ALMOST KENMOTSU PSEUDO-METRIC MANIFOLDS*

BY

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Abstract. In this paper, we introduce the geometry of almost Kenmotsu pseudo-metric manifolds, emphasizing the analogies and differences with respect to the Riemannian case. After giving some fundamental formulas and properties of almost Kenmotsu pseudo-metric manifolds, some classification theorems of such manifolds being locally symmetric or satisfying some nullity conditions are investigated.

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1. Introduction

Manifolds known as contact metric manifolds and Sasakian manifolds have been introduced and studied in [3, 22, 23]. We also refer the reader to the recent monograph [5] for a wide and detailed overview of the results in this field. However, the classical papers related to contact metric manifolds are assumed to have a Riemannian metric, we also notice that the contact manifolds furnished with a pseudo-Riemannian metric and Sasakian pseudo-Riemannian manifolds are introduced in [1, 15, 24]. For that reason, CALVARUSO and PERRONE [8] introduced the geometry of contact pseudo-metric manifolds and classified contact pseudo-metric manifolds of constant sectional curvature. In particular, the contact Lorentzian manifolds are investigated by CALVARUSO [7]. For more details on semi-Riemannian geometry with its applications we refer the reader to [20].

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On the other hand, since Kenmotsu in [18] introduced the notion of Kenmotsu manifolds which can be regarded as an analogy of almost contact metric manifolds, many authors studied such manifolds under various geometric conditions. For instance, Jun, De and Pathak [17] obtained some results for Kenmotsu manifolds being Ricci semi-symmetric or Weyl semi-symmetric. Also, De, Yildiz and Yaliniz [9] studied $\phi$-recurrent Kenmotsu manifolds. It is well known that a Kenmotsu manifold is always an almost Kenmotsu manifold, but the converse is not necessarily true. Recently, Dileo and Pastore obtained some results for almost Kenmotsu manifolds with conditions of local symmetries, $\eta$-parallelism and nullity distributions in [12], [13] and [14], respectively. Also, Dileo [10, 11] studied almost $\alpha$-Kenmotsu manifolds by generalizing some classical results of almost Kenmotsu manifolds. For more results related to almost Kenmotsu manifolds we refer the readers to [16, 19, 21].

Since then, up to our knowledge, a systematic study of general almost Kenmotsu pseudo-metric manifolds has not been undertaken yet. The aim of this paper is to start this study, providing some technical apparatus needed for further investigations.

This paper is organized as the following way. In Section 2, we provide some basic formulas and properties of almost Kenmotsu pseudo-metric manifolds by giving the definition of such manifolds. Section 3 is devoted to investigating the relationship between pseudo-Riemannian metrics of different signature associated to the same almost contact structure. Later, the classification theorems for almost Kenmotsu pseudo-metric manifolds being locally symmetric are obtained in Section 4. In Section 5, we classify almost Kenmotsu pseudo-metric manifolds satisfying nullity conditions, that is, the characteristic vector field belongs to the $(k, \mu)'$-nullity or $(k, \mu)$-nullity distribution. Finally, it is worth pointing out that some of our results extend the corresponding results obtained for the Riemannian case in [12, 14, 18].

2. Almost Kenmotsu pseudo-metric manifolds

Firstly, we give some basic notions of almost Kenmotsu pseudo-metric manifolds. An almost contact structure [5] on a $(2n+1)$-dimensional smooth manifold $M^{2n+1}$ is a triple $(\phi, \xi, \eta)$, where $\phi$ is a $(1, 1)$-tensor, $\xi$ a global vector field and $\eta$ a 1-form, such that

$$\phi^2 = -\text{Id} + \eta \otimes \xi, \quad \eta(\xi) = 1,$$

(2.1)
which implies that $\phi(\xi) = 0$, $\eta \circ \phi = 0$ and $\text{rank}(\phi) = 2n$. It follows from [8] that a pseudo-Riemannian metric $g$ on $M^{2n+1}$ is said to be compatible with the almost contact structure $(\phi, \xi, \eta)$ if

$$g(\phi X, \phi Y) = g(X, Y) - \varepsilon \eta(X)\eta(Y), \quad (2.2)$$

where $\varepsilon = \pm 1$. Following [8], a smooth manifold $M^{2n+1}$ furnished with an almost contact structure $(\phi, \xi, \eta)$ and a compatible pseudo-Riemannian metric $g$ is said to be an almost contact pseudo-metric manifold, denoted by $(M^{2n+1}, \phi, \xi, \eta, g)$. A simple computation shows that $g(X, \phi Y) = -g(\phi X, Y)$ and $\eta(X) = \varepsilon g(\xi, X)$, in particular, $g(\xi, \xi) = \varepsilon$.

The fundamental 2-form $\Phi$ is defined by $\Phi(X, Y) = g(X, \phi Y)$ for any vector fields $X$ and $Y$ on $M^{2n+1}$. An almost contact pseudo-metric manifold such that $d\eta = 0$ and $d\Phi = 2\eta \wedge \Phi$ is said to be an almost Kenmotsu pseudo-metric manifold. It is well known that the normality of almost contact structure is expressed by the vanishing of the tensor $N_\phi = [\phi, \phi] + 2d\eta \otimes \xi$, where $[\phi, \phi]$ is the Nijenhuis tensor of $\phi$. Notice that a Kenmotsu manifold is a normal almost Kenmotsu pseudo-metric manifold $M^{2n+1}$ with $\varepsilon = 1$ and the metric of $M^{2n+1}$ being Riemannian. When an almost Kenmotsu pseudo-metric manifold $M^{2n+1}$ has a normal almost contact structure, we say that $M^{2n+1}$ is a Kenmotsu pseudo-metric manifold. Throughout this paper, we shall denote by $\nabla$ and $\Gamma(TM)$ the Levi-Civita connection of pseudo-Riemannian metric $g$ and the Lie algebra of all tangent vector fields on $M^{2n+1}$ respectively. Using the well known Koszul formula, then the following Lemma can be proved by a direct calculation.

**Lemma 2.1** ([8]). Let $(\phi, \xi, \eta)$ be an almost contact structure and $g$ a compatible pseudo-Riemannian metric on a smooth manifold $M^{2n+1}$. Then,

$$2g((\nabla_X \phi)Y, Z) = 3d\Phi(X, \phi Y, \phi Z) - 3d\Phi(X, Y, Z) + g(N_\phi(Y, Z), \phi X) + \varepsilon N(Y, Z)\eta(X) + 2\varepsilon d\eta(\phi Y, X)\eta(Z) - 2\varepsilon d\eta(\phi Z, X)\eta(Y),$$

for any tangent vector fields $X, Y, Z \in \Gamma(TM)$ and $N(X, Y) = (\mathcal{L}_\phi X)Y - (\mathcal{L}_\phi Y)X$, where $\mathcal{L}_X$ denotes the Lie derivative in the direction of $X$.

**Proposition 2.2.** Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be an almost Kenmotsu pseudo-metric manifold. Then

$$2g((\nabla_X \phi)Y, Z) = 2g(\varepsilon g(\phi X, Y)\xi - \eta(Y)\phi X, Z) + g(N_\phi(Y, Z), \phi X), \quad (2.3)$$

for any tangent vector fields $X, Y, Z \in \Gamma(TM)$. 
Proof. By the definition of tensor $N$, it is easy to get

\[ N(Y, Z) = (\phi Y)(\eta(Z)) - \eta([\phi Y, Z]) - (\phi Z)(\eta(Y)) + \eta([\phi Z, Y]) = 2d(\phi Y, Z) - 2d(\phi Z, Y). \]

Thus, using the definition of almost Kenmotsu pseudo-metric manifolds, then the proof follows from the Lemma 2.1.

Let $h = \frac{1}{2} L_\phi$ and $h' = h \circ \phi$, putting $X = \xi$ and $Y = \xi$ in (2.3), and noticing that $g(N_{\phi}(\xi, Z), \phi X) = -g([\xi, Z] + \phi[\xi, \phi Z], \phi X) = -2g(h X, Z)$, then we obtain the following two equations

\begin{align*}
\nabla \phi &= 0, \\
\nabla X \xi &= -\phi^2 X + h' X,
\end{align*}

for any $X \in \Gamma(TM)$. Also, as in the Riemannian case, using the above two equations we can prove that $h$ is self-adjoint, $h\phi = \phi h$, $h'\phi = -\phi h'$, $h^2 = h'^2$ and $h\xi = \text{tr}(h) = 0$.

Proposition 2.3. Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be an almost Kenmotsu pseudo-metric manifold. Then $\text{div} \xi = 2n$, $\text{div} \eta = -2n \varepsilon$, $\text{Ric}(\xi, \xi) = -2n - \text{tr} h^2$.

Proof. We may consider a local $\phi$-basis $\{\xi, E_1, \ldots, E_{2n}\} = \{\xi, e_1, \ldots, e_n, \phi e_1, \ldots, \phi e_n\}$ on $M^{2n+1}$. Using (2.5) and the fact that $h$ is anti-commutative with respective to $\phi$, then we have

\[ \text{div} \xi = \text{tr} \nabla \xi = \sum_{i=1}^{n} \varepsilon_i g(\nabla e_i \xi, e_i) + \sum_{i=1}^{n} \varepsilon_i g(\nabla \phi e_i \xi, \phi e_i) = \sum_{i=1}^{n} \varepsilon_i g(e_i + h' e_i, e_i) + \sum_{i=1}^{n} \varepsilon_i g(\phi e_i + h' \phi e_i, \phi e_i) = 2n, \]

where $g(e_i, e_i) = g(\phi e_i, \phi e_i) = \varepsilon_i = \pm 1$. Thus, from (2.6) we get $\text{div} \eta = -\text{tr} \nabla \eta = -\text{div} \xi = -2n \varepsilon$.

Now, we denote by $R$ and $\ell$ the curvature tensor of $M^{2n+1}$ and $R(\cdot, \xi)\xi$, respectively. By using (2.5) then we have

\begin{align*}
\ell(X) &= \nabla_X \nabla_\xi \xi - \nabla_\xi \nabla_X \xi - \nabla_{[X, \xi]} \xi \\
&= -\nabla_\xi (X - \eta(X) \xi + h' X) + \eta([X, \xi]) - [X, \xi] - h' \nabla_X \xi + h'_\xi X \\
&= - (\nabla_\xi h') X - 2h' X + \phi^2 X - h^2 X.
\end{align*}
Applying (1.1)-type tensor $\phi$ on (2.7), we get $\phi \ell(X) = -(\nabla_{\xi} h)X - 2hX - \phi X - \phi h^2 X$. Replacing $X$ by $\phi X$ in the above equation, together with (2.7) we have

$$\ell(X) - \phi \ell(\phi X) = 2(\phi^2(X) - h^2(X)).$$

Again consider the local $\phi$-basis $\{\xi, e_1, \ldots, e_n, \phi e_1, \ldots, \phi e_n\}$ on $M^{2n+1}$. The sectional curvatures of non-degenerate planes spanned by $\{\xi, e_i\}$ and $\{\xi, \phi e_i\}$ respectively are given by $K(\xi, e_i) = \epsilon \epsilon_i R(\xi, e_i, \xi, e_i) = \epsilon \epsilon_i g(\ell(e_i), e_i)$, $K(\xi, \phi e_i) = \epsilon \epsilon_i R(\xi, \phi e_i, \xi, \phi e_i) = -\epsilon \epsilon_i g(\phi \ell(\phi e_i), e_i)$, where $\epsilon_i = g(e_i, e_i) = g(\phi e_i, \phi e_i) = \pm 1$ for all indices $i = 1, \ldots, n$. Thus, it follows from (2.8) and the above two equations that

$$\text{Ric}(\xi, \xi) = \sum_{i=1}^n R(\xi, e_i, \xi, e_i) + \sum_{i=1}^n R(\xi, \phi e_i, \xi, \phi e_i) = \epsilon \sum_{i=1}^n (K(\xi, e_i)$$

$$+ K(\xi, \phi e_i)) e_i = \sum_{i=1}^n g(\ell(e_i) - \phi \ell(\phi e_i), e_i) = -2n - \text{tr} h^2.$$

Then the proof is complete. \(\square\)

**Theorem 2.4.** An almost Kenmotsu pseudo-metric manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ is a Kenmotsu pseudo-metric manifold if and only if

$$\nabla_X \phi Y = \epsilon g(\phi X, Y)\xi - \eta(Y)\phi X,$$

for any $X, Y \in \Gamma(TM)$.

**Proof.** Suppose that $M^{2n+1}$ is a Kenmotsu pseudo-metric manifold, then (2.9) follows from Proposition 2.2 and the vanishing of tensor $N_{\phi}$. Conversely, let $M^{2n+1}$ be an almost Kenmotsu pseudo-metric manifold. Putting $Y = \xi$ in (2.9) yields $\nabla_X \xi = -\phi^2 X$, then we have

$$d\eta(X, Y) = \epsilon X (g(\xi, Y)) - \epsilon Y (g(\xi, X)) - \eta([X, Y]) = \epsilon g(\nabla_X \xi, Y)$$

$$+ \epsilon g(\xi, \nabla_X Y) - \epsilon g(\nabla_Y \xi, X) - \epsilon g(\xi, \nabla_Y X) - \eta([X, Y])$$

$$= -\epsilon g(\phi^2 Y, X) + \epsilon g(\phi^2 Y, X) = 0.$$

On the other hand, by using (2.9) and a straightforward computation we have

$$d\Phi(X, Y, Z) = g(Y, \nabla_X \phi Z) - g(X, \nabla_Y \phi Z) + g(\nabla_Z X, \phi Y)$$

$$+ g(X, \nabla_Z \phi Y) + g([X, Z], \phi Y) - g([Y, Z], \phi X)$$

$$= 2\eta(X) g(Y, \phi Z) - 2\eta(Y) g(X, \phi Z) + 2\eta(Z) g(X, \phi Y)$$

$$= 2\eta \wedge \Phi(X, Y, Z).$$
Finally, it follows from (2.5) and $\nabla_X \xi = -\phi^2 X$ that $h = h' = 0$. Thus, using (2.2) and (2.9) we obtain

$$N_\phi(X, Y) = \phi^2 [X, Y] + [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y]$$

$$= -\phi(\nabla_X \phi Y - \phi \nabla_Y X) + \phi(\nabla_Y \phi X - \phi \nabla_X X)$$

$$+ (\nabla_{\phi X} \phi Y - \phi \nabla_{\phi X} Y) - (\nabla_{\phi Y} \phi X - \phi \nabla_{\phi Y} X)$$

$$= -\phi(\varepsilon g(\phi X, Y)\xi - \eta(Y)\phi X) + \phi(\varepsilon g(\phi Y, X)\xi - \eta(X)\phi Y)$$

$$+ (\varepsilon g(\phi^2 X, Y)\xi - \eta(Y)\phi^2 X) - (\varepsilon g(\phi^2 Y, X)\xi - \eta(X)\phi^2 Y) = 0,$$

for any $X, Y \in \Gamma(TM)$. Then we complete the proof.

Throughout the paper, we denote by $\mathcal{D}$ the distribution orthogonal to $\xi$, that is, $\mathcal{D} = \text{Im}(\phi) = \ker(\eta)$ and it is said to be the contact distribution. Thus, we obtain the following theorem.

**Theorem 2.5.** Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be an almost Kenmotsu pseudo-metric manifold with $h = 0$. Then $M^{2n+1}$ is locally a warped product space $M_\xi \times_f M'$, where $M_\xi$ is an open interval with coordinate $t$, $M'$ is a $2n$-dimensional indefinite almost Kähler manifold and $f = ce^t$ for some positive constant $c$.

**Proof.** Using $h = 0$ then it follows from (2.5) that $\nabla_X \xi = X$ for any $X \in \mathcal{D}$. We denote by $M'$ and $\nabla'$ the integral manifold of the contact distribution $\mathcal{D}$ and the Levi-Civita connection of the metric of $M'$, respectively. Then the second fundamental form $B(X, Y)$ of semi-Riemannian immersion $M' \to M^{2n+1}$ is given by $g(B(X, Y), \xi) = g(\nabla_X Y - \nabla_Y X, \xi) = -g(Y, \nabla_X \xi) = -g(X, Y)$, for any $X, Y \in \mathcal{D}$. Thus, $M'$ is a totally umbilical submanifold of $M^{2n+1}$ and the mean curvature vector field is given by $H = -\varepsilon \xi$. It follows from (2.5) that $\nabla_\xi ^\mathcal{D} = 0$, then $M^{2n+1}$ is locally a warped product space $M_\xi \times_f M'$, where $M_\xi$ is an integral curve of $\xi$. It follows from [2] that the mean vector field $H$ is $\pi$-related to $-\frac{1}{\varepsilon}\text{grad} f$, where the projection $\pi : M_\xi \times_f M' \to M_\xi$ is a semi-Riemannian submersion. This means that $\varepsilon f \xi = \text{grad} f$, locally, we may write $\xi = \frac{\partial}{\partial t}$ with coordinate $t$. Following Definition 47 of [20] and the fact $g(\xi, \xi) = \varepsilon$, we see that $\text{grad}(f) = \varepsilon \frac{\partial f}{\partial t} \frac{\partial}{\partial t}$. The above arguments implies that $f = ce^t$, where $c$ is a positive constant. Finally, we denote by $J$ the restriction of $\phi$ on $\mathcal{D}$, then a simple computation gives that $(M', J)$ is a $2n$-dimensional indefinite almost Kähler manifold.
3. Deformations of almost contact pseudo-metric structure

In this section, following [8] we now investigate the relationship between pseudo-Riemannian metrics of different signature associated to the same almost contact pseudo-metric structure. Let \((\phi, \xi, \eta)\) be an almost contact pseudo-metric structure on a smooth manifold \(M^{2n+1}\) and \(g\) a compatible pseudo-Riemannian metric with \(g(\xi, \xi) = \varepsilon\). It is easy to check that the following pseudo-Riemannian metric (see Section 3 of [8])

\[
\overline{g} = g - 2\varepsilon \eta \otimes \eta
\]

is still compatible with the same almost contact structure \((\phi, \xi, \eta)\) and \(\xi = \overline{g}(\xi, \xi) = -\varepsilon\).

**Proposition 3.1.** Let \((M^{2n+1}, \phi, \xi, \eta, g)\) be an almost Kenmotsu pseudo-metric manifold. Then \((M^{2n+1}, \phi, \xi, \eta, \overline{g})\) is also an almost Kenmotsu pseudo-metric manifold.

**Proof.** We denote by \(\overline{\Phi}\) the fundamental 2-form with respective to \((M^{2n+1}, \phi, \xi, \eta, \overline{g})\), then it is easy to see that the equation \(\overline{\Phi}(X, Y) = \overline{g}(X, \phi Y)\) holds. Moreover, by using \(\overline{\Phi}(X, Y) = \overline{g}(X, \phi Y) - 2\varepsilon \eta(X)\eta(\phi Y) = \Phi(X, Y)\), then we get \(d\overline{\Phi}(X, Y) = 2\eta \wedge \overline{\Phi}(X, Y)\), for any \(X, Y \in \Gamma(TM)\). Thus, the proof is complete. \(\Box\)

Now we denote by \(\overline{\nabla}\) and \(\overline{R}\) the semi-Riemannian connection and the curvature tensor of \(\overline{g}\) respectively on an almost Kenmotsu pseudo-metric manifold, then we get the following result.

**Proposition 3.2.** Let \((M^{2n+1}, \phi, \xi, \eta, g)\) be an almost Kenmotsu pseudo-metric manifold and \(\overline{g}\) the pseudo-Riemannian metric described by (3.1). Then

\[
\overline{\nabla}_X Y = \nabla_X Y - 2\varepsilon g(\phi^2 X - h' X, Y)\xi,
\]

\[
\overline{R}(X, Y) Z = R(X, Y) Z + 2\varepsilon g(\nabla_Y \xi, Z) \nabla_X \xi - 2\varepsilon g(\nabla_X \xi, Z) \nabla_Y \xi
\]

\[+ 2\varepsilon g(\eta(X) \nabla_Y \xi - \eta(Y) \nabla_X \xi + 2\varepsilon g(\phi X, \phi Y) \xi
\]

\[+ (\nabla_X h') Y - (\nabla_Y h') X, Z) \xi,
\]

for any \(X, Y, Z \in \Gamma(TM)\).

**Proof.** It follows from Koszul formula that

\[
\overline{g}(\overline{\nabla}_X Y, Z) = g(\nabla_X Y, Z) - 2\varepsilon (d\eta(X, Z)\eta(Y) + d\eta(Y, Z)\eta(X))
\]

\[ - 2\varepsilon (d\eta(X, Y)\eta(Z) + X(\eta Y)\eta(Z)) = g(\nabla_X Y, Z) - 2\varepsilon X(\eta Y)\eta(Z).
\]
On the other hand, by using (3.1) in (3.4) we see that

\[(3.5) \quad g(\overline{\nabla}_XY - \nabla_XY, Z) = 2\varepsilon\eta(Z)(\eta(\overline{\nabla}_XY) - X(\eta(Y))).\]

Then, (3.2) follows from the (3.4) and (3.5). Moreover, taking into account (3.2), we have

\[(3.6) \quad R(X, Y)Z = \overline{\nabla}_X(\nabla_YZ - 2\varepsilon g(\phi^2Y - h'Y, Z)\xi)
- \nabla_Y(\nabla_XZ - 2\varepsilon g(\phi^2X - h'X, Z)\xi)
+ \nabla_X(\nabla_[X,Y]Z - 2\varepsilon g(\phi^2[X, Y] - h'[X, Y], Z)\xi)
= R(X, Y)Z - 2\varepsilon g(\phi^2Y - h'Y, Z)\nabla_X\xi + 2\varepsilon g(\phi^2X - h'X, Z)\nabla_Y\xi
+ 2\varepsilon (-X(g(\phi^2Y - h'Y, Z)) + Y(g(\phi^2X - h'X, Z)))\xi
+ 2\varepsilon g(\phi^2X - h'X, \nabla_YZ)
+ g(\phi^2[X, Y] - h'[X, Y], Z)\xi = R(X, Y)Z + 2\varepsilon((\nabla_Y\phi^2)X
- (\nabla_Xh')Y + (\nabla_Xh')Y - (\nabla_Yh')X, Z)\xi
+ 2\varepsilon g(\phi^2X - h'X, Z)\nabla_Y\xi - 2\varepsilon g(\phi^2Y - h'Y, Z)\nabla_X\xi.\]

Substituting (2.5) into the above equation yields (3.3).

As seen in [8] that the $\xi$-sectional and $\phi$-sectional curvatures of a contact pseudo-metric manifold are defined and studied, similarly, we define the following two kinds of sectional curvatures on an almost Kenmotsu pseudo-metric manifold.

**Definition 3.3.** Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be an almost Kenmotsu pseudo-metric manifold. For any $X \in \Gamma(TM)$, we call $K(\xi, X)$ and $K(\phi X)$ the $\xi$-sectional and $\phi$-sectional curvatures of $X$ which are defined respectively as following

\[
K(\xi, X) = \frac{R(X, \xi, X, \xi)}{\varepsilon g(X, X) - (\eta(X))^2} \quad \text{and} \quad K(\phi X) = \frac{R(X, \phi X, X, \phi X)}{g(X, X)(g(X, X) - \varepsilon(\eta(X))^2)}.
\]

Now we denote by $\overline{K}(\xi, X)$ and $\overline{K}(X, \phi X)$ the $\xi$-sectional curvature and $\phi$-sectional curvature on an almost Kenmotsu pseudo-metric manifold with respective to $\overline{g}$, then we get the following result.
Proposition 3.4. Let \((M^{2n+1}, \phi, \xi, \eta, g)\) be an almost Kenmotsu pseudo-metric manifold and \(\overline{g}\) the pseudo-Riemannian metric defined by (3.1). Then

\[
\overline{R}(\xi, X) = -K(\xi, X),
\]

(3.7)

\[
\overline{R}(X, \phi X) = \frac{g(X, X)}{\alpha} K(X, \phi X) + 2 \varepsilon g(\phi^2 X, X)^2 - 2g(h' X, X)^2
\]

\[
\alpha(\alpha + \varepsilon(\eta(X))^2),
\]

(3.8)

\[
\overline{Ric}(\xi, X) = \text{Ric}(\xi, X) - 4n \eta(X) - 2\text{tr}(\nabla_X h') + 2\text{tr}(\nabla_X h')X,
\]

(3.9)

for any \(X \in \Gamma(TM)\), where \(\alpha = g(X, X) - 2\varepsilon(\eta(X))^2\).

Proof. It follows from (3.3) that \(\overline{R}(X, \xi, \xi, X) = g(R(X, \xi, \xi, X) - 2\varepsilon \eta(X))\eta(R(X, \xi, \xi)) = g(R(X, \xi, \xi)), X)\),

since we have \(\eta(R(X, \xi, \xi)) = 0\) following from (2.7). Noticing that \(\tau = -\varepsilon\),

then by a direct calculation we get (3.7). Similarly, from (3.3) we have

\[
\overline{R}(X, \phi X, X, \phi X) = -g(\overline{R}(X, \phi X, X, \phi X) + 2\varepsilon\eta(\overline{R}(X, \phi X, X))\eta(\phi X)
\]

\[= -g(R(X, \phi X)X, \phi X) + 2\varepsilon g(\nabla_{\phi X} \xi, X)\nabla_{\phi X} \xi - g(\nabla_{\phi X} \xi, X)\nabla_{\phi X} \xi, \phi X)\]

\[= -g(R(X, \phi X, X, \phi X) + 2\varepsilon g(\phi^2 X, X)^2 - 4\varepsilon g(h^2 X, X)^2.
\]

Thus, (3.8) follows from the above equation. Moreover, by using (3.3), then a straightforward computation gives

\[
\overline{Ric}(\xi, X) = \sum_{i=1}^{n} \varepsilon_i \overline{R}(\xi, e_i, \xi, e_i) + \sum_{i=1}^{n} \varepsilon_i \overline{R}(\xi, \phi e_i, \xi, \phi e_i)
\]

\[= \sum_{i=1}^{n} \varepsilon_i g(\overline{R}(X, e_i) e_i, \xi) - 2\varepsilon\eta(\overline{R}(X, e_i)e_i))
\]

\[+ \sum_{i=1}^{n} \varepsilon_i g(\overline{R}(X, \phi e_i) \phi e_i, \xi) - 2\varepsilon\eta(\overline{R}(X, \phi e_i) \phi e_i))
\]

\[= -\sum_{i=1}^{n} \varepsilon_i g(R(X, e_i)e_i, \xi) - 2\sum_{i=1}^{n} \varepsilon_i g(\eta(X)\nabla_{e_i} \xi + (\nabla_X h')e_i - (\nabla_{\phi e_i} h')X, e_i)
\]

\[+ \sum_{i=1}^{n} \varepsilon_i g(R(X, \phi e_i) \phi e_i, \xi) - 2\sum_{i=1}^{n} \varepsilon_i g(\eta(X)\nabla_{\phi e_i} \xi + (\nabla_X h')\phi e_i - (\nabla_{\phi e_i} h')X, \phi e_i).
\]

Substituting (2.5) into the above equation, then we obtain (3.9). \qed
4. Local symmetry

In this section we investigate locally symmetric almost Kenmotsu pseudo-metric manifolds, providing some classification theorems for such manifolds. We firstly give the following three properties for almost Kenmotsu pseudo-metric manifolds, omitting the proofs since that the proofs are similar to the Riemannian case which can be seen in [12].

**Proposition 4.1.** Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be an almost Kenmotsu pseudo-metric manifold. Then

\begin{align}
R(X, Y)\xi &= \eta(X)(Y + h'Y) - \eta(Y)(X + h'X) + (\nabla_X h')Y - (\nabla_Y h')X,
\end{align}

for any $X, Y \in \Gamma(TM)$.

**Proposition 4.2.** Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be an almost Kenmotsu pseudo-metric manifold such that the integral manifolds of $D$ are indefinite Kähler manifolds. Then $M^{2n+1}$ is a Kenmotsu pseudo-metric manifold if and only if $\nabla \xi = -\phi^2$.

**Proposition 4.3.** Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be a locally symmetric almost Kenmotsu pseudo-metric manifold, then we have $\nabla \xi \eta = 0$.

Next, we present the pseudo-metric version of the corresponding result shown in [19].

**Theorem 4.4.** Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be an almost Kenmotsu pseudo-metric manifold. Then, the integral manifolds of $D$ are indefinite almost Kähler manifolds with mean curvature vector field $H = -\varepsilon \xi$. Furthermore, the integral manifolds of $D$ are totally umbilical submanifolds if and only if $h = 0$.

**Proof.** Denote by $M'$ and $\nabla'$ the integral manifold of $D$ and the Levi-Civita connection of the metric of the integral manifold, respectively. Then we consider the semi-Riemannian immersion $M' \to M^{2n+1}$, denoting by $B(X, Y)$ the second fundamental form for any $X, Y \in \Gamma(TM)$. By using (2.5), it is easy to see $g(B(X, Y), \xi) = g(\nabla_X Y - \nabla_Y X, \xi) = -g(Y, X + h'X)$, which implies that $B(X, Y) = -\varepsilon g(Y, X + h'X)\xi$. Thus, we know that $M'$ is totally umbilical submanifold of $M^{2n+1}$ if and only if $h = 0$. Moreover, we know that $H = -\varepsilon \xi$ by a straightforward calculation.  

\[ \square \]
Theorem 4.5. Let \((M^{2n+1}, \phi, \xi, \eta, g)\) be a locally symmetric almost Kenmotsu pseudo-metric manifold. Then, the following two statements are equivalent:

1. \(M^{2n+1}\) is a Kenmotsu pseudo-metric manifold.
2. \(h = 0\).

Furthermore, if one of the above conditions holds, then \(M^{2n+1}\) has constant sectional curvature \(K = -\varepsilon\) and hence the integral manifold of \(\mathcal{D}\) is an indefinite Kähler manifold with vanishing sectional curvature.

Proof. (1) \(\Rightarrow\) (2). Let \(M^{2n+1}\) be a Kenmotsu pseudo-metric manifold, it follows from Theorem 2.4 that \(\nabla_X \xi = -\phi^2 X\) for any \(X \in \Gamma(TM)\). Noticing that \(h' = 0\) if and only \(h = 0\), thus, comparing the equation (2.5) with \(\nabla_X \xi = -\phi^2 X\) we know \(h = 0\).

(2) \(\Rightarrow\) (1). Suppose that \(M^{2n+1}\) is an almost Kenmotsu pseudo-metric manifold with \(h = 0\), then from Proposition 4.1 we have

\[
(\nabla_Z R)(X, Y, \xi) = \nabla_Z R(X, Y)\xi - R(\nabla_Z X, Y)\xi \\
- R(X, \nabla_Z Y)\xi - R(X, Y)\nabla_Z \xi \\
= \nabla_Z (\eta(X)Y - \eta(Y)X) + \eta(Y)\nabla_Z X - \eta(\nabla_Z X)Y \\
+ \eta(\nabla_Z Y)X - \eta(X)\nabla_Z Y + R(X, Y)\phi^2 Z \\
= -R(X, Y)Z + \varepsilon g(X, Z)Y - \varepsilon g(Y, Z)X,
\]

for any \(X, Y, Z \in \Gamma(TM)\). Since that \(M^{2n+1}\) is locally symmetric, i.e., \(\nabla R = 0\), then from (4.2) we know that \(M^{2n+1}\) has constant sectional curvature \(K = -\varepsilon\). Denote by \(M'\) the integral manifold of the contact distribution \(\mathcal{D}\), then the second fundamental from of the semi-Riemannian immersion \(M' \to M^{2n+1}\) can be shown by \(g(B(X, Y), \xi) = g(\nabla_X Y, \xi) = -g(X, Y)\), which means that \(M'\) is a totally umbilical submanifold of \(M^{2n+1}\) with the second fundamental form given by \(B(X, Y) = \varepsilon g(X, Y)\xi\). We denote by \(R'\) the Riemannian curvature tensor of \(M'\), then a direct calculation gives

\[
R(X, Y)Z = \nabla_X (\nabla_Y Z - \varepsilon g(Y, Z)\xi) - \nabla_Y (\nabla_X Z - \varepsilon g(X, Z)\xi) \\
- \nabla_X (\nabla_Z [X, Y] - \varepsilon g([X, Y], Z)\xi) \\
= R'(X, Y)Z + \varepsilon g(X, Z)Y - \varepsilon g(Y, Z)X,
\]

for any \(X, Y, Z \in \mathcal{D}\). Thus, from (4.2) and (4.3) we know that \(M'\) is flat and the sectional curvature of \(M'\) vanishes. This means that \(M'\) is a flat
indefinite Kähler manifold. Noting that $\nabla \xi = -\phi^2$ if and only if $h = 0$, then, applying Proposition 4.2 we prove that $M^{2n+1}$ is a Kenmotsu pseudo-metric manifold.

Moreover, if (1) or (2) holds for a locally symmetric almost Kenmotsu pseudo-metric manifold, the above arguments show that $M^{2n+1}$ has constant sectional curvature $K = -\varepsilon$, and hence the integral manifold of $\mathcal{D}$ is an indefinite Kähler manifold with vanishing sectional curvature.

The following result follows from Theorem 4.5.

**Corollary 4.6.** An almost Kenmotsu pseudo-metric manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ is a Kenmotsu pseudo-metric manifold if and only if $h = 0$ and the integral manifolds of $\mathcal{D}$ are indefinite Kähler manifolds.

Theorem 4.1 in [8] implies that a contact pseudo-metric manifold of constant sectional curvature $K$ is a Sasakian pseudo-metric manifold with $K = -\varepsilon$, which generalizes the related result in [5]. Now, we give the Kenmotsu version of this result for almost Kenmotsu pseudo-metric manifolds.

**Theorem 4.7.** An almost Kenmotsu pseudo-metric manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ of constant sectional curvature $K$ is a Kenmotsu pseudo-metric manifold and $K = -\varepsilon$.

**Proof.** Let $M^{2n+1}$ be an almost Kenmotsu pseudo-metric manifold of constant sectional curvature, this implies that $M^{2n+1}$ is locally symmetric. Then from Proposition 4.3 we have $\nabla_{\xi} h = 0$. Also, being constant sectional curvature, we have $R(X, Y)\xi = \varepsilon K(\eta(Y)X - \eta(X)Y)$. Comparing the above equation with (4.1) gives

\begin{equation}
(\varepsilon K + 1)(\eta(X)Y - \eta(Y)X) - \eta(Y)h'X + \eta(X)h'Y + (\nabla_X h')Y - (\nabla_Y h')X = 0,
\end{equation}

for any $X, Y \in \Gamma(TM)$. Noticing that $(\nabla_{\xi} h')X = \phi((\nabla_{\xi} h)X) = 0$ and $(\nabla_{\xi} h')\xi = -h'X - h^2X$, then by replacing $Y$ by $\xi$ in (4.4) we have

\begin{equation}
(\varepsilon K + 1)(\eta(X)\xi - \eta(\xi)X) - 2h'X - h^2X = 0.
\end{equation}

In particular, letting $X \in \mathcal{D}$ be an eigenvector field of $h$ with respective to the eigenvalue $\mu$. Thus, (4.5) becomes that $(\varepsilon K + 1 + \mu^2)X - 2\mu \phi X = 0$. Since that $X$ and $\phi X$ are linearly independent, then we obtain that $\varepsilon K + 1 + \mu^2 = 2\mu = 0$, that is, $K = -\varepsilon$. Thus, we complete the proof.

Before closing this section, we give the following result concerning $\xi$-sectional curvature of a vector filed under the condition of local symmetry.
Proposition 4.8. Let \((M^{2n+1}, \phi, \xi, \eta, g)\) be a locally symmetric almost Kenmotsu pseudo-metric manifold:

1. If \(X\) is eigenvector of \(h\) with respective to the eigenvalue \(\lambda\), then 
   \[ K(\xi, X) = -\varepsilon(1 + \lambda^2). \]

2. If \(X\) is eigenvector of \(h'\) with respective to the eigenvalue \(\mu\), then 
   \[ K(\xi, X) = -\varepsilon(1 + \mu^2). \]

3. \(\text{Ric}(\xi, \xi) = -2\sum_{i=1}^{n}(1 + \lambda_i^2),\) where \(\{\pm \lambda_i : 1 \leq i \leq n\}\) are all eigenvalues of \(h\) on \(D\).

**Proof.** Noticing the Proposition 4.3, from (2.7) we have \(R(X, \xi)\xi = \phi^2 X - 2h'X - h^2 X\). Thus, we obtain \(R(X, \xi, X, \xi) = -(1 + \lambda^2)g(X, X)\), where \(X\) is eigenvector of \(h\) with respective to the eigenvalue \(\lambda\), which proves (1). Similarly, it follows from (2.5) that \(R(X, \xi)\xi = -(1 + \mu)X\), where \(X\) is eigenvector of \(h'\) with respective to the eigenvalue, then (2) holds. Finally, consider a local \(\phi\)-basis \(\{\xi, e_1, \ldots, e_n, \phi e_1, \ldots, \phi e_n\}\) on \(M^{2n+1}\) satisfying \(he_i = \lambda_i e_i\), then we obtain

\[ \text{Ric}(\xi, \xi) = \varepsilon \sum_{i=1}^{n}(K(\xi, e_i) + K(\xi, \phi e_i)) = -2\sum_{i=1}^{n}(1 + \lambda_i^2). \]

Then the proof is complete. \(\square\)

5. Nullity distributions

After Blair [4] introduced the notion of \((k, \mu)\)-nullity distribution, Boeckx and Cito in [6] studied contact metric manifolds for which the tensor \(h\) is \(\eta\)-parallel, i.e., \(g((\nabla_X h)Y, Z) = 0\) for any vector fields \(X, Y, Z\) orthogonal to \(\xi\). Moreover, under the condition \(h \neq 0\), they prove that the tensor \(h\) is \(\eta\)-parallel if and only if the characteristic vector field \(\xi\) belongs to the \((k, \mu)\)-nullity distribution for some constants \(k\) and \(\mu\), i.e., the Riemannian curvature tensor satisfies

\[ R(X, Y)\xi = k(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY), \tag{5.1} \]

for any vector fields \(X\) and \(Y\). Following [14], the characteristic vector field \(\xi\) is said to belong to the \((k, \mu)'\)-nullity distribution, if

\[ R(X, Y)\xi = k(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)h'X - \eta(X)h'Y), \tag{5.2} \]
for any vector fields $X$ and $Y$. In this section, we classify almost Kenmotsu pseudo-metric manifolds satisfying the $(k, \mu)$-nullity or $(k, \mu)′$-nullity condition (for Riemannian case we refer the reader to [14] for some related results).

A complicated but straightforward calculation implies the following theorem, noting that (5.3) still holds for Riemannian case (see Section 2 of [14]).

**Proposition 5.1.** Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be an almost Kenmotsu pseudo-metric manifold. Then

$$
g(R(X, \xi)\phi Y, \phi Z) - g(R(X, \xi)Y, Z)
= 2(\nabla h X)(Y, Z) + 2(\phi Y)g(Z, X - \phi h X) - 2(\eta Y)g(Y, X - \phi h X),
$$

for any $X, Y, Z \in \Gamma(TM)$.

**Theorem 5.2.** Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be an almost Kenmotsu pseudo-metric manifold. Suppose that $\xi$ belongs to $(k, \mu)$-nullity distribution, then $k = -1$ and $h = 0$. Moreover, $M^{2n+1}$ is locally a warped product space $M_\xi \times_f M'$, where $M_\xi$ is an open interval with coordinate $t$, $M'$ is a 2n-dimensional indefinite almost Kähler manifold and $f = ce^t$ for some positive constant $c$.

**Proof.** Let $X \in \mathcal{D}$ and $Y = \xi$, from (5.1) we have $R(X, \xi)\xi = kX + \mu h X$. Substituting this equation into (2.8) yields that $h^2 X = -(k + 1)X$. In particular, letting $X$ be the eigenvector field of $h$ with respect to the eigenvalue $\lambda$, then it follows from (2.7) that $R(X, \xi)\xi = -k(X - \lambda^2 X) + 2(\phi h)X - 2\lambda^2 X$. Thus, we have $\lambda X = \lambda X \in \mathcal{D}$. Taking scalar product with $\phi X$ on both sides of the above equation we have $\lambda = 0$, which implies that $h = 0$. Then, the proof follows from Theorem 2.5.

**Theorem 5.3.** Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be an almost Kenmotsu pseudo-metric manifold. Suppose that $\xi$ belongs to the $(k, \mu)′$-nullity distribution and $h' \neq 0$, then the integral manifolds of contact distribution $\mathcal{D}$ are indefinite Kähler manifold.

**Proof.** Letting $X, Y \in \mathcal{D}$, then it follows from (5.2) that $R(X, Y)\xi = 0$. Thus, from (5.3) we know that $(\nabla h X)(Y, Z) = 0$, for any $X, Y, Z \in \mathcal{D}$. 
On the other hand, from Proposition 2.2 we have that $2g(\nabla hX \phi Y, Z) + g(N_\phi(Y, Z), h'X) = 0$, for any $X, Y, Z \in \mathcal{D}$. Thus, it is easy to see $g(N_\phi(Y, Z), h'X) = 0$, for any $X, Y, Z \in \mathcal{D}$. As $\xi$ belongs to the $(k, \mu)'$-nullity distribution, then using (5.2) in (2.8) gives $h'^2X = -(k + 1)X$. Noting the hypothesis $h' \neq 0$ ($\Rightarrow k \neq -1$), then it follows that $g(N_\phi(Y, Z), X) = 0$, for any $X, Y, Z \in \mathcal{D}$. Which implies that $N_\phi = 0$, i.e., the integral manifolds of $\mathcal{D}$ are indefinite Kähler manifolds.

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REFERENCES