain is studied. The queue and system size distributions are obtained together with several performance measures of the model. In Section 4 the generating function of the vacation times taking into account possible changes in the remaining vacation times is obtained. Section 5 is devoted to the study of the busy period. In Section 6 the generating functions of the sojourn time of a customer in the server, the queue and the system, as well as some associated performance measures, are provided. Finally, numerical results and conclusions are discussed.

2. Markov chain

At time m^+ (the instant immediately after time slot m) the system can be described by the process

$$X_m = (C_m, \xi_m, N_m),$$

where C_m denotes the state of the server 0, 1, or 2 according to whether the server is free, busy, or on vacation, respectively, and N_m is the number of customers in the queue. If $C_m = 1$, then ξ_m represents the remaining service time of the customer being served. If $C_m = 2$, then ξ_m represents the remaining vacation time. When $N(t) = 0$, neither C_m nor ξ_m is needed in the state description.

The input stream of customers into the system is described by means of a Bernoulli process with a as the probability that an arrival occurs in a slot. An arriving customer that finds the server busy, displaces with probability θ the customer that is currently being served to the first place in the queue and starts immediately its service, and with complementary probability $\bar{\theta} = 1 - \theta$ it becomes a negative customer, that is, expels out of the system the customer that is in the server without any other additional effect on the system. The negative customers have no effect on the system if the server is free. Once a service has been completed, the server can choose to initiate, with probability p, a period of vacation, or with complementary probability $\overline{p} = 1 - p$ to remain on duty.

During a vacation period, changes in the remaining vacation times can take place. Specifically, in each slot, with probability ν a change in the remaining vacation times occurs, and with complementary probability $\overline{\nu}$ no change takes place. The one-step transition probabilities fully describe how these changes work. It has been supposed that, if in a slot a vacation period begins, no change in the vacation times will happen, which seems a reasonable assumption.

The service and vacation times are independent and distributed with arbitrary distributions $\{s_i\}_{i=1}^{\infty}$ and $\{v_i\}_{i=1}^{\infty}$, respectively. Hence s_i and v_i are the probabilities that a service or a vacation time lasts i slots. Their generating functions (GFs), will be denoted by

$$S(x) = \sum_{i=1}^{\infty} s_i x^i$$

and

$$V(x) = \sum_{i=1}^{\infty} v_i x^i,$$

respectively. Furthermore,

$$S_k = \sum_{j=k}^{\infty} s_j$$

and

$$V_k = \sum_{j=k}^{\infty} v_j$$

will signify the probabilities that the service and vacation times last no less than k slots, respectively.

3. Steady state probabilities

Let π_0 be the stationary probability that, at the moment noted by m^+, that is, after a potential arrival, there are no customers in the system, $\pi_{1,i,k}$, $i \geq 1$, $k \geq 0$, the stationary probability that there are k customers in the queue and that the customer just being served needs i more slots to finish its service, and $\pi_{2,i,k}$, $i \geq 1$, $k \geq 0$, the stationary probability that there are k customers in the queue and the server is on vacation with a remaining vacation time of i slots.

The one-step transition probabilities are given by

$$\begin{aligned}
p_{(0)(0)} &= \overline{a} + a\,\bar{\theta},\\
p_{(1,1,0)(0)} &= \overline{a} + a\,\bar{\theta},\\
p_{(1,j,0)(0)} &= a\,\bar{\theta}, \quad j \geq 2,\\
p_{(2,1,0)(0)} &= \overline{a} + a\,\bar{\theta}.
\end{aligned}$$

If $i \geq 1$, $k \geq 0$ then

$$\begin{aligned}
p_{(0)(1,i,0)} &= a\,\theta s_i,\\
p_{(1,1,k)(1,i,k)} &= a\,\theta\,\overline{p}\,s_i,\\
p_{(1,1,k+1)(1,i,k)} &= (\overline{a} + a\,\bar{\theta})\,\overline{p}\,s_i,\\
p_{(1,i+1,k)(1,i,k)} &= \overline{a},\\
p_{(1,j,k-1)(1,i,k)} &= a\,\theta\,s_i, \quad j \geq 2,\ k \geq 1,\\
p_{(1,j,k+1)(1,i,k)} &= a\,\bar{\theta}\,s_i, \quad j \geq 2,\\
p_{(2,1,k)(1,i,k)} &= a\,\theta s_i,\\
p_{(2,1,k+1)(1,i,k)} &= (\overline{a} + a\,\bar{\theta})\,s_i,\\
p_{(1,1,k-1)(2,i,k)} &= a\,p\,v_i, \quad k \geq 1,\\
p_{(1,1,k)(2,i,k)} &= \overline{a}\,p\,v_i,\\
p_{(2,i+1,k-1)(2,i,k)} &= a\,\overline{\nu}, \quad k \geq 1,\\
p_{(2,i+1,k)(2,i,k)} &= \overline{a}\,\overline{\nu},\\
p_{(2,j,k-1)(2,i,k)} &= a\,\nu\,v_i, \quad j \geq 2,\ k \geq 1,\\
p_{(2,j,k)(2,i,k)} &= \overline{a}\,\nu\,v_i, \quad j \geq 2.
\end{aligned}$$

The system of equilibrium equations (SEE) for the stationary distribution is given by

$$\begin{aligned}\pi_0 &= (\overline{a}+a\bar{\theta})\,\pi_0 + (\overline{a}+a\bar{\theta})\overline{p}\,\pi_{1,1,0}\\ &\quad + a\bar{\theta}\sum_{j=2}^{\infty}\pi_{1,j,0} + (\overline{a}+a\bar{\theta})\,\pi_{2,1,0}, \qquad (1)\\ \pi_{1,i,k} &= \delta_{0,k}a\theta s_i\,\pi_0 + a\theta\overline{p}s_i\,\pi_{1,1,k}\\ &\quad + (\overline{a}+a\bar{\theta})\overline{p}s_i\,\pi_{1,1,k+1} + \overline{a}\,\pi_{1,i+1,k}\\ &\quad + (1-\delta_{0,k})a\theta s_i\sum_{j=2}^{\infty}\pi_{1,j,k-1}\\ &\quad + a\bar{\theta}s_i\sum_{j=2}^{\infty}\pi_{1,j,k+1} + a\theta s_i\,\pi_{2,1,k}\\ &\quad + (\overline{a}+a\bar{\theta})s_i\,\pi_{2,1,k+1}, \quad i\ge 1,\ k\ge 0, \qquad (2)\\ \pi_{2,i,k} &= (1-\delta_{0,k})apv_i\,\pi_{1,1,k-1} + \overline{a}pv_i\,\pi_{1,1,k}\\ &\quad + (1-\delta_{0,k})a\overline{\nu}\,\pi_{2,i+1,k-1} + \overline{a}\,\overline{\nu}\,\pi_{2,i+1,k}\\ &\quad + (1-\delta_{0,k})a\nu v_i\sum_{j=2}^{\infty}\pi_{2,j,k-1}\\ &\quad + \overline{a}\nu v_i\sum_{j=2}^{\infty}\pi_{2,j,k}, \quad i\ge 1,\ k\ge 0, \qquad (3)\end{aligned}$$

where $\overline{a} = 1-a$ and $\delta_{i,j}$ denotes Kronecker's delta.

The normalization condition is

$$\pi_0 + \sum_{i=1}^{\infty}\sum_{k=0}^{\infty}\pi_{1,i,k} + \sum_{i=1}^{\infty}\sum_{k=0}^{\infty}\pi_{2,i,k} = 1.$$

In order to solve Eqns. (1)–(3), the following generating functions are introduced:

$$\varphi_1(x,z) = \sum_{i=1}^{\infty}\sum_{k=0}^{\infty}\pi_{1,i,k}x^i z^k,$$
$$\varphi_2(x,z) = \sum_{i=1}^{\infty}\sum_{k=0}^{\infty}\pi_{2,i,k}x^i z^k,$$

along with the auxiliary generating functions

$$\varphi_{1,i}(z) = \sum_{k=0}^{\infty}\pi_{1,i,k}z^k, \quad i\ge 1,$$
$$\varphi_{2,i}(z) = \sum_{k=0}^{\infty}\pi_{2,i,k}z^k, \quad i\ge 1.$$

With the aim of finding the GFs $\varphi_l(x,z)$, $l=1,2$, the auxiliary generating functions $\varphi_{l,i}(z)$, $l=1,2$, will be firstly obtained, which is accomplished by multiplying Eqns. (2) and (3) by z^k, summing over k and taking into account Eqn. (1):

$$\begin{aligned}\varphi_{1,i}(z) &= \overline{a}\varphi_{1,i+1}(z) + \frac{A(z)}{z}s_i\,\varphi_{1,1}(z)\\ &\quad + \frac{1-a\theta(1-z)}{z}s_i\,\varphi_{2,1}(z)\\ &\quad + \frac{\theta z^2+\bar{\theta}}{z}as_i\,\varphi_1(1,z)\\ &\quad - a\theta\frac{1-z}{z}s_i\,\pi_0, \quad i\ge 1, \qquad (4)\\ \varphi_{2,i}(z) &= (\overline{a}+az)\overline{\nu}\varphi_{2,i+1}(z) + (\overline{a}+az)\\ &\quad \times \big[pv_i\varphi_{1,1}(z) + \nu v_i\varphi_2(1,z)\\ &\quad - \nu v_i\varphi_{2,1}(z)\big], \quad i\ge 1, \qquad (5)\end{aligned}$$

where $A(z) = \overline{a}\,\overline{p} + a\theta z(\overline{p}-z) - a\bar{\theta}p$.

Since

$$\varphi_l(x,z) = \sum_{i=1}^{\infty}\varphi_{l,i}(z)x^i, \quad l=1,2,$$

it is readily obtained that

$$\begin{aligned}& z\frac{x-\overline{a}}{x}\varphi_1(x,z)\\ &\quad = [A(z)S(x) - \overline{a}z]\varphi_{1,1}(z)\\ &\qquad + [1-a\theta(1-z)]S(x)\varphi_{2,1}(z)\\ &\qquad + [\theta z^2+\bar{\theta}]aS(x)\varphi_1(1,z)\\ &\qquad - a\theta(1-z)S(x)\pi_0, \qquad (6)\\ & \frac{x-(\overline{a}+az)\overline{\nu}}{x}\varphi_2(x,z)\\ &\quad = (\overline{a}+az)\Big[pV(x)\varphi_{1,1}(z)\\ &\qquad + \nu V(x)\varphi_2(1,z) - [\nu V(x)+\overline{\nu}]\varphi_{2,1}(z)\Big]. \qquad (7)\end{aligned}$$

The unknown GFs $\varphi_l(1,z)$, $l=1,2$, that are inserted in Eqns. (6) and (7), respectively, can be found by choosing $x=1$ in both the equations, getting

$$\begin{aligned}a(1-z)[\theta z-\bar{\theta}]\varphi_1(1,z) &= [A(z)-\overline{a}z]\varphi_{1,1}(z)\\ &\quad + [1-a\theta(1-z)]\varphi_{2,1}(z)\\ &\quad - a\theta(1-z)\pi_0, \qquad (8)\\ a(1-z)\varphi_2(1,z) &= (\overline{a}+az)[p\varphi_{1,1}(z)\\ &\quad - \varphi_{2,1}(z)]. \qquad (9)\end{aligned}$$

Now, substituting the above equations into (6) and (7) yields

$$(1-z)(\theta z-\bar{\theta})\frac{x-\overline{a}}{x}\varphi_1(x,z) = \Big[[A(z)-\overline{a}(\theta z^2+\bar{\theta})]S(x) - \overline{a}(1-z)(\theta z-\bar{\theta})]\Big]\varphi_{1,1}(z) + [1-a\theta(1-z)]S(x)\varphi_{2,1}(z) - a\theta_1(1-z)S(x)\pi_0, \quad (10)$$

$$a(1-z)\frac{x-(\overline{a}+az)\overline{\nu}}{x}\varphi_2(x,z) = (\overline{a}+az)\Big[[1-(\overline{a}+az)\overline{\nu}]pV(x)\varphi_{1,1}(z) - [\nu V(x)+a(1-z)\overline{\nu}]\varphi_{2,1}(z)\Big]. \quad (11)$$

By letting $x=\overline{a}$ and $x=(\overline{a}+az)\overline{\nu}$ in Eqns. (10) and (11), respectively, a system of equations is obtained in which only the auxiliary GFs $\varphi_{l,1}(z)$, $l=1,2$, and the unknown constant π_0 are present:

$$a\theta(1-z)S(\overline{a})\pi_0 = \Big[[A(z)-\overline{a}(\theta z^2+\bar{\theta})]S(\overline{a}) - \overline{a}(1-z)(\theta z-\bar{\theta})\Big]\varphi_{1,1}(z) + [1-a\theta(1-z)](\overline{a})\varphi_{2,1}(z), \quad (12)$$

$$pV((\overline{a}+az)\overline{\nu})\varphi_{1,1}(z) = \frac{1}{(1-(\overline{a}+az)\overline{\nu})}\Big[\nu V((\overline{a}+az)\overline{\nu}) + a(1-z)\overline{\nu}\Big]\varphi_{2,1}(z). \quad (13)$$

Solving the above system gives

$$\varphi_{1,1}(z) = \frac{a\theta(1-z)S(\overline{a})}{D(z)} \times \Big[\nu V((\overline{a}+az)\overline{\nu})+a(1-z)\overline{\nu}\Big]\pi_0, \quad (14)$$

$$\varphi_{2,1}(z) = \frac{a\theta(1-z)S(\overline{a})}{D(z)} \times [1-(\overline{a}+az)\overline{\nu}]V((\overline{a}+az)\nu)\,p\pi_0, \quad (15)$$

where

$$D(z) = [\nu V((\overline{a}+az)\overline{\nu})+a(1-z)\overline{\nu}] \times \Big[[A(z)-\overline{a}(\theta z^2+\bar{\theta})]S(\overline{a}) - \overline{a}(1-z)(\theta z-\bar{\theta})\Big] + [1-a\theta(1-z)] \times [1-(\overline{a}+az)\overline{\nu}]V((\overline{a}+az)\overline{\nu})pS(\overline{a}).$$

Inserting (14) and (15) into (10) and (11), the final expressions for $\varphi_l(x,z)$, $l=1,2$ are obtained:

$$\varphi_1(x,z) = \frac{S(x)-S(\overline{a})}{(x-\overline{a})D(z)} \times \Big[xa\theta(1-z)\big[[\nu V((\overline{a}+az)\overline{\nu}) + a(1-z)\overline{\nu}]\Big]\,\overline{a}\pi_0, \quad (16)$$

$$\varphi_2(x,z) = \frac{V(x)-V((\overline{a}+az)\overline{\nu})}{(x-(\overline{a}+az)\overline{\nu})D(z)} \times \Big[xa\theta(1-z)[1-(\overline{a}+az)\overline{\nu}]S(\overline{a}) \times \overline{\nu}(\overline{a}+az)\Big]\,p\pi_0. \quad (17)$$

The normalization condition, which can be written as $\pi_0+\varphi_1(1,1)+\varphi_2(1,1)=1$, allows us to find the unknown constant π_0:

$$\pi_0 = \frac{-D'(1)}{\nu V(\overline{\nu})[\overline{a}\bar{\theta}+\theta S(\overline{a})] - a\overline{\nu}\bar{\theta}pS(\overline{a})(1-V(\overline{\nu}))},$$

where

$$D'(1) = a\overline{\nu}pS(\overline{a})[1-V(\overline{\nu})] + \nu V(\overline{\nu})\Big[\overline{a}\theta[1-S(\overline{a})]+\theta[\overline{a}-S(\overline{a})]-\overline{a}\Big].$$

Since $\pi_0>0$, the inequality $D'(1)<0$ is a necessary condition for the system's stability. Applying Foster's theorem, it can be shown that this condition is also sufficient for the stability of the system.

The above results are summarized in the following theorem.

Theorem 1. *If $D'(1)<0$, the stationary distribution of the Markov chain $\{X_m,\ m\in\mathbb{N}\}$ has the following generating functions:*

$$\varphi_1(x,z) = \frac{S(x)-S(\overline{a})}{(x-\overline{a})D(z)} \times \Big[xa\theta(1-z)\big[[\nu V((\overline{a}+az)\overline{\nu}) + a(1-z)\overline{\nu}]\Big]\,\overline{a}\pi_0,$$

$$\varphi_2(x,z) = \frac{V(x)-V((\overline{a}+az)\overline{\nu})}{(x-(\overline{a}+az)\overline{\nu})D(z)} \times \Big[xa\theta(1-z)[1-(\overline{a}+az)\overline{\nu}]S(\overline{a}) \times \overline{\nu}(\overline{a}+az)\Big]\,p\pi_0,$$

where

$$D(z) = [\nu V((\overline{a}+az)\overline{\nu})+a(1-z)\overline{\nu}] \times \Big[[A(z)-\overline{a}(\theta z^2+\bar{\theta})]S(\overline{a}) - \overline{a}(1-z)(\theta z-\bar{\theta})\Big] + [1-a\theta(1-z)] \times [1-(\overline{a}+az)\overline{\nu}]V((\overline{a}+az)\overline{\nu})pS(\overline{a})$$

and

$$\pi_0 = \frac{-D'(1)}{\nu V(\overline{\nu})[\overline{a}\bar{\theta} + \theta S(\overline{a})] - a\overline{\nu}\bar{\theta}pS(\overline{a})(1 - V(\overline{\nu}))}.$$

Corollary 1.

1. *The probability generating function of the number of customers in the queue (i.e., of the variable N) is given by*

$$\begin{aligned}\Psi(z) &= \pi_0 + \varphi_1(1,z) + \varphi_2(1,z)\\ &= \Big[1 + \frac{\theta(1-z)}{D(z)}\Big[\overline{a}[1 - S(\overline{a})]\\ &\quad \times [\nu V((\overline{a} + az)\overline{\nu}) + a(1-z)\overline{\nu}]\\ &\quad + [1 - V((\overline{a} + az)\overline{\nu})](\overline{a} + az)\overline{\nu}pS(\overline{a})\Big]\Big]\,\pi_0.\end{aligned}$$

2. *The probability generating function of the number of customers in the system (i.e., of the variable L) is given by*

$$\begin{aligned}\Phi(z) &= \pi_0 + z\,\varphi_1(1,z) + \varphi_2(1,z)\\ &= \Big[1 + \frac{\theta(1-z)}{D(z)}\Big[\overline{a}z[1 - S(\overline{a})]\\ &\quad \times [\nu V((\overline{a} + az)\overline{\nu}) + a(1-z)\overline{\nu}]\\ &\quad + [1 - V((\overline{a} + az)\overline{\nu})](\overline{a} + az)\overline{\nu}pS(\overline{a})\Big]\Big]\,\pi_0.\end{aligned}$$

Corollary 2.

1. *The mean number of customers in the queue is given by*

$$\begin{aligned}E[N] &= \Psi'(1)\\ &\quad - \frac{\theta}{2D'(1)\nu S(\nu)[\overline{a}\bar{\theta} + \theta S(\overline{a})]}\\ &\quad \times \Big[\big[[1 - S(\overline{a})]\overline{a}\nu V(\overline{\nu})\\ &\quad + S(\overline{a})[1 - V(\overline{\nu})]a\overline{\nu}p\big]D''(1)\\ &\quad - 2a\overline{\nu}D'(1)\big[[1 - S(\overline{a})][\nu V'(\overline{\nu}) - 1]\overline{a}\\ &\quad + S(\overline{a})[1 - V(\overline{\nu}) - \overline{\nu}V'(\overline{\nu})]ap\big]\Big],\end{aligned}$$

where

$$\begin{aligned}D''(1) &= 2a\overline{\nu}\Big[[\nu V'(\overline{\nu}) - 1][\overline{a}\theta[1 - S(\overline{a})]\\ &\quad + \theta[\overline{a} - S(\overline{a})] - \overline{a}]\\ &\quad + apS(\overline{a})\big[\theta[1 - V(\overline{\nu})] - \overline{\nu}V'(\overline{\nu})\big]\Big].\end{aligned}$$

2. *The mean number of customers in the system is given by*

$$E[L] = \Phi'(1) = E[N] + \frac{\overline{a}\theta[1 - S(\overline{a})]}{\overline{a}\bar{\theta} + \theta S(\overline{a})}.$$

4. Generating function of the vacation times subjected to possible changes

In this section we study the distribution of the remaining vacation times taking into account possible changes in the remaining vacation times.

Let v_k^* be the probability that the remaining vacation times subjected to possible changes lasts exactly k slots. The probabilities v_k^* satisfy the following recursive formulas:

$$\begin{aligned}v_0^* &= 0,\\ v_k^* &= \overline{\nu}^{k-1}v_k + \sum_{i=1}^{k}\overline{\nu}^{i-1}V_i\nu\, v_{k-i}^*, \quad k \ge 1.\end{aligned}$$

The GF $V^*(x) = \sum_{k=0}^{\infty} v_k^* x^k$ is given by

$$V^*(x) = \frac{(1 - \overline{\nu}x)V(\overline{\nu}x)}{1 - x[1 - \nu V(\overline{\nu}x)]}, \tag{18}$$

and its corresponding mean is

$$(V^*)'(1) = \frac{1 - V(\overline{\nu})}{\nu V(\overline{\nu})}\,\overline{\nu}. \tag{19}$$

Let us note that for $\nu = 0$ the above formulas coincide with $V(x)$ and $V'(1)$, respectively.

It is easy to check that, if the distribution of the vacation times is of geometrical type with mean b, then we have $(V^*)'(1) = b$. In fact, the converse is also true, which provides a characterization of the vacation times distributions such that the vacation mean times subjected to possible changes are constant. This remark will be of interest in Section 7 with numerical results.

5. Busy period

A busy period is defined to begin with the arrival of a customer to an empty system and to end when the system next becomes empty. In this section we discuss the busy period of an auxiliary system, which will be useful in order to find the customer delay in the original system.

This auxiliary system differs from the original one by the fact that the customer that enters the system goes directly to the server, interrupting the service of the customer that is currently being served, if any. Therefore, the possibility that any customer enters the system during the vacation times is not considered. The probability that the busy period lasts $k \ge 0$ slots is denoted by h_k .

Then

$$\begin{aligned}h_0 &= 0,\\ h_k &= \overline{a}^{k-1}(\overline{a}+a\bar{\theta})\overline{p}s_k + a\bar{\theta}\overline{a}^{k-1}S_{k+1}\\ &\quad + a\theta\overline{p}\sum_{i=1}^{k}\overline{a}^{k-1}s_i h_{k-i}\\ &\quad + a\theta p\sum_{i=1}^{k}\overline{a}^{i-1}s_i\sum_{j=1}^{k-i}v_j^* h_{k-i-j}\\ &\quad + (\overline{a}+a\bar{\theta})p\sum_{i=1}^{k}\overline{a}^{i-1}s_i v_{k-i}^*\\ &\quad + a\theta\sum_{i=1}^{k}\overline{a}^{i-1}S_{i+1}\sum_{j=1}^{k-i}h_j h_{k-i-j}, \quad k\geq 1.\end{aligned}$$

The GF

$$h(x) = \sum_{k=0}^{\infty} h_k x^k$$

of the busy period has the following form:

$$\begin{aligned}h(x) &= \frac{(\overline{a}+a\bar{\theta})\overline{p}}{\overline{a}}S(\overline{a}x) + \frac{a\bar{\theta}}{\overline{a}}\frac{\overline{a}x - S(\overline{a}x)}{1-\overline{a}x}\\ &\quad + \frac{a\theta\overline{p}}{\overline{a}}S(\overline{a}x)h(x) + \frac{a\theta p}{\overline{a}}S(\overline{a}x)V^*(x)h(x)\\ &\quad + \frac{(\overline{a}+a\bar{\theta})p}{\overline{a}}S(\overline{a}x)V^*(x) + \frac{a\theta}{\overline{a}}\frac{\overline{a}x - S(\overline{a}x)}{1-\overline{a}x}.\end{aligned}$$

The above expression shows that the GF $h = h(x)$ satisfies the quadratic equation

$$f(h) = 0, \tag{20}$$

where

$$\begin{aligned}f(h) &= a\theta(\overline{a}x - S(\overline{a}x))\,h^2\\ &\quad + (1-\overline{a}x)\big[a\theta S(\overline{a}x)[\overline{p}+pV^*(x)] - \overline{a}\big]\,h\\ &\quad + (1-\overline{a}x)(\overline{a}+a\bar{\theta})S(\overline{a}x)[\overline{p}+pV^*(x)]\\ &\quad + a\bar{\theta}(\overline{a}x - S(\overline{a}x)).\end{aligned}$$

Let us note that for any $x\in(0,1)$ we have

$$\begin{aligned}&a\theta\big[\overline{a}x - S(\overline{a}x)\big] > 0,\\ &f(0) = (1-\overline{a}x)(\overline{a}+a\bar{\theta})S(\overline{a}x)[\overline{p}+pV^*(x)]\\ &\qquad + a\bar{\theta}[\overline{a}x - S(\overline{a}x)] > 0,\\ &f(1) = (1-\overline{a}x)p[V^*(x)-1] < 0.\end{aligned}$$

The above relations show that, for any $x\in(0,1)$, Eqn. (20) has two solutions, $h(x)$ and $h^*(x)$, satisfying the inequalities $0 < h(x) < 1 < h^*(x)$ and given by

$$\begin{aligned}h(x) &= \frac{(1-\overline{a}x)\big[\overline{a} - a\theta S(\overline{a})[\overline{p}+pV^*(x)]\big]}{2a\theta\big[\overline{a}x - S(\overline{a}x)\big]} - u(x),\\ h^*(x) &= \frac{(1-\overline{a}x)\big[\overline{a} - a\theta S(\overline{a})[\overline{p}+pV^*(x)]\big]}{2a\theta\big[\overline{a}x - S(\overline{a}x)\big]} + u(x),\end{aligned}$$

where

$$\begin{aligned}u(x) &= \Big[(1-\overline{a}x)^2\big[\overline{a} - a\theta S(\overline{a}x)[\overline{p}+pV^*(x)]\big]^2\\ &\quad - 4a\theta(\overline{a}x - S(\overline{a}x))\big[(1-\overline{a}x)(\overline{a}+a\bar{\theta})\\ &\quad \times S(\overline{a}x)(\overline{p}+pV^*(x)) + a\bar{\theta}(\overline{a}x - S(\overline{a}x))\big]\Big]^{\frac{1}{2}}\\ &\quad \times \Big[2a\theta(\overline{a}x - S(\overline{a}x))\Big]^{-1}.\end{aligned}$$

For $x=1$ we have $f(1)=0$, which implies that at least one of the two solutions takes the value 1 for $x=1$.

The GF of the busy period is defined by the first (minimal) solution $h(x)$. It only remains to check that $h(1)=1$; with this aim it will be shown that $h^*(1)>1$:

$$\begin{aligned}&\Big[a^2[a\theta S(\overline{a}) - \overline{a}]^2 - 4a\theta[\overline{a} - S(\overline{a})]\\ &\qquad \times[a(\overline{a}+a\bar{\theta})S(\overline{a}) + a\bar{\theta}(\overline{a} - S(\overline{a}))]\Big]^{\frac{1}{2}}\\ &\qquad > 2a\theta(\overline{a} - S(\overline{a})) + a[a\theta S(\overline{a}) - \overline{a}]\\ &\qquad = a\Big(\overline{a}(\theta-\bar{\theta}) - \theta S(\overline{a})(1+\overline{a})\Big),\end{aligned}$$

but the right-hand side of the above inequality is negative if the stability condition (21), as will be seen in the next paragraph, is fulfilled. Therefore, $h^*(1)>1$ and $h(1)=1$. In consequence, the GF of the busy period is $h(x)$.

The mean length of the busy period is given by

$$\overline{h} = h'(1) = \frac{\overline{a}[1-S(\overline{a})] + apS(\overline{a})(V^*)'(1)}{a\big(\overline{a}(\bar{\theta}-\theta) + \theta S(\overline{a})(1+\overline{a})\big)}.$$

Observe that the denominator of the expression of $\overline{h}$ is positive if the condition (20) is fulfilled.

6. Sojourn times

6.1. Sojourn time of a customer in the server. In this paragraph the distribution of the time that a customer spends in the server is found. The probability that the sojourn time of a customer in the server lasts exactly k slots is denoted by b_k. The distribution $\{b_k : k\geq 0\}$ is given by

$$\begin{aligned}b_0 &= 0,\\ b_k &= \overline{a}^{k-1}s_k + \overline{a}^k S_{k+1}a\bar{\theta}\\ &\quad + \sum_{i=1}^{k}\overline{a}^{i-1}S_{i+1}a\theta b_{k-i}, \quad k\geq 1.\end{aligned}$$

The corresponding GF is

$$b(x) = \sum_{k=0}^{\infty} b_k x^k = \frac{(1-\overline{a}x)S(\overline{a}x) + a\bar{\theta}(\overline{a}x - S(\overline{a}x))}{\overline{a}(1-\overline{a}x) - a\theta(\overline{a}x - S(\overline{a}x))}.$$

The mean sojourn time of a customer in the server is given by

$$\bar{b} = b'(1) = \frac{\overline{a}[1 - S(\overline{a})]}{a[\overline{a}\bar{\theta} + \theta S(\overline{a})]}.$$

Let us note that the load of the auxiliary system considered in Section 5 is expressed by

$$\rho = a\theta\bar{b},$$

and the stability condition for this model is $\rho < 1$, which can be written as

$$\overline{a}(\theta - \bar{\theta}) - \theta S(\overline{a})(1 + \overline{a}) < 0. \tag{21}$$

6.2. Sojourn time of a customer in the system. In this section the distribution of the period of time that a customer spends in the system from the beginning of its service till the moment of its departure is found. Let g_k be the probability that this period of time lasts exactly k slots. Therefore,

$$\begin{aligned} g_0 &= 0, \\ g_k &= \overline{a}^{k-1} s_k + a\bar{\theta} a^{k-1} S_{k+1} \\ &\quad + \sum_{i=1}^{k} \overline{a}^{i-1} S_{i+1} a\theta \sum_{j=1}^{k-i} h_j g_{k-i-j}, \quad k \geq 0. \end{aligned}$$

The corresponding GF is

$$g(x) = \sum_{k=0}^{\infty} g_k x^k = \frac{(1 - \overline{a}x)S(\overline{a}x) + a\bar{\theta}(\overline{a}x - S(\overline{a}x))}{\overline{a}(1 - \overline{a}x) - a\theta[\overline{a}x - S(\overline{a}x)]h(x)},$$

and the mean time of the analysed period of time is expressed by

$$\bar{g} = g'(1) = \frac{\overline{a}[1 - S(\overline{a})] + a\theta[\overline{a} - S(\overline{a})]\overline{h}}{a[\overline{a}\bar{\theta} + a\theta S(\overline{a})]}.$$

The stationary distribution of the waiting time that a customer spends in the queue till the beginning of its service has the following GF:

$$\begin{aligned} w(x) &= \pi_0 + \varphi_1(1,1) + \left[\overline{a} + a\bar{\theta} + a\theta h(x)\right] \\ &\quad \times V^*(x) \sum_{j=1}^{\infty} \sum_{k=0}^{\infty} \pi_{2,j,k} h^k(x) \\ &= \pi_0 + \varphi_1(1,1) + \left[\overline{a} + a\bar{\theta} + a\theta h(x)\right] \\ &\quad \times V^*(x) \varphi_2(1, h(x)), \end{aligned}$$

where

$$\begin{aligned} \varphi_2(1, h(x)) &= \left[1 - V\left((\overline{a} + ah(x)\overline{\nu})\right)\right] p\pi_0 \\ &\quad \times \frac{(1 - h(x))(\overline{a} + ah(x))a\theta\overline{\nu}S(\overline{a})}{D(h(x))}. \end{aligned}$$

The mean waiting time is given by

$$\begin{aligned} \bar{w} &= w'(1) \\ &= \Bigg[[a\theta\overline{h}\nu v(\overline{\nu}) + 1 - v(\overline{\nu})] \frac{1 - v(\overline{\nu})}{\nu v(\overline{\nu})} \\ &\quad - \frac{\overline{h}}{2D'(1)} \Big(2D'(1)[a\overline{\nu}v(\overline{\nu}) - [1 - v(\overline{\nu})a] \\ &\quad + [1 - v(\overline{\nu})]D''(1)] \Big) \frac{aS(\overline{a})\theta\overline{\nu}p}{\nu v(\overline{\nu})[\overline{a}\bar{\theta} + \theta S(\overline{a})]} \Bigg]. \end{aligned}$$

The GF $u(x)$ of the sojourn time of a customer in the system is expressed by

$$u(x) = g(x)\, w(x).$$

The mean sojourn time of a customer in the system is given by

$$\bar{u} = u'(x) = \bar{g} + \bar{w}.$$

7. Numerical results

In this section some numerical examples of the performance measures obtained in Section 3 are presented. The following plots corroborate what the analytical results and intuition say. Of course, the value of the parameters is chosen under the stability condition. Two important performance descriptors will be considered: the probability that the system is empty and the mean queue size. It is assumed that the service time distribution is geometrical with mean $10/9$ and the generating function of the vacation times is given by $V(x) = x^2$.

From the stability condition, we find the value

$$\theta^*(a, p, \nu) = \frac{\overline{a}\nu V(\overline{\nu}) - a\overline{\nu}pS(\overline{a})[1 - V(\overline{\nu})]}{[\overline{a}(1 - S(\overline{a})) + \overline{a} - S(\overline{a})]\nu V(\overline{\nu})}, \tag{22}$$

such that the system is stable if and only if $\theta < \theta^*$. Hence, the domain of the functions whose plots are represented against θ will be $[0, \theta^*)$. From the stability condition we also find the value

$$p^*(a, \theta, \nu) = \frac{\left[\overline{a} - \theta[\overline{a}(1 - S(\overline{a})) + \overline{a} - S(\overline{a})]\right]\nu V(\overline{\nu})}{a\overline{\nu}S(\overline{a})[1 - V(\overline{\nu})]},$$

such that the system is stable if and only, if $p < p^*$ and, consequently, the domain of the functions, whose plots are represented against p will be $[0, p^*)$.

In Fig. 1 the probability that the system is empty is plotted against the parameter θ. As is to be expected, π_0 decreases with increasing values of θ and, when the variable θ approaches the stability's abscissa, that is, as $\theta \to \theta^*$, the system becomes unstable and hence the probability that the system is empty tends to

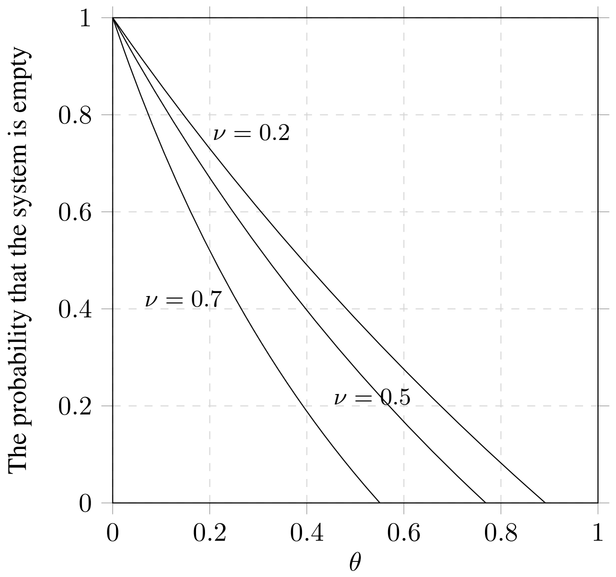

Fig. 1. $a = 0.7$, $p = 0.2$, $\theta^*_{\nu=0.2} = 0.8915$, $\theta^*_{\nu=0.5} = 0.7688$, $\theta^*_{\nu=0.7} = 0.5506$.

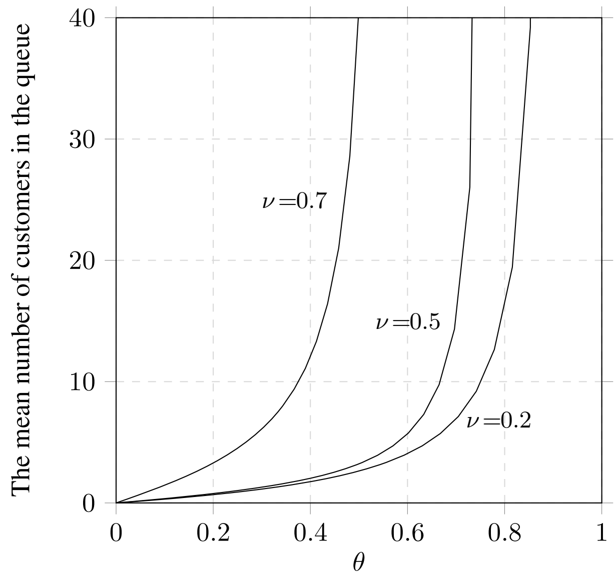

Fig. 2. $a = 0.7$, $p = 0.2$, $\theta^*_{\nu=0.2} = 0.8915$, $\theta^*_{\nu=0.5} = 0.7688$, $\theta^*_{\nu=0.7} = 0.5506$.

zero. Besides, let us observe that, if $\theta = 0$, only negative customers enter in the system, and, therefore, the probability that the system is empty is one. The three curves presented in Fig. 1 correspond to three values of the parameter ν, specifically to $\nu = 0.2, 0.5, 0.7$, and, as expected, π_0 decreases for increasing values of ν.

Figure 2 illustrates how the parameters θ and ν affect the mean queue size. As expected, the curves show that $E[N]$ is increasing as a function of θ. In addition, the mean number of customers in the queue diverges when the parameter θ approximates θ^*. Let us observe that, when $\theta = 0$, no customers enter in the queue and, consequently, $E[N] = 0$. With respect to the parameter ν, the curves show that $E[N]$ increases when ν increases.

The behaviour of $E[N]$ plotted against the parameter p has been omitted since it is similar to the drawing in Fig. 2.

In Fig. 3, the probability that the system is empty is plotted against the parameter p. It can be observed that π_0 is a decreasing function of p, which is evident because, when the probability that the server takes a vacation increases, the probability that the system is empty decreases. Again, when the variable p approaches the stability's abscissa p^*, the system becomes unstable and the probability that the system is empty tends to zero.

If the vacation times are governed by a geometric distribution with mean $1/\overline{s}$, and θ^* is plotted as a function of s, $s = 1-\overline{s}$, the graphics shown in Fig. 4 are obtained. It can be observed that θ^* is decreasing as a function of s, which is not surprising since when s increases the vacation mean time also increases. In Fig. 4 three curves are presented corresponding to $p = 0.2$ and three values of the arrival rate $a = 0.5, 0.7, 0.9$, and, as intuition tells us, θ^* decreases for increasing values of a. With respect to the curve obtained for $a = 0.5$, it can be noted that for $s < 0.775$ and arbitrary θ the system is stable, for $s \in [0.775, 0.9052)$ and $\theta < \theta^*(s)$ the system is stable, and for $s \geq 0.9052$ the system becomes unstable for any value of θ. A similar analysis can be carried out with respect to the other two curves.

Let us note that the function $\theta^*(a, p, \nu)$ in (22) can be written in the form

$$\theta^*(a, p, \nu) = \frac{\overline{a} - apS(\overline{a})(V^*)'(1)}{\overline{a}(1 - S(\overline{a})) + \overline{a} - S(\overline{a})}$$

and, if the distribution of the vacation times is geometric, it turns out, as noted in Section 4, that the stability's abscissa θ^* is independent of the parameter ν. The same can be said of p^* and of all the descriptors of the system that only depend on ν thorough $(V^*)'(1)$, which is the case of some characteristics of the model as important as π_0 and the stability condition.

8. Conclusions and research results

In this paper a $Geo/G/1/\infty$ queueing system with changes in the vacation times was studied. A customer that arrives to the system may opt to follow an LCFS discipline or to become a negative customer. Once a service is finished the server can decide, with a certain probability, to take a vacation period, or with complementary probability to be ready for new services. A novel aspect of the paper is the consideration of changes

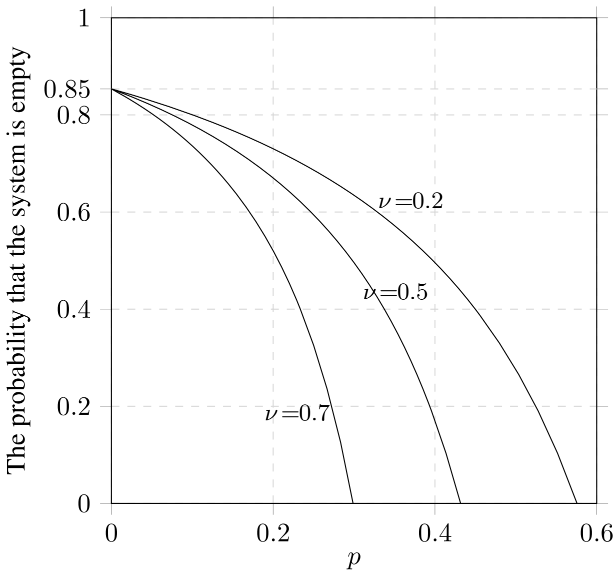

Fig. 3. $a = 0.7$, $\theta = 0.2$, $p^*_{\nu=0.2} = 0.5756$, $p^*_{\nu=0.5} = 0.4317$, $p^*_{\nu=0.7} = 0.2989$.

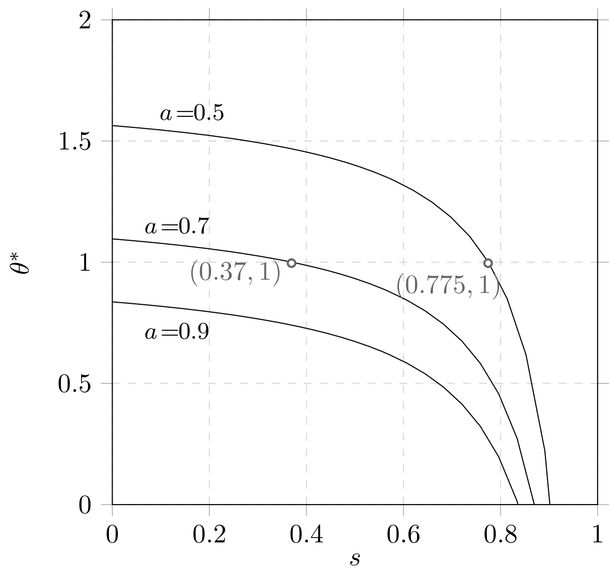

Fig. 4. $p = 0.2$, $\theta^*_{a=0.5}(0.9052) = 0$, $\theta^*_{a=0.7}(0.8701) = 0$, $\theta^*_{a=0.9}(0.8363) = 0$.

in the vacation times, which gives the model a more realistic approach to real life problems. A thorough study of the system was carried out, which made it possible to obtain the generating function of the vacation times taking into account possible changes in the remaining vacation times, and also the generating functions for the number of customers in the queue and in the system.

For the study of the sojourn time of a customer in the system it was necessary to consider a busy period of an auxiliary system in which the customer that enters in the system goes directly to the server, interrupting the service of the customer that is in the server obtaining service, which means that no customer enters the system during a vacation period. Making use of this busy period, the generating function of the sojourn time of a customer in the queue and in the system is given.

An important topic that was studied in the paper is the special character of the geometric distributions with respect to changes in the remaining vacation times, as shown in the numerical results. The characterization of distributions with vacation mean times subjected to possible changes was analyzed, which yields that the only distributions with such a property are the geometric ones. In the section with numerical examples we analyzed the reasons why some important characteristics of the system, if the vacation times are geometric, are independent of the parameter ν.

Another important aspect of the paper is the presence of negative and triggered customers, which, as said in Introduction, enlarges the field of applications of the model.

It is worth remarking that the widely used method based on generating functions was the basic tool that allowed a treatable analytical development of this complex model.

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References

Alfa, A. (2010). *Queueing Theory for Telecommunications 1: Discrete Time Modelling of a Single Node System*, Springer, New York, NY.

Artalejo, J. (2000). G-networks: A versatile approach for work removal in queueing networks, *European Journal of Operational Research* **126**(2): 233–249.

Atencia, I. (2014). A discrete-time system with service control and repairs, *International Journal of Applied Mathematics and Computer Science* **24**(3): 471–484, DOI: 10.2478/amcs-2014-0035.

Atencia, I. (2015). A discrete-time system queueing system with server breakdowns and changes in the repair times, *Annals of Operations Research* **235**(1): 37–49.

Atencia, I. and Moreno, P. (2004). The discrete-time $Geo/Geo/1$ queue with negative customers and disasters, *Computers and Operations Research* **31**(9): 1537–1548.

Atencia, I. and Moreno, P. (2005). A single-server G-queue in discrete-time with geometrical arrival and service process, *Performance Evaluation* **59**(1): 85–97.

Bocharov, P.P., D'Apice, C., Pechinkin, A.V. and Salerno, S. (2004). *Queueing Theory: Modern Probability and Statistics*, VSP, Utrecht/Boston, MA.

Bruneel, H. and Kim, B. (1993). *Discrete-time Models for Communication Systems Including ATM*, Kluwer Academic Publishers, New York, NY.

Brush, A., Bongshin, L., Ratul, M., Sharad, A., Stefan, S. and Colin, D. (2011). Home automation in the wild: Challenges and opportunities, *ACM Conference on Computer-Human Interaction, Vancouver, Canada*, pp. 2115–2124.

Cascone, A., Manzo, P., Pechinkin, A. and Shorgin, S. (2011). A $Geom/G/1/n$ system with LIFO discipline without interrupts and constrained total amount of customers, *Automation and Remote Control* **72**(1): 99–110.

Chao, X., Miyazawa, M. and Pinedo, M. (1999). *Queueing Networks: Customers, Signals and Product Form Solutions*, John Wiley and Sons, Chichester.

Cooper, R. (1981). *Introduction to Queueing Theory*, 2nd Edn., North-Holland, New York, NY.

Fiems, D. and Bruneel, H. (2013). Discrete-time queueing systems with Markovian preemptive vacations, *Mathematical and Computer Modelling* **57**(3–4): 782–792.

Fiems, D., Steyaert, B. and Bruneel, H. (2002). Randomly interrupted $GI/G/1$ queues: Service strategies and stability issues, *Annals of Operations Research* **112**(1): 171–183.

Gelenbe, E. and Labed, A. (1998). G-networks with multiple classes of signals and positive customers, *European Journal of Operational Research* **108**(1): 293–305.

Guzmán-Navarro, F. and Merino-Córdoba, S. (2015). *Gestión de la energía y gestión técnica de edificios*, 1st Edn., RA-MA, Málaga.

Harrison, P.G., Patel, N.M. and Pitel, E. (2000). Reliability modelling using G-queues, *European Journal of Operational Research* **126**(2): 273–287.

Hochendoner, P., Curtis, O. and Mather, W.H. (2014). A queueing approach to multi-site enzyme kinetics, *Interface Focus* **4**(3): 1–9, DOI: 10.1098/rsfs.2013.0077.

Hunter, J. (1983). *Mathematical Techniques of Applied Probability*, Academic Press, New York, NY.

Kim, T., Bauer, L., Newsome, J., Perrig, A. and Walker, J. (2010). Challenges in access right assignment for secure home networks, *HotSec 2010, CA, USA*, pp. 1–6.

Kleinrock, L. (1976). *Queueing Theory*, John Wiley and Sons, New York, NY.

Krishnamoorthy, A., Gopakumar, B. and Viswanath Narayanan, V. (2012). A retrial queue with server interruptions, resumption and restart of service, *Operations Research International Journal* **12**(2): 133–149.

Krishnamoorthy, A., Pramod, P. and Chakravarthy, S. (2013). A note on characterizing service interruptions with phase-type distribution, *Journal of Stochastic Analysis and Applications* **31**(4): 671–683.

Krishnamoorthy, A., Pramod, P. and Chakravarthy, S. (2014). A survey on queues with interruptions, *TOP* **22**(1): 290–320.

Lakatos, L., Szeidl, L. and Telek, M. (2013). *Introduction to Queueing Systems with Telecommunication Applications*, Springer, New York, NY.

Lucero, S. and Burden, K. (2010). *Home Automation and Control*, ABI Research, New York, NY.

Meykhanadzhyan, L.A., Milovanova, T.A., Pechinkin, A.V. and Razumchik, R.V. (2014). Stationary distribution in a queueing system with inverse service order and generalized probabilistic priority, *Informatika Primenenia* **8**(3): 28–38, (in Russian).

Milovanova, T.A. and Pechinkin, A.V. (2013). Stationary characteristics of the queueing system with LIFO service, probabilistic priority, and hysteric policy, *Informatika Primenenia* **7**(1): 22–35, (in Russian).

Moreno, P. (2006). A discrete-time retrial queue with unreliable server and general service lifetime, *Journal of Mathematical Sciences* **132**(5): 643–655.

Oliver, C.I. and Olubukola, A.I. (2014). $M/M/1$ multiple vacation queueing systems with differentiated vacations, *Modelling and Simulation in Engineering* **2014**(1): 1–6.

Park, H.M., Yang, W.S. and Chae, K.C. (2009). The $Geo/G/1$ queue with negative customers and disasters, *Stochastic Models* **25**(4): 673–688.

Pechinkin, A. and Svischeva, T. (2004). The stationary state probability in the $BMAP/G/1/r$ queueing system with inverse discipline and probabilistic priority, *24th International Seminar on Stability Problems for Stochastic Models, Jurmala, Latvia,* pp. 141–174.

Piórkowski, A. and Werewka, J. (2010). Minimization of the total completion time for asynchronous transmission in a packet data-transmission system, *International Journal of Applied Mathematics and Computer Science* **20**(2): 391–400, DOI: 10.2478/v10006-010-0029-z.

Takagi, H. (1993). *Queueing analysis: A Foundation of Performance Evaluation, Discrete-Time Systems*, North-Holland, Amsterdam.

Tian, N. and Zhang, Z. (2002). The discrete-time $GI/Geo/1$ queue with multiple vacations, *Queueing Systems* **40**(3): 283–294.

Tian, N. and Zhang, Z. (2006). *Vacation Queueing Models: Theory and Applications*, Springer, New York, NY.

Walraevens, J., Steyaert, B. and Bruneel, H. (2006). A preemptive repeat priority queue with resampling: Performance analysis, *Annals of Operations Research* **146**(1): 189–202.

Wieczorek, R. (2010). Markov chain model of phytoplankton dynamics, *International Journal of Applied Mathematics and Computer Science* **20**(4): 763–771, DOI: 10.2478/v10006-010-0058-7.

Woodward, M.E. (1994). *Communication and Computer Networks: Modelling with Discrete-Time Queues*, IEEE Computer Society, London.

Xeung, W., Jin, D., Dae, W. and Kyung, C. (2007). The $Geo/G/1$ queue with disasters and multiple working vacations, *Stochastic Models* **23**(4): 537–549.

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