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EQUIVALENT DESCRIPTIONS OF A DISCRETE–TIME FRACTIONAL–ORDER LINEAR SYSTEM AND ITS STABILITY DOMAINS

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Two description forms of a linear fractional-order discrete system are considered. The first one is by a fractional-order difference equation, whereas the second by a fractional-order state-space equation. In relation to the two above-mentioned description forms, stability domains are evaluated. Several simulations of stable, marginally stable and unstable unit step responses of fractional-order systems due to different values of system parameters are presented.

Keywords: fractional calculus, linear discrete-time system, stability domain.

1. Introduction

Fractional calculus (Oldham and Spanier, 1974; Miller and Ross, 1993; Samko and Marichev, 1993; Oustaloup, 1995; Podlubny, 1999; Ostalczyk, 2008; Kaczorek, 2011) has become a recognized mathematical tool in many scientific areas. One can mention some successful applications in dynamic system identification (Ostalczyk, 2008) and the synthesis of PID (Podlubny, 1999; Valério and Costa, 2006) or CRONE (Oustaloup, 1991; 1995; 1999) controllers in closed-loop dynamical systems. A main advantage of such controllers is that they have additional parameters, i.e., differentiation and integration orders, to reshape the transient characteristics of the designed closed-loop system. The closed-loop system stability is the first requirement of a synthesis (Dzieliński and Sierociuk, 2008; Guermah *et al.*, 2010). Thus a simple and readable criterion may be helpful.

There are equivalent (under some assumptions) definitions of the Fractional-Order (FO) derivative. The so-called Riemann–Liouville left-sided derivative of order α of a real function $f(t)$ having n continuous derivatives for $t \leq t_0$ is defined as the following integral:

$$
{}_{t_0}D_t^{(\alpha)} y(t) = \frac{1}{\Gamma(n-\alpha)} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \left[\int_{t_0}^{t} \frac{y(\tau)}{(t-\tau)^{\alpha-n+1}} \, \mathrm{d}\tau \right], \quad (1)
$$

where $n = [\alpha] + 1$, $[\alpha]$ is the integer part of α, $[t_0, t]$ is the differentiation range, Γ is the Euler gamma function.

One can prove that (1) is equivalent to the Grünwald–Letnikov form

$$
{}_{t_0}D_t^{(\alpha)} y(t) = \lim_{h \to 0^+} \left[\frac{{}_{k_0}\Delta_k^{(\alpha)} f(kh)}{h^\alpha} \right], \quad (2)
$$

where

$$
{}_{k_0}\Delta_k^{(\alpha)} f(kh) = \sum_{i=k_0}^{k} a_{i-k_0}^{(\alpha)} y(kh + k_0 h - ih) \quad (3)
$$

is the Grünwald–Letnikov backward difference, $[t_0, t] = [k_0 h, kh]$ is the differentiation range, h is the differentiation step.

$$
a_i^{(\alpha)} = \begin{cases} 1 & \text{for } i = 0, \\ (-1)^i \dfrac{\alpha(\alpha-1)\ldots(\alpha-i+1)}{i!} & \text{for } i = 1, 2, \ldots. \end{cases} \quad (4)
$$

In the numerical evaluation of the FO derivative (2), h is finite and constant. To simplify the notation, one can assume $h = 1$ and omit it in the formula. Here one should emphasise that left (k_0) and right (k) subscripts in the difference sign Δ denote a differentiation range (a fixed summation range), whereas (k) in the function y denotes its discrete variable. Here, one should care about notation because, in general,

$$ {}_{k_0}\Delta_k^{(\alpha)} y(kh) \neq {}_{k_0}\Delta_{k+1}^{(\alpha)} y(kh) \neq {}_{k_0}\Delta_k^{(\alpha)} y[(k+1)h]. \tag{5} $$

It will be further assumed that

$$ k_0 = 0, \tag{6} $$

and all the fractional orders considered are rational numbers, i.e., they can be expressed as a ratio of positive integers

$$ \alpha = \frac{1}{d} n = \nu n \quad (d, n \in \mathbb{Z}_+, \quad \alpha, \nu \in \mathbb{R}_+), \tag{7} $$

with

$$ 0 < \frac{1}{d} = \nu < 1. \tag{8} $$

Greek letters are reserved for non-integer numbers.

2. Equivalent descriptions of the FO linear dynamical system

In this section the FO commensurate state-space description of the FO linear single-input single-output discrete-time system is discussed. A relationship between this description and the FO difference equation is established.

2.1. Commensurate FO state-space description.

Any linear time-invariant FO Differential Equation (FODE) with orders satisfying the condition (7) can be represented by the commensurate state-space equations

$$ {}_0\Delta_{k+1}^{(\nu)} \mathbf{x}[(k+1)h] = \mathbf{A}\mathbf{x}(kh) + \mathbf{b}u(kh), \tag{9} $$

and

$$ y(kh) = \mathbf{c}\mathbf{x}(kh), \tag{10} $$

where

$$ {}_0\Delta_{k+1}^{(\nu)} \mathbf{x}[(k+1)h] = \begin{bmatrix} {}_0\Delta_{k+1}^{(\nu)} x_1[(k+1)h] \\ {}_0\Delta_{k+1}^{(\nu)} x_2[(k+1)h] \\ \vdots \\ {}_0\Delta_{k+1}^{(\nu)} x_n[(k+1)h] \end{bmatrix}. \tag{11} $$

It is well-known that there exists a similarity transformation matrix $\mathbf{T}_F$ transforming the state matrix $\mathbf{A}$ in (9) to the Frobenius canonical form $\mathbf{A}_F$ (Kailath, 1980),

$$ \begin{aligned} \mathbf{A}_F &= \mathbf{T}_F \mathbf{A} (\mathbf{T}_F)^{-1} \\ &= \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ -A_0 & -A_1 & -A_2 & -A_3 & \cdots & -A_{n-1} \end{bmatrix}, \end{aligned} $$

$$ \mathbf{b}_F = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{c}_F = [B_0 \cdots B_m 0 \cdots 0]. \tag{12} $$

Another similarity transformation of the state vector (11) represented by a matrix $\mathbf{T}_D$ transforms a state matrix in the formula (9) to the diagonal form $\mathbf{A}_D$,

$$ {}_0\Delta_{k+1}^{(\nu)} \mathbf{x}[(k+1)h] = \mathbf{A}_D \mathbf{x}(kh) + \mathbf{B}_D u(kh), \tag{13} $$

$$ y(kh) = \mathbf{c}_D \mathbf{x}(kh), \tag{14} $$

where

$$ \mathbf{x}(kh) = \mathbf{T}\bar{\mathbf{x}}(kh), \tag{15} $$

$$ \mathbf{A}_D = \begin{bmatrix} \bar{p}_1 & 0 & \cdots & 0 \\ 0 & \bar{p}_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \bar{p}_n \end{bmatrix}. \tag{16} $$

Here, without loss of generality, one may assume that all eigenvalues are distinct. Because the Jordan canonical forms of the state matrix of the system (9) with different configurations of Jordan blocks of one multiple eigenvalue lead to the same characteristic polynomial, in the system stability analysis the multieigenvalue case may be considered similarly.

2.2. FO difference equation description.

From equations derived from the state-space description (9) with (12) we obtain

$$ {}_0\Delta_{k+1}^{(\nu)} x_{i-1}[(k+1)h] = x_i(kh) \quad \text{for} \quad i = 2, \ldots, n, \tag{17} $$

and

$$ {}_0\Delta_{k+1}^{(\nu j)} x_{i-1}[(k+1)h] = {}_0\Delta_{k+1}^{[(j-1)\nu]} x_i[(k+1)h] \quad \text{for} \quad i = 2, \ldots, n, \tag{18} $$

and, after simple rearrangements,

$$\sum_{i=0}^{n} A_{i0}\Delta_{k+i}^{(i\nu)} y_{i-1}[(k+i)h] = \sum_{i=0}^{m} B_{i0}\Delta_{k+i}^{(i\nu)} u_{i-1}[(k+i)h], \quad (19)$$

where $A_n = 1$.

One should note that this FO difference equation contains shifted functions and the shift coincides with the differentiation order. Also the differentiation range is the same as the function shift. This remark essentially simplifies stability analysis.

3. Stability domains of systems described by the FO difference equation and state-space form

Here the stability of FO discrete linear systems is considered (Dzieliński and Sierociuk, 2008; Kaczorek, 2011; Matignon, 1996). The stability of the system described by the FO difference equation (15) is analysed in Subsection 3.1. The stability domains are evaluated in Subsection 3.2.

3.1. FO linear difference equation and state-space form $\mathcal{Z}$-transforms. Application of the one-sided $\mathcal{Z}$-transform to Eqn. (19) under the assumption of zero initial conditions yields

$$\sum_{i=0}^{n} A_i z^i (1-z^{-1})^{\nu i} Y(z) = \sum_{i=0}^{m} B_i z^i (1-z^{-1})^{\nu i} U(z). \quad (20)$$

From the state-space equations (14) and (15), after the $\mathcal{Z}$-transform, we obtain

$$\prod_{i=0}^{n} \left[z(1-z^{-1})^{\nu} - \bar{p}_i\right] Y(z) = B_m \prod_{i=0}^{m} \left[z(1-z^{-1})^{\nu} - r_i\right] U(z). \quad (21)$$

An analogous procedure performed on Eqn. (14) gives

$$\mathbf{X}(z) = \mathrm{diag}\left\{\frac{1}{z(1-z^{-1})^{\nu} - \bar{p}_i}\right\}_{i=1,\dots,n} \mathbf{b}_D U(z). \quad (22)$$

The equalities (21) and (22) form the system characteristic polynomial containing information about the system stability. Thus

$$z(1-z^{-1})^{\nu} - \bar{p}_i = 0 \quad (23)$$

may be expressed as

$$z^{1-\nu}(z-1)^{\nu} - \bar{p}_i = 0 \quad (24)$$

and further

$$\prod_{j=1}^{\infty} (z - b_j) = 0. \quad (25)$$

The characteristic polynomial (26) is stable if and only if b_j are settled in the interior of a unit circle defined by $|z| = 1$ (Ogata, 1987).

3.2. Stability domains of the system described by FO state-space equations. Defining the one-to-one transformation

$$p_d(\theta) = e^{j\theta}\left(1 - e^{-j\theta}\right)^{\frac{1}{d}} \quad \text{for} \quad \theta \in [0, 2\pi), \quad (26)$$

we obtain system stability regions in the space of parameters $\bar{p}_j$. For different orders (8) defined by the integer $d = 1, 2, \dots, 10$, the corresponding stability domains are plotted in Fig. 1. As $d \to \infty$, stability domains tend to

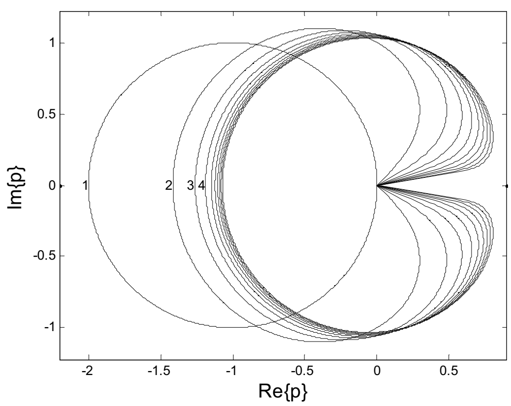

Fig. 1. Stability domains for increasing values of d.

a unit circle except for a real positive axis. This domain (evaluated for $d = 100$) is presented in Fig.2.

On the other hand, for $\nu = 1$ (the case of a classical integer order system) from (23) we get

$$z - (1 + \bar{p}_i) = 0, \quad (27)$$

which explains the left shift of the unit circle present in Fig. 1. The formula (26) can be also expressed in the form

$$p_d(\theta) = \left(2\sin\left(\frac{\theta}{2}\right)\right)^{\frac{1}{d}} e^{j\theta + \frac{\pi-\theta}{2d}} \quad \text{for} \quad \theta \in [0, 2\pi). \quad (28)$$

Then

$$\lim_{d\to\infty} p_d(\theta) = e^{j\theta}. \quad (29)$$

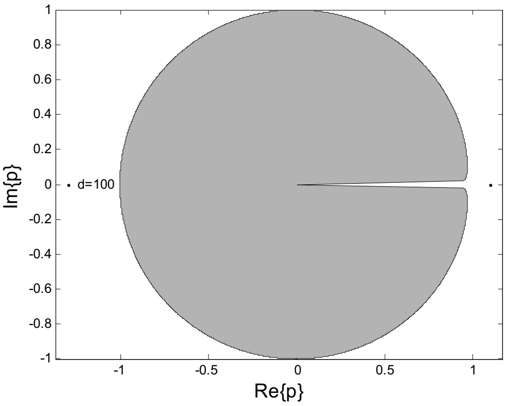

Fig. 2. Stability domains for $d = 100$.

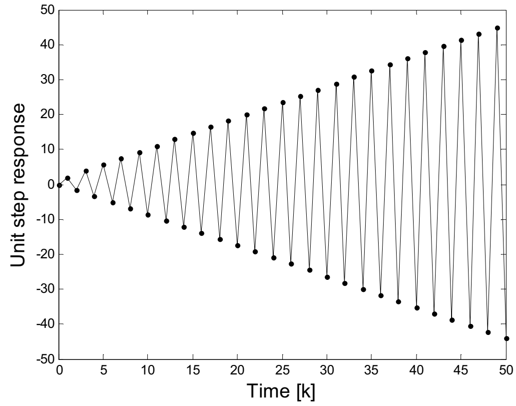

Fig. 3. Unit step response of the system (30) with the coefficients (32).

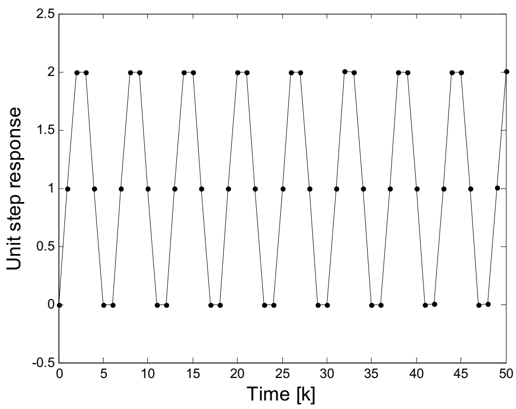

Fig. 4. Unit step response of the system (30) with the coefficients (34).

4. FO discrete linear system unit step response

Now several unit step responses of FO discrete linear systems are numerically evaluated. All the presented FO systems are characterised by the same FO $\nu = 0.5$. First, the system

$$
{}_0\Delta_{k+2}^{\left(\frac{2}{2}\right)}\, y(k+2) + A_1\, {}_0\Delta_{k+1}^{\left(\frac{1}{2}\right)}\, y(k+1) + A_0 y(k) = A_0 \mathbf{1}(k), \quad (30)
$$

where $\mathbf{1}(k)$ denotes a discrete unit step function, and the condition $B_0 = A_0$ preserve the steady-state response level equal to 1. In the following figures black dots indicate response values which are connected by thin lines to provide better clarity of the response shape. In Figs. 3 and 4, critically stable responses are presented. The first one is characterised by

$$
\bar{p}_1 = \bar{p}_2 = -\sqrt{2} \quad (31)
$$

or, equivalently, by

$$
A_1 = 2\sqrt{2}, \qquad A_0 = 4. \quad (32)
$$

The second system is characterised by

$$
\bar{p}_1 = j, \qquad \bar{p}_2 = -j \quad (33)
$$

or

$$
A_1 = 0, \qquad A_1 = 2. \quad (34)
$$

One should note that in both the cases considered the poles (31) and (33) lie precisely on boundary of the stability domain. Next, the unit step responses for two asymptotically stable systems are presented. In Figs. 3 and 4 responses related to the coefficients

$$
\bar{p}_1 = -1.4 + j0.1, \qquad \bar{p}_2 = -1.4 - j0.1 \quad (35)
$$

$$
A_1 = 2.8, \qquad A_0 = 1.97 \quad (36)
$$

and

$$
\bar{p}_1 = -0.2, \qquad \bar{p}_2 = -0.4 \quad (37)
$$

$$
A_1 = 0.6, \qquad A_0 = 0.2 \quad (38)
$$

are displayed, respectively.

Next, we consider again Eqn. (15) with $n = 4$, $m = 0$, $\nu = 0.5$ and $B_0 = A_0$. All parameters $\bar{p}_j$, $j = 1, 2, 3, 4$ are on boundary of the stability domain,

$$
\begin{aligned}
\bar{p}_1 &= -1.284110014049142 \\
&\quad + j0.5318957833982609, \\
\bar{p}_2 &= \bar{p}_1^*, \\
\bar{p}_3 &= -1.130235782084677 \\
&\quad + j0.7551994054009926, \\
\bar{p}_4 &= \bar{p}_3^*
\end{aligned} \quad (39)
$$

where (*) denotes the complex conjugate. The unit step

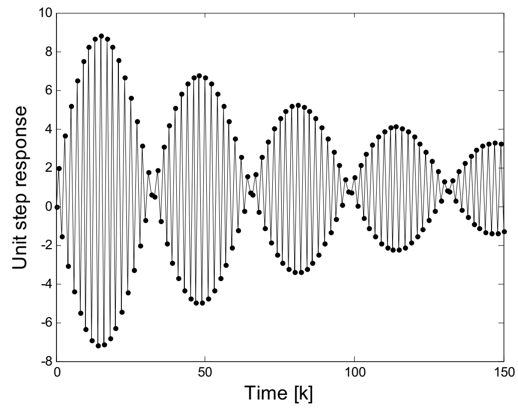

Fig. 5. Unit step response of the system (30) with the coefficients (36).

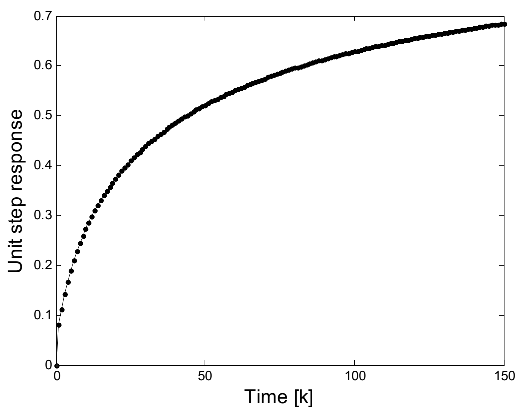

Fig. 6. Unit step response of the system (30) with the coefficients (38).

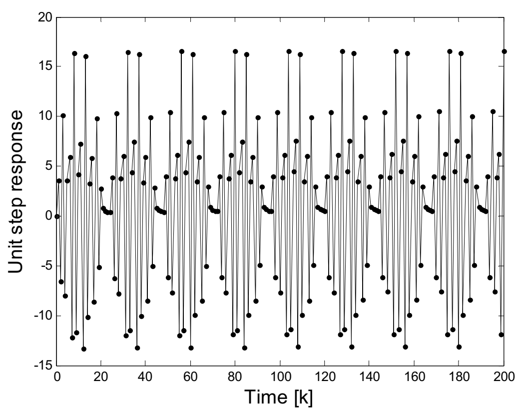

Fig. 7. Unit step response of the system (15) with the coefficients (39).

response of the system (15) with the coefficients (39) is presented in Fig. 7.

Applying now a diminishing factor $\upsilon_1 = 0.99$ to all coefficients (39), i.e., taking $\upsilon_1 \bar{p}_j, \quad j = 1, 2, 3, 4,$ we get an asymptotically stable system. For an increasing factor $\upsilon_2 = 1.01$, the system (39) with $\upsilon_2 \bar{p}_j$, $j = 1, 2, 3, 4$ is unstable. Stable and unstable system responses are presented in Figs. 8 and 9, respectively.

Finally, one can mention that the systems considered above are FO first and second order systems, due to the highest orders of the difference equations (15).

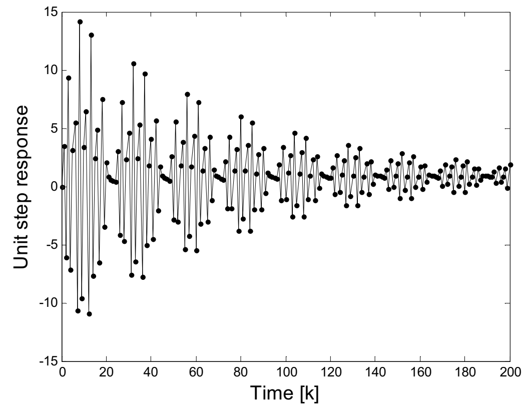

Fig. 8. Unit step response of the system (15) with the coefficients (39) multiplied by $\upsilon_1 = 0.99$.

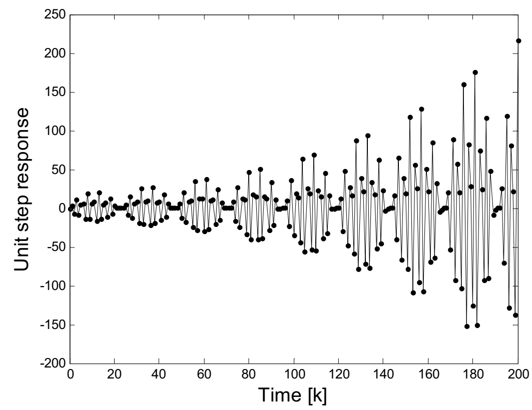

Fig. 9. Unit step response of the system (15) with the coefficients (39) multiplied by $\upsilon_1 = 1.01$.

5. Conclusions

The transformation (26) proposed in this paper allows quick and precise graphical and numerical evaluation of the stability domains of FO linear discrete systems. One should note that an approximation of the FO discrete system by an integer order system may be inadequate, especially when the system is on the stability limit. Its simplicity and visibility may be useful in a robust digital controller in FO closed-loop control system synthesis due to plant parameter changes leading to different closed-loop system poles configurations. The presented transient responses of FO stable systems revealing new shapes of waves may be helpful in the FO generator or digital filter synthesis.

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