GENERALIZED SEQUENCE SPACES IN 2-NORMED SPACES DEFINED BY IDEAL AND A MODULUS FUNCTION

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Abstract. The main objective of this paper is to define some new kind of generalized convergent sequence spaces with respect to a modulus function, and difference operator $\Delta^m$, $m \geq 1$ in a 2-normed space. We also examine some topological properties of the resulting sequence spaces. Finally, we have introduced a new class of generalized convergent sequences with the help of an ideal and difference sequences in the same space.

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1. Introduction and background

The notion of statistical convergence has been introduced by Fast [9] in 1951 and later developed by Connor [1], Fridy [10], Maddox [21], Šalát [30] and many others. Furthermore, Kostyrko et al. [19] presented a very interesting generalization of statistical convergence called as $I$-convergence. The detailed history and development in this regard can be found in [2], [3], [4] and [12].

In 1960, Gähler [11] initially introduced the concept of 2-normed space as a generalization of normed linear space. Recently, many authors have started to study summability, sequence spaces in these nonlinear spaces (see, for instance [13]). In [28], Sahiner discussed ideal summability in these spaces and defined a new type of sequence spaces.
Kizmaz [18] introduced the notion of difference sequence spaces as follows \( \Delta(X) = \{ x = (x_k) : (\Delta x_k) \in X \} \), for \( X = \ell_\infty, c \) and \( c_0 \), where \( \Delta x = (x_k - x_{k+1}) \).

Continuing on this way, the notion was further generalized by Et and Çolak [5] by introducing the sequence spaces as follows \( \Delta^m(X) = \{ x = (x_k) : (\Delta^m x_k) \in X \} \), for \( X = \ell_\infty, c \) and \( c_0 \), where \( m \in \mathbb{N} \), \( \Delta^m x = \sum_{v=0}^{m} (-1)^v (\frac{m!}{v!}) x_{k+v} \). More applications of the difference sequences can be seen in [6], [7], [23] and [33].

The following inequality will be used throughout the paper. Let \( p = (p_k) \) be a positive sequence of real numbers with \( 0 < p_k \leq \sup_k p_k = H \), \( C = \max(1, 2^{H-1}) \). Then for \( a_k, b_k \in \mathbb{C} \), we have

\[ |a_k + b_k|^{p_k} \leq C \{ |a_k|^{p_k} + |b_k|^{p_k} \} , \quad \text{for all } k \in \mathbb{N}. \]

We recall [25] that a modulus function \( f \) is a function from \([0, \infty)\) to \([0, \infty)\) such that

(i) \( f(x) = 0 \) if and only if \( x = 0 \),
(ii) \( f(x + y) \leq f(x) + f(y) \) for all \( x, y \geq 0 \),
(iii) \( f \) is increasing,
(iv) \( f \) is continuous from right at 0.

It follows that \( f \) must be continuous everywhere on \([0, \infty)\). A modulus function may be bounded or unbounded. Subsequently, modulus function was used to define sequence spaces by Gürdal [15], Pehlivan [26] and Savas [31].

Let \( \lambda = (\lambda_n) \) be a non-decreasing sequence of positive numbers such that \( \lambda_{n+1} \leq \lambda_n + 1 \), \( \lambda_1 = 1 \), \( \lambda_n \to \infty \) as \( n \to \infty \) and \( I_n = \{ n - \lambda_n + 1, n \} \).

In [24], Mursaleen introduced the idea of \( \lambda \)-statistical convergence by extending the concept of \([V, \lambda]\) summability of [20]. Further, Savas [32] unified the two approaches and gave a new concepts of \( I \)-statistical convergence, \( I-S_\lambda \)-convergence and \( I-[V, \lambda] \) convergence.

Quite recently, many authors including [8], [22], [27] and [31] have constructed some sequence spaces by using modulus function, difference sequences and investigate their properties. In the present work, we also construct some sequence spaces defined by a modulus function, generalized difference sequences with the help of an ideal in a 2-normed space.

2. Preliminaries

Throughout the paper, \( \mathbb{N} \) will denote the set of all positive integers.

Let \( (X, \| \cdot \|) \) be a normed linear space. We recall that a sequence \( x = (x_k) \in X \) is called statistically convergent to \( L \in X \) if for each \( \epsilon > 0 \), the set \( A(\epsilon) = \{ k \in \mathbb{N} : \| x_k - L \| \geq \epsilon \} \) having its natural density zero.
A family of sets $\mathcal{I} \subset \mathcal{P}(\mathbb{N})$ is called an ideal in $\mathbb{N}$ if and only if:

(i) $\emptyset \in \mathcal{I}$; (ii) For each $A, B \in \mathcal{I}$ we have $A \cup B \in \mathcal{I}$; (iii) For $A \in \mathcal{I}$ and $B \subset A$ we have $B \in \mathcal{I}$.

A non-empty family of sets $\mathcal{F} \subset \mathcal{P}(\mathbb{N})$ is called a filter on $\mathbb{N}$ if and only if:

(i) $\emptyset \notin \mathcal{F}$; (ii) For each $A, B \in \mathcal{F}$ we have $A \cap B \in \mathcal{F}$; (iii) For $A \in \mathcal{F}$ and $B \supset A$ we have $B \in \mathcal{F}$.

An ideal $\mathcal{I}$ is called non-trivial if $\mathcal{I} \neq \emptyset$ and $\mathbb{N} \notin \mathcal{I}$.

It immediately implies that $\mathcal{I} \subset \mathcal{P}(\mathbb{N})$ is a non-trivial ideal if and only if the class $\mathcal{F} = \mathcal{F}(\mathcal{I}) = \{N - A : A \in \mathcal{I}\}$ is a filter on $\mathbb{N}$. The filter $\mathcal{F} = \mathcal{F}(\mathcal{I})$ is called the filter associated with the ideal $\mathcal{I}$.

A non-trivial ideal $\mathcal{I} \subset \mathcal{P}(\mathbb{N})$ is called an admissible ideal in $\mathbb{N}$ if and only if it contains all singletons i.e., if it contains $\{\{n\} : n \in \mathbb{N}\}$. Throughout the paper, $\mathcal{I}$ is considered as a non-trivial admissible ideal.

Let $X$ be a real vector space of dimension $d$, where $2 \leq d < \infty$. A 2-norm on $X$ is a function $\|.,.\| : X \times X \to \mathbb{R}$, which satisfies: (i) $\|x, y\| = 0$, if and only if $x$ and $y$ are linearly dependent, (ii) $\|x, y\| = \|y, x\|$, (iii) $\|\alpha x, y\| = |\alpha| \|x, y\|, \alpha \in \mathbb{R}$ and (iv) $\|x, y + z\| \leq \|x, y\| + \|x, z\|$. Then the pair $(X, \|.,.\|)$ is called 2-normed space.

Using the above terminology, GÜRDAL [14] defined $\mathcal{I}$-convergence in 2-normed space which were further investigated in [17], [29] and [34].

Let $\mathcal{I} \subset \mathcal{P}(\mathbb{N})$ be a non-trivial ideal in $\mathbb{N}$ and $(X, \|.,.\|)$ is a 2-normed space. A sequence $x = (x_k)$ in $X$ is said to be $\mathcal{I}$-convergent to $L \in X$ if for each $\epsilon > 0$ and nonzero $z \in X$, the set $A(\epsilon) = \{k \in \mathbb{N} : \|x_k - L, z\| \geq \epsilon\} \in \mathcal{I}$. In this case, we write $\mathcal{I} - \lim_{k \to \infty} \|x_k, z\| = \|L, z\|$. 

**Definition 2.1** ([8]). A sequence $x = (x_k)$ is said to be $\lambda^m_X$-statistically convergent to the number $L$ if, for every $\epsilon > 0$, $\lim_{n \to \infty} \frac{1}{n} \sum_{k \in I_n} \{k \in I_n : \|\Delta^m x_k - L\| \geq \epsilon\} = 0$. In this case, we write $S_\lambda(\Delta^m, X) - \lim_{k \to \infty} x_k = L$.

Recently, SAVAŞ ET AL. [32] combined the ideas of $\lambda$-statistical convergence and ideal convergence to introduce new concepts of $\mathcal{I} - S_\lambda$-convergence, $\mathcal{I} - [V, \lambda]$ convergence and later some pioneer works have been extended in this direction by numerous authors such as [2] and [16].

**Definition 2.2** ([32]). A sequence $x = (x_k)$ is said to be $\mathcal{I} - [V, \lambda]$ summable to $L$, if for any $\delta > 0$, $\{n \in \mathbb{N} : \frac{1}{n} \sum_{k \in I_n} \|x_k - L\| \geq \delta\} \in \mathcal{I}$, where $I_n = [n - \lambda_n + 1, n]$.

**Definition 2.3** ([32]). A sequence $x = (x_k)$ is said to be $\mathcal{I} - \lambda$-statistically convergent or $\mathcal{I} - S_\lambda$ convergent to $L$, if for every $\epsilon > 0$ and $\delta > 0$, 

...
\( r \in \mathbb{N} : \frac{1}{r} \{ k \in I_n : \|x_k - L\| \geq \epsilon \} \geq \delta \} \in I \). In this case, we write \( x_k \to L(I - S_{\lambda}) \) or \( I - S_{\lambda} \lim_{k \to \infty} x_k = L \).

The following well-known lemma is required for establishing a very important result in our article.

**Lemma 2.1.** Let \( f \) be a modulus function and let \( 0 < \delta < 1 \). Then for each \( x > \delta \) we have \( f(x) \leq \frac{2f(1)}{\delta} \).

### 3. Main results

In this section, we introduce a certain type of sequence spaces using modulus function and generalized difference operator \( \Delta^m \) in a 2-normed space, where \( S^2_X \) stands for the space of all sequences defined over 2-normed space \((X, \| \cdot \|)\).

**Definition 3.1.** Let \( I \subseteq 2^\mathbb{N} \) be an admissible ideal, \( f \) be a modulus function and \( p = (p_k) \) be a bounded sequence of positive (strictly) real numbers, then for each \( \epsilon > 0 \) and \( z \in X \), we define the following sequence spaces:

\[
V^f_0[\| \cdot \|, \| \cdot \|, \Delta^m, \lambda, p]
= \left\{ x = (x_k) \in S^2_X : \{ n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in I_n} [f(\|\Delta^m x_k, z\|)]^{p_k} \geq \delta \} \in I \right\},
\]

\[
V^f[\| \cdot \|, \| \cdot \|, \Delta^m, \lambda, p]
= \left\{ x = (x_k) \in S^2_X : \{ n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in I_n} [f(\|\Delta^m x_k, z\|)]^{p_k} \geq \delta \} \in I \right\}, \text{ for } L > 0,
\]

\[
V^f[\| \cdot \|, \| \cdot \|, \Delta^m, \lambda, p]_\infty
= \left\{ x = (x_k) \in S^2_X : \{ n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in I_n} [f(\|\Delta^m x_k, z\|)]^{p_k} \geq K \} \in I \right\}, \text{ for } K > 0.
\]

We can write it as \( x = (x_k) \in V^f[\| \cdot \|, \| \cdot \|, \Delta^m, \lambda, p] \) or \( x_k \to L(V^f[\| \cdot \|, \| \cdot \|, \Delta^m, \lambda, p]) \).

**Remark 3.1.** If we take \( f(x) = x \) in the above definition, then we obtain \( V^x[\| \cdot \|, \| \cdot \|, \Delta^m, \lambda, p] \) instead of \( V^x[\| \cdot \|, \| \cdot \|, \Delta^m, \lambda, p] \) and \( V^x[\| \cdot \|, \| \cdot \|, \Delta^m, \lambda, p] \) respectively.
Theorem 3.1. \( V_f^T[||.,||, \Delta^m, \lambda, p]_0, \ V_f^T[||.,||, \Delta^m, \lambda, p] \) and \( V_f^T[||.,||, \Delta^m, \lambda, p]_\infty \) are linear spaces over \( \mathbb{C} \).

Proof. We will prove the assertion for \( V_f^T[||.,||, \Delta^m, \lambda, p]_0 \) only and the others can be proved similarly.

We assume \( x = (x_k), y = (y_k) \in V_f^T[||.,||, \Delta^m, \lambda, p]_0 \) and \( \alpha, \beta \in \mathbb{C} \). Then for any \( \delta > 0 \) and for each \( z \in X \), the sets

\[
A_\delta(\lambda) = \left\{ n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in I_n} [f(\|\alpha \Delta^m x_k + \beta \Delta^m y_k, z\|)]^{p_k} \geq \frac{\delta}{2} \right\},
\]

\[
B_\delta(\lambda) = \left\{ n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in I_n} [f(\|\Delta^m x_k, z\|)]^{p_k} \geq \frac{\delta}{2} \right\}
\]

belong to \( \mathcal{I} \).

Since \( f \) be a modulus function and \( ||.,|| \) is a 2-norm function, then the following inequality holds

\[
\frac{1}{\lambda_n} \sum_{k \in I_n} [f(\|\alpha \Delta^m x_k + \beta \Delta^m y_k, z\|)]^{p_k} \leq \frac{1}{\lambda_n} \sum_{k \in I_n} [f(\|\alpha \Delta^m x_k, z\|) + f(\|\beta \Delta^m y_k, z\|)]^{p_k} \leq C.(M_\alpha)^H \frac{1}{\lambda_n} \sum_{k \in I_n} [f(\|\Delta^m x_k, z\|)]^{p_k}
\]

\[
+ C.(M_\beta)^H \frac{1}{\lambda_n} \sum_{k \in I_n} [f(\|\Delta^m y_k, z\|)]^{p_k}, \quad \text{by (1)}
\]

where \( M_\alpha, M_\beta \) are positive integers such that \( |\alpha| \leq M_\alpha \) and \( |\beta| \leq M_\beta \). For given \( \delta > 0 \) and for all \( z \in X \), we have the following containment

\[
\left\{ n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in I_n} [f(\|\alpha \Delta^m x_k + \beta \Delta^m y_k, z\|)]^{p_k} \geq \delta \right\}
\]

\[
\subseteq \left\{ n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in I_n} [f(\|\Delta^m x_k, z\|)]^{p_k} \geq \frac{\delta}{2C.(M_\alpha)^H} \right\}
\]

\[
\cup \left\{ n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in I_n} [f(\|\Delta^m y_k, z\|)]^{p_k} \geq \frac{\delta}{2C.(M_\beta)^H} \right\}.
\]

By using (2) and (3), the set \( \left\{ n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in I_n} [f(\|\alpha \Delta^m x_k + \beta \Delta^m y_k, z\|)]^{p_k} \geq \delta \right\} \in \mathcal{I} \). This completes the proof. \( \square \)
Theorem 3.2. Let $p = (p_k)$ be a sequence of strictly positive real numbers, then for $m \geq 1$ the inclusion $V_f[[\cdot, \cdot], \Delta^{m-1}, \lambda, p]_{0,\infty} \subset V_f[[\cdot, \cdot], \Delta^m, \lambda, p]_{0,\infty}$ is strict.

Proof. We will prove the result for $V_f[[\cdot, \cdot], \Delta^{m-1}, \lambda, p]_{0}$ only. The others can be proved similarly.

Suppose $x \in V_f[[\cdot, \cdot], \Delta^{m-1}, \lambda, p]_{0}$, by definition for every $\delta > 0$ and $z \in X$, we have

$$\left\{ n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in I_n} [f(\|\Delta^{m-1}x_k, z\|)]^{p_k} \geq \delta \right\} \in \mathcal{I}.$$ 

By the property of modulus function, we have

$$\frac{1}{\lambda_n} \sum_{k \in I_n} [f(\|\Delta^m x_k, z\|)]^{p_k} \leq \frac{1}{\lambda_n} \sum_{k \in I_n} [f(\|\Delta^{m-1} x_k, z\|)]^{p_k} + f(\|\Delta^{m-1} x_{k+1}, z\|)]^{p_k} \leq C, \frac{1}{\lambda_n} \sum_{k \in I_n} [f(\|\Delta^{m-1} x_{k+1}, z\|)]^{p_k}$$

by (1).

Now for given $\delta > 0$, we have

$$\left\{ n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in I_n} [f(\|\Delta^m x_k, z\|)]^{p_k} \geq \delta \right\}$$

$$\subseteq \left\{ n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in I_n} [f(\|\Delta^{m-1} x_k, z\|)]^{p_k} \geq \frac{\delta}{2C} \right\}$$

$$\cup \left\{ n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in I_n} [f(\|\Delta^{m-1} x_{k+1}, z\|)]^{p_k} \geq \frac{\delta}{2C} \right\},$$

for each $z \in X$.

Since $x \in V_f[[\cdot, \cdot], \Delta^{m-1}, \lambda, p]_{0}$, it follows that the sets on the right hand side in the above containment belong to $\mathcal{I}$. Hence $x \in V_f[[\cdot, \cdot], \Delta^m, \lambda, p]_{0}$. To show that the inclusion is strict, we give the following example:

We take $f(x) = x$, $\lambda_n = n$ and consider a sequence $x = (x_k) = k^{m-1}$, then $x \in V_f[[\cdot, \cdot], \Delta^{m-1}, \lambda, p]_{0}$ but does not belong to $V_f[[\cdot, \cdot], \Delta^{m-1}, \lambda, p]_{0}$ for $p_k = 1$, $k \in \mathbb{N}$. This shows that the inclusion is strict. \hfill $\Box$

Theorem 3.3. Let $f'$, $f''$ are modulus functions. If $\limsup_{t \to \infty} \frac{f'(t)}{f''(t)} = P > 0$, then $V_f^T[[\cdot, \cdot], \Delta^m, \lambda, p] \subset V_{f''}^T[[\cdot, \cdot], \Delta^m, \lambda, p].$
Proof. Let \( \limsup_{t \to \infty} \frac{f'(t)}{f''(t)} = P \), then there exists a constant \( M > 0 \) such that \( f'(t) \geq Mf''(t) \), for all \( t \geq 0 \). Therefore for each \( z \in X \), we have

\[
\frac{1}{\lambda_n} \sum_{k \in I_n} [f'(||\Delta^m x_k - L, z||)]^{p_k} \geq (M)^H \frac{1}{\lambda_n} \sum_{k \in I_n} [f''(||\Delta^m x_k - L, Z||)]^{p_k}.
\]

Then for every \( \delta > 0 \) and \( z \in X \), we have following relationship

\[
\left\{ n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in I_n} [f''(||\Delta^m x_k - L, Z||)]^{p_k} \geq \delta \right\} 
\subseteq \left\{ n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in I_n} [f'(||\Delta^m x_k - L, z||)]^{p_k} \geq \delta, (K)^H \right\}.
\]

Since \( x \in V_f^I[\| \cdot \|, \Delta^m, \lambda, p] \), it follows that the set on left side of the above containment belong to \( I \). Which gives that \( x \in V_f^I[\| \cdot \|, \Delta^m, \lambda, p] \).

**Theorem 3.4.** If \( f, f' \) and \( f'' \) are modulus functions, then:

(i) \( V_f^I[\| \cdot \|, \Delta^m, \lambda, p] \subset V_{f'f'}^I[\| \cdot \|, \Delta^m, \lambda, p] \),

(ii) \( V_f^I[\| \cdot \|, \Delta^m, \lambda, p] \cap V_{f''}^I[\| \cdot \|, \Delta^m, \lambda, p] \subset V_{f'f''}^I[\| \cdot \|, \Delta^m, \lambda, p] \).

Proof. (i) Let \( x = (x_k) \in V_f^I[\| \cdot \|, \Delta^m, \lambda, p] \), then for every \( \epsilon > 0 \) and for some \( L > 0 \) such that

\[
\left\{ n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in I_n} [f'(||\Delta^m x_k - L, z||)]^{p_k} \geq \epsilon \right\} \in I,
\]

for each \( z \in X \). For given \( \epsilon > 0 \), we choose \( \delta \in (0, 1) \) such that \( f(t) < \epsilon \) for all \( 0 < t < \delta \). On the other hand, we have

\[
\frac{1}{\lambda_n} \sum_{k \in I_n} [f' \circ f'(||\Delta^m x_k - L, z||)]^{p_k}
= \frac{1}{\lambda_n} \sum_{k \in I_n, |f'(||\Delta^m x_k - L, z||)|^{p_k} < \delta} [f \circ f'(||\Delta^m x_k - L, z||)]^{p_k}
+ \frac{1}{\lambda_n} \sum_{k \in I_n, |f'(||\Delta^m x_k - L, z||)|^{p_k} \geq \delta} [f \circ f'(||\Delta^m x_k - L||)]^{p_k}
\leq (\epsilon)^H + \max(1, (2f(1))^{1/H}) \frac{1}{\lambda_n} \sum_{k \in I_n} [f'(||\Delta^m x_k - L, z||)]^{p_k} \text{ by Lemma 2.1.}
\]
By using (5), we obtain $x \in V_f^{\mathcal{I}}[\|\cdot\|, \Delta^m, \lambda, p]$.

(ii) The result of the theorem is proved by using the following inequality

$$\frac{1}{\lambda_n} \sum_{k \in I_n} [(f' + f'')(\|\Delta^m x_k - L, z\|)]^{p_k} \leq \frac{C}{\lambda_n} \sum_{k \in I_n} [f'(\|\Delta^m x_k - L, z\|)]^{p_k}$$

$$+ \frac{C}{\lambda_n} \sum_{k \in I_n} [f''(\|\Delta^m x_k - L, z\|)]^{p_k},$$

where $\sup_k p_k = H$ and $C = \max(1, 2^{H-1})$. 

**Theorem 3.5.** Let $f$ be a modulus function and $p = (p_k)$ be a sequence of positive real numbers, then $V_f^{\mathcal{I}}[\|\cdot\|, \Delta^m, \lambda, p] \subseteq V_f^{\mathcal{I}}[\|\cdot\|, \Delta^m, \lambda, p]$.

**Proof.** This can be proved by using the techniques similar to those used in Theorem 3.4 (i).

**Theorem 3.6.** Let $f$ be a modulus function. If $\limsup_{t \to \infty} \frac{f(t)}{t} = M > 0$, then $V_f^{\mathcal{I}}[\|\cdot\|, \Delta^m, \lambda, p] \subseteq V_f^{\mathcal{I}}[\|\cdot\|, \Delta^m, \lambda, p]$.

**Proof.** Suppose $x = (x_k) \in V_f^{\mathcal{I}}[\|\cdot\|, \Delta^m, \lambda, p]$ and $\limsup_{t \to \infty} \frac{f(t)}{t} = M > 0$, then there exists a constant $K > 0$ such that $f(t) \geq Kt$, for all $t \geq 0$. Which implies that

$$\frac{1}{\lambda_n} \sum_{k \in I_n} [f(\|\Delta^m x_k - L, z\|)]^{p_k} \geq (K)^H \frac{1}{\lambda_n} \sum_{k \in I_n} \|\Delta^m x_k - L, z\|^{p_k},$$

for each $z \in X$. Which gives the result.

**Theorem 3.7.** If $0 < p_k \leq q_k$ and $(\frac{q_k}{p_k})$ be bounded, then $V_f^{\mathcal{I}}[\|\cdot\|, \Delta^m, \lambda, q] \subset V_f^{\mathcal{I}}[\|\cdot\|, \Delta^m, \lambda, p]$.

**Proof.** The proof of this theorem is omitted.

4. $\mathcal{S}^\Delta^m_\lambda[\|\cdot\|, \mathcal{I}]$-convergence

In this section, we define a new class of generalized statistical convergent sequences with the help of an ideal and difference sequences. Furthermore, we also establish a strong connection between this convergence and the sequence space $V_f^{\mathcal{I}}[\|\cdot\|, \Delta^m, \lambda, p]$. 
Definition 4.1. Let $I \subseteq \mathcal{P}(\mathbb{N})$ be a non-trivial ideal and $\lambda = (\lambda_n)$ be a non-decreasing sequence. A sequence $x = (x_k) \in X$ is said to be $S_\lambda^m(I)$-convergent to a number $L$ provided that for every $\epsilon > 0$, $\delta > 0$ and $z \in X$, the set $\{n \in \mathbb{N} : \frac{1}{\lambda_n} \max \{k \mid k \in I_n : \|\Delta^m x_k - L, z\|\} \geq \epsilon\} \subseteq I$. In this case, we write $S_\lambda^m(I) - \lim_{k \to \infty}^{} \|x_k, z\| = \|L, z\|$. Let $S_\lambda^m(||, ||, I)$ denotes the set of all $S_\lambda^m(I)$-convergent sequences in $X$.

Theorem 4.1. Let $f$ be a modulus function and $0 < \inf_k p_k = h \leq p_k \leq \sup_k p_k = H < \infty$, then $V_f^T(||, ||, \Delta^m, \lambda, p) \subset S_\lambda^m(||, ||, I)$.

Proof. Suppose $x \in V_f^T(||, ||, \Delta^m, \lambda, p)$ and $\epsilon > 0$ be given. Then for each $z \in X$, we obtain

$$
\frac{1}{\lambda_n} \sum_{k \in I_n} [f(\|\Delta^m x_k - L, z\|)]^{p_k} = \frac{1}{\lambda_n} \sum_{k \in I_n\|\Delta^m x_k - L, z\| \geq \epsilon} [f(\|\Delta^m x_k - L, z\|)]^{p_k} + \frac{1}{\lambda_n} \sum_{k \in I_n\|\Delta^m x_k - L, z\| < \epsilon} [f(\|\Delta^m x_k - L, z\|)]^{p_k} \geq \frac{1}{\lambda_n} \sum_{k \in I_n\|\Delta^m x_k - L, z\| \geq \epsilon} [f(\|\Delta^m x_k - L, z\|)]^{p_k} \geq \frac{1}{\lambda_n} \min \{[f(\epsilon)]^h, [f(\epsilon)]^H\} \geq K \frac{1}{\lambda_n} \min \{k \in I_n : \|\Delta^m x_k - L, z\| \geq \epsilon\},
$$

where $K = \min \{[f(\epsilon)]^h, [f(\epsilon)]^H\}$. Then for every $\delta > 0$ and $z \in X$, we have

$$
\left\{ n \in \mathbb{N} : \frac{1}{\lambda_n} \max \{k \mid k \in I_n : \|\Delta^m x_k - L, z\| \geq \epsilon\} \geq \delta \right\} \subseteq \left\{ n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in I_n} [f(\|\Delta^m x_k - L, z\|)]^{p_k} \geq K. \delta \right\}.
$$

Since $x_k \to L(V_f^T(||, ||, \Delta^m, \lambda, p))$ so that $S_\lambda^m(I) - \lim_{k \to \infty}^{} \|x_k, z\| = \|L, z\|$. \qed

Theorem 4.2. Let $p = (p_k)$ be a sequence of strictly positive real numbers and $f$ be a bounded modulus function. If $0 < \inf_k p_k = h \leq p_k \leq \sup_k p_k = H < \infty$, then $S_\lambda^m(||, ||, I) \subset V_f^T(||, ||, \Delta^m, \lambda, p)$.

Proof. Using the same technique of [8, Theorem 3.5], it is easy to prove this theorem. \qed
Theorem 4.3. If $f$ be bounded and $0 < \inf_k p_k = h \leq p_k \leq \sup_k p_k = H < \infty$, then $S_{\Delta}^m(\|., .\|, \mathcal{I}) = V_f^Z[\|., .\|, \Delta^m, \lambda, p]$ if and only if $f$ is bounded.

Proof. This part can be obtained by combining Theorems 4.1 and 4.2.

Conversely. Suppose $f$ is unbounded defined by $f(k) = k$ for all $k \in \mathbb{N}$. We take a fixed set $A \in \mathcal{I}$, where $\mathcal{I}$ be an admissible ideal and define $x = (x_k)$ as follows:

$$x_k = \begin{cases} k^{m+1}, & \text{for } n - \lfloor \sqrt[n]{\lambda_n} \rfloor + 1 \leq k \leq n \notin A, \\ k^{m+1}, & \text{for } n - \lambda_n + 1 \leq k \leq n, \in A, \\ 0, & \text{otherwise.} \end{cases}$$

For given $\epsilon > 0$ and for each $z \in X$, we have $\lim_{n \to \infty} \frac{1}{X_n} \{ k \in I_n : \|\Delta^m x_k - 0, z\| \geq \epsilon \} \leq \frac{\lfloor \sqrt[n]{\lambda_n} \rfloor}{\lambda_n} \to 0$ for $n \notin A$. Hence for $\delta > 0$, there exists a positive integer $n_0$ such that $\frac{1}{X_n} \{ k \in I_n : \|\Delta^m x_k - 0, z\| \geq \epsilon \} < \delta$ for $n \notin A$ and $n \geq n_0$. Now, we have $\{ n \in \mathbb{N} : \frac{1}{X_n} \{ k \in I_n : \|\Delta^m x_k - 0, z\| \geq \epsilon \} \geq \delta \} \subset \{ A \cup (1, 2, \cdots, n_0 - 1) \}$. Since $\mathcal{I}$ be an admissible ideal. It follows that $S_{\Delta}^m(\mathcal{I}) - \lim_{k \to \infty} \|x_k, z\| \to 0$ for each $z \in X$.

On the other hand, if we take $p_k = 1$, for all $k = 1, 2, \cdots$, then $x_k \notin V_f^Z[\|., .\|, \Delta^m, \lambda, p]$. This contradicts the fact $S_{\Delta}^m(\|., .\|, \mathcal{I}) = V_f^Z[\|., .\|, \Delta^m, \lambda, p]$, so our supposition is wrong.

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