1. INTRODUCTION

Today piezoelectric transducers are applied in many different industries (Tzou, 1999; Przybyłowicz, 1999; Liu et al., 2014, Szpiro et al., 2018). They are used as measuring and controlling elements in, more or less complex, systems for working process execution. Design of a transducer depends on its manufacturing method and intended use. Normally, they are made of one or more piezoelectric layers (with electrodes placed on their entire length) and a non-piezoelectric component. Their working principle is based on the conversion of electric energy to mechanical, or vice versa (Busch-Vishniac, 1999). Electromechanical characteristics – relation between deflection of a transducer and applied electrical and mechanical loads, is defined based on constitutive equations. These equations bound together geometrical properties, material properties and physical parameters, such as force, displacement and electric field. Developing and solving of such simultaneous equations, in case of piezoelectric benders, is very difficult. Material and geometrical heterogeneity of the global structure of bender and diverse boundary conditions force the use of some reductions.

Electromechanical characteristics of piezoelectric transducers were dealt with by many scientists. Smits et al. (1991), by using energetic methods, formed and solved physical equations for a transducer made of two layers of even length (piezoelectric bimorph). Then Wang and Cross (1999) extended and solved the issue of a triple segment, while Xiang and Shi (2008) – a multi-layer one. In papers (Park and Moon, 2005; Dunsch and Breguet, 2008; Rahmoun and Osmond, 2010; Raeisifard et al., 2014, Mieczkowski, 2016, Nguyen et al., 2018), results of piezoelectric bimorph testing were given, wherein their design featured piezoelectric layer of length other than that of a non-piezoelectric component. The paper (Mieczkowski, 2017) analysed static and dynamic characteristics of a three-layer actuator, made of a non-piezoelectric component and two piezoelectric layers of the same length and thickness. Durability and utility features of piezoelectric transducer were analysed in paper (Mieczkowski, 2018).

![Fig. 1. Three-layer piezoelectric converter; 1,3 – piezoelectric elements, 2 – beam](image-url)
2. CONSTITUTIVE EQUATIONS

2.1. Basic assumptions

To define constitutive equations, the method (described in detail in papers [Mieczkowski, 2016; Mieczkowski, 2017]), which allows for implementation to a homogeneous beam two types of modules, hereinafter referred to as PBS (piezoelectric bimorph segment) and PTS (piezoelectric triple segment) was used. This allows for including in electromechanical characteristics of the segment a local change in stiffness and strain, which are caused by the transverse piezoelectric effect (Curie et al., 1880). In order to reduce the mathematical model, the following assumptions were made:

a) bending of the element takes place according to the Euler’s hypothesis, and radii of curvature of the deflected components are identical

b) in connection plane of piezoelectric and non-piezoelectric components, there is no intermediate layer (the presence of adhesive layers and electrodes on the bottom and top surface of the piezoelectric material is omitted), and no sliding occurs,

c) in piezoelectric layers, transverse piezoelectric effect 1–3 takes place, causing pure bending.

2.2. General equation for strain of piezoelectric triple segment

Fragment of three-layer piezoelectric converter, as given in Fig. 2, shall now be examined. In the structure of the transducer, two piezoelectric components (of different thickness and length) and a non-piezoelectric element can be distinguished. Moment $M(x)$ is induced by mechanical operating load. Its value and distribution is also dependant on boundary conditions related to the mounting of the converter. In the bender, also electrical load $M_e$ will occur, caused by piezoelectric effect (present in piezoelectric components).

![Fig. 2. Fragment of three-layer piezoelectric converter](image)

In the analysed structure, subjected to bending, it is possible to determine the five characteristic ranges, related to a change in load and stiffness. Within the $x_1<x<x_2$ and $x_2+L_3<x<x_1+L_1$ ranges, there is a piezoelectric bimorph segment PBS generating electric moment $M_{eB}$. Between the PBS segments (the $x_2<x<x_3$, L3 segment), there is piezoelectric triple segment PTS, which generates electric moment $M_{eT}$ and has flexural stiffness $E_{idT}$. The other two ranges are within the homogeneous beam with stiffness $E_{idB}$.

Taking into account the above structural variability and load conditions, general differential equation for strain of bender can be noted as follows:

$$\frac{\partial^2 y}{\partial x^2} = M_{eB} Y_b \left( H[x - x_1] - H[x - x_2] \right) + M_{eT} Y_T \left( H[x - x_2] - H[x - (x_2 + L_3)] \right) + M(x)/E_{idT}$$

where: $H[x - x_1]$ – Heaviside function,

$$Y_b = \frac{E_b J_o (M_e + M(x)) - E_o J_o M(x)}{E_o E_b J_o J_{tot}}$$

$E_p, E_o$ – Young’s modules of piezoelectric and non-piezoelectric element, $J_o, J_{tot}$ – moments of inertia (described hereinafter).

As the determining of the mechanical moment $M(x)$ in general does not pose any problems, determining electric load $M_e$, generated by piezoelectric segments, is very burdensome and requires solving the two-dimensional problem of bending the multilayer structure, while taking piezoelectric effect into account.

2.3. Electric load $M_e$ generated by piezoelectric segments

2.3.1. Piezoelectric triple segment PTS

Three-layer structure of piezoelectric triple segment PTS (Fig. 3), with constant width $b$, made of two piezoelectric components (1) and (3) and a non-piezoelectric element (2), shall now be investigated.

The PTS is not subjected to any mechanical load, and longitudinal forces $N_i$ and bending moments $M_i$, occurring on individual layers are the result of the applied voltage $V$. Based on the equilibrium equation of forces condition, the following can be noted:

$$N_1 + N_2 + N_3 = 0.$$  

The sum of moments in relation to interface must be zero, therefore:

$$M_1 + M_2 + M_3 - \frac{N_2 t_2}{2} - N_3 \left( t_2 + \frac{t_3}{2} \right) + \frac{N_1 t_1}{2} = 0.$$  

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According to the adopted Euler’s hypothesis, bending moments can be notified as follows:

\[
\begin{align*}
M_1 &= \frac{\varepsilon_p l_3}{\rho}
, \\
M_2 &= \frac{\varepsilon_p l_b}{\rho}
, \\
M_3 &= \frac{\varepsilon_p l_3}{\rho}
, \\
\end{align*}
\] (4)

Substituting relations (4) to (3) and providing simple mathematical transformations resulted in the following:

\[
\frac{1}{\rho} = \frac{(N_2+2N_3)\varepsilon_2-N_1\varepsilon_1+N_3\varepsilon_3}{2(E_p/\rho + E_p/l_3 + E_p/l_3)}

(5)
\]

Including the relation between radius of curvature \( \rho \) and deflection \( w(x) \):

\[
\frac{1}{\rho} = \frac{\partial^2w}{\partial x^2}

(6)
\]

differential equation for bending of converter can be notified as follows:

\[
\frac{\partial^2w}{\partial x^2} = \frac{(N_2+2N_3)\varepsilon_2-N_1\varepsilon_1+N_3\varepsilon_3}{2(E_p/\rho + E_p/l_3 + E_p/l_3)}

(7)
\]

Constitutive equations for lower and upper converter layers, including the piezoelectric effect in layers 1 and 3 gave the following:

\[
\begin{align*}
\frac{\partial^2 u_{x1}}{\partial x^2} &= \frac{N_1}{E_p A_2} - d_{31}\left(\frac{\partial V}{\partial t}\right) \\
\frac{\partial^2 u_{x2}}{\partial x^2} &= \frac{N_2}{E_p A_2} \\
\frac{\partial^2 u_{x3}}{\partial x^2} &= \frac{N_3}{E_p A_2} + d_{31}\left(\frac{\partial V}{\partial t}\right)
\end{align*}

(8)
\]

where: \( A_2, A_3, A_3 \) – layers cross sectional areas, \( d_{31} \) – piezoelectric constant.

\[
\begin{align*}
N_1 &= -\frac{bd_3 A_p V (\alpha_1 + \alpha_2)(t_1^2 + t_2^2 + t_3^2 - t_1 t_3)(t_1 + t_2 + t_3))}{\alpha_1 + \alpha_2} \\
N_2 &= -\frac{-3bd_3 A_p V t_2 (t_1 - t_3)(t_2 + t_1 + t_3)(t_2 + t_1 + t_3)}{\alpha_1 + \alpha_2} \\
N_3 &= \frac{bd_3 A_p V (\alpha_3 + \alpha_2)(t_2^2 + t_3^2 + t_2 t_3)(t_2 + t_3 + t_3^2)\alpha_1 + \alpha_2}{\alpha_1 + \alpha_2}
\end{align*}

(11)
\]

where: \( \alpha_1 = E_p t_2^2 t_3^2 - 2E_p t_2 t_3 (t_1^3 + t_3^3 + t_3^2 t_1 + t_1 t_3 + 3t_2(t_1^2 + t_2^2)), \alpha_2 = E_p t_1^4 - 2t_4^3 t_3 + t_3^4 - 6t_2^2 t_3 (2t_2 + t_3) - 2t_2 t_3 (6t_2^2 + 6t_2 t_3 + t_3^2), \alpha_3 = E_p t_2^2 t_3^2 + E_p t_3^2 (t_3 + 3t_3^2 + t_3^3 + t_3^4)
\]

The differential equation for bending of PTS, in the \( M_{oT} \) moment function, can be notified as follows:

\[
\frac{\partial^2 w}{\partial x^2} = -\frac{M_{oT}(1 + t_2/t_3)}{E_p/\rho t_3}

(12)
\]

Basing on the comparison of relations (7) and (12), it is possible to determine the bending moment \( M_{oT} \), which is the result of the piezoelectric effect:

\[
M_{oT} = \frac{E_p/\rho t_3 (N_1 t_3 - N_2 t_2 - 2N_3 t_2 - N_3 t_3)}{2(E_p/\rho + E_p/l_3 + E_p/l_3)(t_1 + t_2 + t_3)}

(13)
\]

The required moment, taking relation (11) into account, can be noted as follows:

\[
M_{oT} = d_{31} E_p^2 V \theta

(14)
\]

where:

\[
\theta = -\frac{b J_{oT} (t_1 + t_3)(t_1 + t_2 + t_3)}{2(E_p/\rho + E_p/l_3 + E_p/l_3)(t_1 + t_2 + t_3)(\alpha_1 + \alpha_2)}
\]

Moments of inertia for particular layers are, respectively:

\[
J_b = \frac{b t_3^2}{12}, J_{p1} = \frac{b t_1^2}{12}, J_{p3} = \frac{b t_3^2}{12}

(15)
\]

The averaging value of the moment of inertia \( J_{oT} \) (16) – calculated in relation to the neutral layer – was determined in the same way as in the paper (Mieczkowski, 2016), using the method of transformation of the cross sectional area (Fertis, 1996).

\[
J_{oT} = \frac{b(\alpha_1 + \alpha_2 + \alpha_3)}{12E_p(t_1 + t_3)}

(16)
\]

\[
\alpha_3 = 6E_p (E_p t_1^2 t_3 t_2 (t_1 + 2t_2 + t_3)^2 + E_p t_2 (t_1^3 + 2t_1^2 t_2 + t_1 t_2^2 + t_2 t_3 + t_3(t_2 + t_3)^2))
\]
2.3.2. Piezoelectric bimorph segment PBS

Solution for the issue of piezoelectric bimorph segment is provided in the paper (Mieczkowski, 2016). Below are the relations describing electric load (17) generated by the PBS and the averaging value of the moment of inertia (18).

\[
M_{EB} = d_{31} E_p^2 V \beta,
\]

\[
J_{oB} = \frac{b \rho_0}{2E_p (\rho t_z + \rho t_1)}
\]

where: \( \beta = \frac{6d_{31} E_p t_z (t_2 + t_1)}{\beta_2}, \beta_2 = E_p^2 t_2^2 + E_p^2 t_1^2 + 2E_p E_p t_2 t_1 (2t_2^2 + 3t_2 t_1 + t_1^2). \)

2.4. Electromechanical characteristics of cantilever transducer subjected to concentrated force \(F\)

This part of the paper is concerned with the application of the proposed method, based on implementing the PBS and PTS segments to a homogeneous beam so as to determine the analytical relations describing the deflection of the transducer with fixed geometry and known boundary conditions.

In the transducer, as shown on Fig. 4, the left side is fixed-mounted, and the right can move freely. The operational load results from the external force \(F\) and electric moments \(M_o\) generated by the applied voltage \(V\). Based on the conditions for equilibrium of forces and moments, values for the reaction in mounting were established, and are as follows: \(R_y = F, R_x = 0, M_F = FL\).

![Fig. 4. Cantilever transducer with PBS and PTS segments](image)

Therefore, mechanical moment \(M(x)\) takes the following form:

\[
M(x) = -M_x + R_x x = -FL + Fx.
\]

Substituting relations (14), (17) and (19) to the general solution, described by the formula (1), upon double integration, gives a relation that describes deflection of the analysed bender (20):

\[
A_1 = 3E_p d_{31} E_p^2 J_{oB} + (\beta J_{ot} H(x - x_1)(x - x_1)^2 - \beta J_{oB} - \theta J_{ot})(H(x - x_1)(x - x_2)^2 + x(L_1 + x_2))
\]

\[
+H(x - x_1)(L_3 - x_1)(L_3 - x_1^2) - \beta J_{ot} H(x - x_1)(x - x_1)^2 + x(L_1 + x_2))
\]

\[
B_1 = J_{ot}(E_p J_{oB} - E_{pJ_{oB}})(H(x - x_1)(x - x_1)^2(3L - x - 2x_1) - H(x - x_1 - L_1)(3L - 2L_1 - x - 2x_1))
\]

\[
+\beta p J_{ot} x^2(x - 3L) + (J_{oB} - J_{ot}E_p J_{oB})(-H(x - x_1)(x - x_2)^2(3L - x - 2x_1) + (2(L_3 + x_2) - 3L)(3L + x_2)^2x)
\]

\[
+H(x - x_2 - L_3)(3L - 2L_3 - x - 2x_2)(L_3 - x + x_2)^2.
\]

Integration constants are determined on the basis of the following boundary conditions:

\[
\frac{dy}{dx}(0) = 0, y(0) = 0.
\]

It is worth noting that, in the obtained solution (20), assumption of even length \((L_1 = L_3)\) and thickness \((t_1 = t_3)\) of both piezoelectric layers, results in obtaining the identical solution as the one presented in the paper (Mieczkowski, 2017). In order to determine the electromechanical behaviour of the converter, most frequently, the FEM-based analyses are carried out (Rahmoun and Osmont, 2010, Szpica, 2015, 2018, Borawski, 2015, 2018). Therefore, in order to validate the method developed, a transducer with the following geometry \(-L_1 \neq L_3, t_1 \neq t_3\) - was modelled following the FEM method with the use of the ANSYS application (Mieczkowski, 2016; Documentation for ANSYS, 2010). In the calculations, the following material data were assumed:

- Young's modulus: \(E_p = 2.0 \times 10^5, E_0 = 4.0 \times 10^5\) N/m²
- Poisson's ratio: \(\nu_p = 0.29, \nu_0 = 0.33\)
- Piezoelectric strain coefficients: \(d_{31} = 2.2 \times 10^{-11}\) C/N, \(d_{32} = 0.3 \times 10^{-11}\) C/N, \(d_{33} = -3.0 \times 10^{-11}\) C/N
- Relative permittivity at constant stress: \((\varepsilon_{33}) = 12\)

The difference between analytical and numerical solutions was approx. 1% for electrical load and 3% for mechanical load. The comparison of both solutions is graphically shown in Fig. 5.

The obtained particular solution (20) allowed one to determine the influence of mutual geometric relations of particular components of the bender on its functional characteristics. Two types of converters were analysed:

1. converter with piezoelectric components of different length \(-L_1 \neq L_3\) (whereas the upper piezoelectric layer is of the same length as beam element \(-L_1 = L_1\)
2. two piezoelectric layers of the same length \(-L_1 = L_3, x_1 = x_2\)
Fig. 5. Deflection of the cantilever converter for \( x_1 = 0, x_2 = 23L, t_1 = 2 \) mm, \( t_2 = 3 \) mm, \( t_3 = 6 \) mm, \( L = 60 \) mm: a) subjected only to an electrical voltage, \( V = 100 \) V, \( F = 0 \); b) subjected only to a force, \( V = 0, F = -10 \) N

Fig. 6. Influence of location and length of the piezoelectric element on the deflection of the transducer under electric load (\( V = 100 \) V, \( F = 0 \)) only: a) \( t_1/t_3 = 2 \), b) \( t_1/t_3 = 1 \), c) \( t_1/t_3 = 0.5 \)

Fig. 7. Influence of location and length of the piezoelectric element on the deflection of the transducer under mechanical load (\( F = -10 \) N, \( V = 0 \)) only: a) \( t_1/t_2 = 2 \), b) \( t_1/t_3 = 1 \), c) \( t_1/t_2 = 0.5 \)

For both variants, the influence of the above parameters on the deflection of the right end of the transducer was analysed:

- relative thickness of particular piezoelectric elements \( t_1/t_3 \)
- relative length of components \( L_{1,3}/L \)
- locations of piezoelectric components (PTS segment: at the left of the converter \( -x_2 = 0 \), in the middle \( -x_2 + L_3/2 = L/2 \), at the right \( -x_2 + L_3 = L \))

In the analysis performed, the following material and geometric data were included: \( E_p = 2.0 \times 10^{12} \) N/m\(^2\), \( E_s = 4.0 \times 10^{12} \) N/m\(^2\), \( d_{31} = 2.2 \times 10^{-11} \) C/N, \( L = 60 \) mm, \( t_2 = 1 \) mm.

The below figures show a graphical representation of the results of tests carried out for the converter with different length of piezoelectric components, subjected to electric moment (Fig. 6) and mechanical load (Fig. 7).

Based on the obtained results (Fig. 6), it can be claimed that when subjecting the element to electric moment only, displacement of the right end of the transducer:

- is increased with the increase in relative thickness of the layers \( t_1/t_3 \)
- is increased with the increase in relative length \( L_3/L \) in case of \( t_1/t_2 \geq 1 \)
is decreased with the increase in relative length $L_3/L$ in case of $t_1/t_3 < 1$

- has the highest value, when the piezoelectric segment is located at the left in case of $t_1/t_3 \geq 1$

- has higher values ($L_3 \neq 0, L_3 \neq L$), when the piezoelectric segment is located at the right in case of $t_1/t_3 < 1$

As expected, when applying mechanical load only (Fig. 7), the increase in relative thickness of layers $t_1/t_3$ causes the increase in the deflection of the transducer. The biggest displacement is achieved, when the piezoelectric segment is located at the right of the transducer. Noticeable also is the decrease in displacement value along with the increase in the $L_3/L$ parameter value.

The results of analyses for the transducer with piezoelectric layers of the same length ($L_1 = L_3, x_1 = x_3$), are given in Fig. 8. Due to the fact that when applying mechanical load, the influence of the considered factors (relative thickness, length and location of piezoelectric layers) on electromechanical characteristics was identical as in the case of the converter with different lengths of piezoelectric components, the results obtained only for electric load are given.

![Graph](image)

**Fig. 8.** Influence of location and length of the piezoelectric element on the deflection of the transducer – with the same length of piezoelectric components – under electric load ($V = 100 \text{ V}, \text{ F} = 0$) only: a) $t_1/t_3 = 2$, b) $t_1/t_3 = 1$, c) $t_1/t_3 = 0.5$

Analysing the obtained results, as given in Fig. 8, it can be claimed that displacement of the right end of the transducer increases with the increase in relative thickness ($t_1/t_3$) and length ($L_1,3/L$) of piezoelectric layers. The highest values of relocation are obtained when piezoelectric segment is located at the left.

3. CONCLUSIONS

Based on the testing performed, it can be noted that:

- the obtained solution for cantilever transducer, assuming the same length and thickness of piezoelectric layers, is compatible with the solution provided in paper (Mieczkowski, 2017)

- analytical solutions conform to the results obtained from the FEM (for electric load, the difference is about 1%, and for mechanical load – 3%)

As for the electromechanical characteristics of the cantilever transducer, it was found that the following factors have significant influence on the conditions of strain of the transducer:

- relative thickness of particular piezoelectric elements $t_1/t_3$

- relative length of components $L_1,3/L$

- locations of piezoelectric components

For transducer with different length of piezoelectric layers, displacement of the end of the transducer is increased with the increase in relative thickness of the layers $t_1/t_3$. Increase in relative length $L_3/L$ results in:

a) when applying electric load:

- increase in displacement $y(L)$, if $t_1/t_3 \geq 1$

- decrease in displacement $y(L)$, if $t_1/t_3 < 1$

b) when applying mechanical load:

- decrease in displacement

The biggest relocation ($L_3 \neq 0, L_3 \neq L$) is achieved when piezoelectric segment is located:

a) when applying electric load:

- at the left in case of $t_1/t_3 \geq 1$

- at the right in case of $t_1/t_3 < 1$

b) when applying mechanical load:

- at the right of the transducer

Analysing the issue of the transducer with uniform length of piezoelectric layers, it was found that:

a) for applying mechanical load, the influence of the considered factors on electromechanical characteristics was identical to the case of the transducer with different lengths of piezoelectric components

b) when applying electric load (induced by the voltage $V$):

- deflection of the right end of the transducer is increased with the increase in relative thickness ($t_1/t_3$) and length ($L_1,3/L$) of piezoelectric layers

- the highest values of displacement are achieved in the case when piezoelectric segment is located at the left

It is therefore concluded that the utility features of the piezoelectric transducers are significantly influenced by their geometrical properties. The increase in thickness and length of piezoelectric layers increases the flexural stiffness and the generated moment $M$. The increase in stiffness results in a decrease in converter deflection. However, the increase in the moment increases its deflection. Therefore, by choosing the location, thickness and length of the piezoelectric components properly, a compromise solution can be obtained, ensuring the greatest deflection of the
transducer. Analytical solutions describing deformations of piezoelectric transducers may be helpful in this.

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