HYPERSURFACES WITH CONSTANT SQUARED NORM OF SECOND FUNDAMENTAL FORM

BY

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Abstract. Let $M^n$ be an $n(n \geq 3)$-dimensional complete connected and oriented hypersurface with constant squared norm of the second fundamental form and two distinct principal curvatures in a real space form $M^{n+1}(c)$. Denote by $S^k(a)$ the $k$-dimensional sphere with radius $a$, where $a$ is a constant parameter, we obtain some nonexistence theorems and some characterizations of the Riemannian products: $R^k \times S^{n-k}(a)$, or $S^k(a) \times S^{n-k}(\sqrt{1-a^2})$, or $S^k(a) \times H^{n-k}(-\sqrt{1+a^2})$ in $M^{n+1}(c)(c = 0, 1, -1)$, where $1 < k \leq n-1$.

Mathematics Subject Classification 2010: 53C42, 53A10.

Key words: hypersurface, mean curvature, second fundamental form, principal curvature.

1. Introduction

Let $M^{n+1}(c)$ be an $(n + 1)$-dimensional connected Riemannian manifold with constant sectional curvature $c$. According to $c > 0$ $c = 0$, or $c < 0$ it is called sphere space, Euclidean space, or hyperbolic space, and denoted by $S^{n+1}(c)$, $\mathbb{R}^{n+1}$, or $H^{n+1}(c)$ respectively. Let $M^n$ be an $n$-dimensional immersed hypersurface in a real space form $M^{n+1}(c)$. Denote by $(h_{ij})$ the second fundamental form, by $H = \frac{1}{n} \sum_{i=1}^{n} h_{ii}$ the mean curvature and by $S = \sum_{i,j=1}^{n} h_{ij}^2$ the squared norm of the second fundamental form of $M^n$.

We notice that if $M^n$ has constant mean curvature and two distinct principal curvatures in $M^{n+1}(c)$ and the squared norm $S$ of the second fundamental form of $M^n$ satisfies some pinching conditions (related to mean curvature $H$), there are many important characteristic results for such hypersurfaces, see [1, 2, 5]. Since $H$ and $S$ are two important rigidity invariants of $M^n$, 


we may naturally ask an inverse question: if $S$ is constant, and $H$ satisfies some pinching conditions (related to $S$), can we obtain any characteristic results?

In this article, we try to study hypersurfaces with constant squared norm of the second fundamental form and two distinct principal curvatures in $M^{n+1}(c)$. Firstly, denote by $S^k(a)$ the $k$-dimensional sphere with radius $a$, where $a$ is a constant parameter, we introduce the well-known standard models of complete hypersurfaces with constant squared norm of the second fundamental form in $M^{n+1}(c)$.

**Example 1.1.** $N_{k,n-k} := R^k \times S^{n-k}(a) \hookrightarrow M^{n+1}(c) (c = 0), 1 \leq k \leq n-1$. We know that $N_{k,n-k}$ has two distinct constant principal curvatures $0$ and $\frac{1}{a^2}$ with multiplicities $k$ and $n-k$, respectively. We easily see that the Riemannian product $R^k \times S^{n-k}(a)$ has constant squared norm of the second fundamental form $S = (n-k)\frac{a^2}{n^2}$, and $a^2 = \frac{n-k}{S^2}$. Thus, the mean curvature of the Riemannian product $R^k \times S^{n-k}(a)$ is $H = \frac{1}{n} \sqrt{(n-k)S}$.

**Example 1.2.** $M_{k,n-k} := S^k(a) \times S^{n-k}(\sqrt{1-a^2}) \hookrightarrow M^{n+1}(c) (c = 1), 1 \leq k \leq n-1$. We know that $M_{k,n-k}$ has two distinct constant principal curvatures $\frac{1}{\sqrt{1-a^2}}$ and $\frac{-a}{\sqrt{1-a^2}}$ with multiplicities $k$ and $n-k$, respectively. We easily see that the Riemannian product $S^k(a) \times S^{n-k}(\sqrt{1-a^2})$ has constant squared norm of the second fundamental form $S = k\frac{1-a^2}{n^2} + (n-k)\frac{a^2}{1-a^2}$ and $a^2 = \frac{2k}{S \pm \sqrt{S^2 - 4k(n-k)+2k}}$. Denote by $H_1$ the mean curvature of $S^{n-1}(a) \times S^1(\sqrt{1-a^2})$, if $a^2 = \frac{2(n-1)}{S \pm \sqrt{S^2 - 4(n-1)+2(n-1)}},$ then

$$H_1 = \frac{\sqrt{n-1}(S \mp \sqrt{S^2 - 4(n-1)-2})}{n \sqrt{2(S \mp \sqrt{S^2 - 4(n-1)})}}.$$  

For the Riemannian product $S^{n-1}(a) \times S^1(\sqrt{1-a^2})$, denote by $\alpha(t)$ and $\beta(t)$ the following functions:

$$\alpha(t) = \frac{2(n-1)}{nt^2 + \sqrt{n^2t^4 - 4(n-1)+2(n-1)}}, \quad \beta(t) = \frac{2(n-1)}{nt^2 - \sqrt{n^2t^4 - 4(n-1)+2(n-1)}}.$$  

$$1$$
where $t^2 \geq \frac{2\sqrt{n-1}}{n}$.

If $\lambda = \frac{\sqrt{1-a^2}}{a} \geq \kappa > \frac{\sqrt{S}}{n}$, we have $a^2 \leq \frac{1}{1+\kappa^2}$ and

$$S = (n-1) \frac{1-a^2}{a^2} + \frac{a^2}{1-a^2} < n\kappa^2.$$  

By a direct and simple calculation, we see that (1.3) holds if and only if $\alpha(\kappa) < a^2 < \beta(\kappa)$, $\kappa^2 > \frac{2\sqrt{n-1}}{n}$. It can be easily checked that $\beta(\kappa) > \frac{1}{1+\kappa^2}$.

Thus, we conclude that $\alpha(\kappa) < a^2 \leq \frac{1}{1+\kappa^2}$, $\kappa \neq 1$ and $\kappa^2 > \frac{2\sqrt{n-1}}{n}$, where $\alpha(\kappa)$ and $\beta(\kappa)$ are the values of functions $\alpha(t)$ and $\beta(t)$ at $t = \kappa$.

If $\lambda = \frac{\sqrt{1-a^2}}{a} \leq \kappa' < \frac{\sqrt{S}}{n}$, we have $a^2 \geq \frac{1}{1+\kappa'^2}$ and

$$S = (n-1) \frac{1-a^2}{a^2} + \frac{a^2}{1-a^2} > n\kappa'^2.$$  

By a direct and simple calculation, we see that (1.4) holds if and only if $a^2 > \beta(\kappa')$ or $a^2 < \alpha(\kappa')$, where $\kappa'^2 \geq \frac{2\sqrt{n-1}}{n}$. It can be easily checked that $\beta(\kappa') > \frac{1}{1+\kappa'^2}$, and if $\kappa' \geq 1$, we have $a^2 < \alpha(\kappa') \leq \frac{1}{1+\kappa'^2}$, a contradiction.

Thus, it must imply that $\kappa' < 1$ and $\alpha(\kappa') > \frac{1}{1+\kappa'^2}$. Therefore, we conclude that $a^2 > \beta(\kappa')$, $\kappa'^2 \geq \frac{2\sqrt{n-1}}{n}$, or $\frac{1}{1+\kappa'^2} \leq a^2 < \alpha(\kappa')$, $\frac{2\sqrt{n-1}}{n} \leq \kappa'^2 < 1$, where $\alpha(\kappa')$ and $\beta(\kappa')$ are the values of functions $\alpha(t)$ and $\beta(t)$ at $t = \kappa'$.

**Example 1.3.** $T_{k,n-k} := S^k(a) \times \mathbb{H}^{n-k}(-\sqrt{1+a^2}) \hookrightarrow M^{n+1}(c) (c = -1)$, $1 \leq k \leq n-1$. We know that $T_{k,n-k}$ has two distinct constant principal curvatures $\frac{\sqrt{1+a^2}}{a}$ and $\frac{a}{\sqrt{1+a^2}}$ with multiplicities $k$ and $n-k$, respectively.

We easily see that the Riemannian product $S^k(a) \times \mathbb{H}^{n-k}(-\sqrt{1+a^2})$ has constant squared norm of the second fundamental form $S = k \frac{a^2}{\sqrt{1+a^2}} + (n-k) \frac{2k}{\sqrt{S^2-4(n-1)^2}}$. Denote by $H_2$ the mean curvature of $S^{n-1}(a) \times \mathbb{H}^1(-\sqrt{1+a^2})$, if $a^2 = \frac{2(n-1)}{S \sqrt{S^2-4(n-1)^2}}$, then

$$H_2 = \frac{\sqrt{n-1}(S \pm \sqrt{S^2-4(n-1)^2} + 2)}{n \sqrt{2}(S \pm \sqrt{S^2-4(n-1)^2})}.$$  

For the Riemannian product $S^{n-1}(a) \times \mathbb{H}^1(-\sqrt{1+a^2})$, denote by $\gamma(t)$ and
\( \delta(t) \) the following functions:

\[
\gamma(t) = \frac{2(n-1)}{nt^2 + \sqrt{n^2t^4 - 4(n-1) - 2(n-1)}},
\]

\[
\delta(t) = \frac{2(n-1)}{nt^2 - \sqrt{n^2t^4 - 4(n-1) - 2(n-1)}},
\]

where \( t^2 \geq \frac{2\sqrt{n-1}}{n} \).

If \( \lambda = \sqrt{\frac{1+a^2}{a}} \geq \kappa > \sqrt{\frac{S}{n}} \), from \( \sqrt{\frac{1+a^2}{a}} \geq \kappa \), we have \( a^2 \geq \frac{1}{\kappa^2-1} \) if \( \kappa < 1 \), \( a^2 \leq \frac{1}{\kappa^2-1} \) if \( \kappa > 1 \), from \( S = (n-1)\frac{1+a^2}{a} + \frac{a^2}{1+a^2} \leq n\kappa^2 \), we have \( \kappa \neq 1 \), \( \gamma(\kappa) < a^2 < \delta(\kappa) \) if \( 2\sqrt{\frac{n-1}{n}} \kappa < 1 \), \( a^2 < \gamma(\kappa) \) or \( a^2 > \delta(\kappa) \) if \( \kappa > 1 \).

It can be easily checked that if \( 2\sqrt{\frac{n-1}{n}} \kappa < 1 \), then \( \gamma(\kappa) > \frac{1}{\kappa^2-1} \); if \( \kappa > 1 \), then \( \gamma(\kappa) < \frac{1}{\kappa^2-1} \) and \( \delta(\kappa) > \frac{1}{\kappa^2-1} \). Thus, we conclude \( \gamma(\kappa) < a^2 < \delta(\kappa) \) if \( 2\sqrt{\frac{n-1}{n}} \kappa < 1 \), \( a^2 < \gamma(\kappa) \) if \( \kappa > 1 \), where \( \gamma(\kappa) \) and \( \delta(\kappa) \) are the values of functions \( \gamma(t) \) and \( \delta(t) \) at \( t = \kappa \).

If \( \lambda = \sqrt{\frac{1+a^2}{a}} \leq \kappa' < \sqrt{\frac{S}{n}} \), from \( \sqrt{\frac{1+a^2}{a}} \leq \kappa' \), we have \( \kappa' > 1 \) and \( a^2 \geq \frac{1}{\kappa'^2-1} \), from \( S = (n-1)\frac{1+a^2}{a} + \frac{a^2}{1+a^2} \geq n\kappa'^2 \), we have \( \gamma(\kappa') < a^2 < \delta(\kappa') \) if \( \kappa' > 1 \). It can be easily checked that if \( \kappa' > 1 \), then \( \gamma(\kappa') > \frac{1}{\kappa'^2-1} \) and \( \delta(\kappa') > \frac{1}{\kappa'^2-1} \). Thus, we conclude that \( \frac{1}{\kappa'^2-1} < a^2 < \delta(\kappa') \) and \( \kappa' > 1 \), where \( \gamma(\kappa') \) and \( \delta(\kappa') \) are the values of functions \( \gamma(t) \) and \( \delta(t) \) at \( t = \kappa' \).

We shall obtain the following nonexistence and characterization Theorems:

**Theorem 1.1.** Let \( M^n \) be an \( n(n \geq 3) \)-dimensional complete connected and oriented hypersurface with nonzero constant squared norm of the second fundamental form and two distinct principal curvatures in \( M^{n+1}(c) \). Then

(1) (i) for \( c = 0 \), the only hypersurfaces with two distinct principal curvatures one of which is zero are the Riemannian products: \( R^k \times S^{n-k}(a) \), \( a^2 = \frac{n-k}{S} \).

(ii) for \( c = \pm 1 \), there are no hypersurfaces with two distinct principal curvatures one of which is zero;

(2) (i) for \( c = 0 \), there are no hypersurfaces with two distinct nonzero principal curvatures and the multiplicities of both principal curvatures are constant and greater than 1;

(ii) for \( c = \pm 1 \), if \( M^n \) has two distinct nonzero principal curvatures and the multiplicities of both principal curvatures are constant and greater
than 1, then \( M^n \) is isometric to one of the Riemannian products \( S^k(a) \times S^{n-k}(\sqrt{1-a^2}) \), \( a^2 = \frac{2k}{S \pm \sqrt{S^2 - 4k(n-k) + 2k}} \), \( c = 1 \), or \( S^k(a) \times H^{n-k}(-\sqrt{1 + a^2}) \), \( a^2 = \frac{2k}{S \pm \sqrt{S^2 - 4k(n-k) - 2k}} \), \( c = -1 \), where \( 1 < k < n - 1 \).

**Theorem 1.2.** Let \( M^n \) be an \( n(n \geq 3) \)-dimensional complete connected and oriented hypersurface in \( M^{n+1}(c) \) with nonzero constant squared norm of the second fundamental form and two distinct nonzero principal curvatures \( \lambda \) and \( \mu \) of multiplicities \( n-1 \) and 1. If \( \lambda \) is bounded from below by a positive constant \( \kappa > \sqrt{\frac{S}{n}} \) or bounded from above by a positive constant \( \kappa' < \sqrt{\frac{S}{n}} \), then

1. For \( c = 0 \), there are no hypersurfaces in \( M^{n+1}(c) \);
2. For \( c = 1 \), if the sectional curvature of \( M^n \) is nonnegative, then \( M^n \) is isometric to \( S^{n-1}(a) \times S^1(\sqrt{1 - a^2}) \), \( a^2 = \frac{2(1-n)}{S \pm \sqrt{S^2 - 4(1-n) + 2(1-n)}} \), \( \alpha(\kappa) < a^2 \leq \frac{1}{1+\kappa^2} \), \( \kappa \neq 1 \) and \( \kappa^2 > \frac{2(n-1)}{n} \), or \( a^2 > \beta(\kappa') \) and \( \kappa^2 \geq \frac{2(n-1)}{n} \), or \( \frac{1}{1+\kappa'^2} \leq a^2 < \alpha(\kappa') \) and \( \frac{2(n-1)}{n} \leq \kappa'^2 < 1 \), where \( \alpha(\kappa), \alpha(\kappa') \) and \( \beta(\kappa') \) are the values of (1.1) and (1.2) at \( \kappa \) and \( \kappa' \);
3. For \( c = -1 \), if the sectional curvature of \( M^n \) is nonnegative, then \( M^n \) is isometric to \( S^{n-1}(a) \times H^1(-\sqrt{1 + a^2}) \), \( a^2 = \frac{2(1-n)}{S \pm \sqrt{S^2 - 4(1-n) - 2(1-n)}} \), \( \gamma(\kappa) < a^2 < \delta(\kappa), \frac{2(n-1)}{n} < \kappa'^2 < 1 \), or \( a^2 < \gamma(\kappa), \kappa > 1 \), or \( \frac{1}{\kappa'^2} < a^2 < \delta(\kappa') \) and \( \kappa' > 1 \), where \( \gamma(\kappa), \delta(\kappa) \) and \( \delta(\kappa') \) are the values of (1.5) and (1.6) at \( \kappa \) and \( \kappa' \).

For \( c = \pm 1 \), if \( M^n \) has two distinct nonzero principal curvatures \( \lambda \) and \( \mu \) of multiplicities \( n - 1 \) and 1, we conclude that \( \mu = \pm \sqrt{S - (n-1)\lambda^2} \).

Putting \( t = \lambda^2 \), if \( \mu = -\sqrt{S - (n-1)\lambda^2} \), we denote by \( P_1(t) \) and \( H_1(t) \) the following functions:

(1.7) \[ P_1(t) = c - \sqrt{t} \sqrt{S - (n-1)t}, \]

(1.8) \[ H_1(t) = (n-1) \sqrt{t} - \sqrt{S - (n-1)t}. \]

If \( \mu = \sqrt{S - (n-1)\lambda^2} \), we denote by \( P_2(t) \) and \( H_2(t) \) the following:

(1.9) \[ P_2(t) = c + \sqrt{t} \sqrt{S - (n-1)t}, \]

(1.10) \[ H_2(t) = (n-1) \sqrt{t} + \sqrt{S - (n-1)t}. \]
If \( c = 1 \), then \( P_1(t) \) has two positive real roots \( t_1 = \frac{s-\sqrt{s^2-4(n-1)}}{2(n-1)} \) and \( t_2 = \frac{s+\sqrt{s^2-4(n-1)}}{2(n-1)} \) and

\[
H_1(t_1) = \frac{\sqrt{n-1}(S-\sqrt{S^2-4(n-1)}-2)}{n\sqrt{2(S-\sqrt{S^2-4(n-1)})}}
\]

\[
H_1(t_2) = \frac{\sqrt{n-1}(S+\sqrt{S^2-4(n-1)}-2)}{n\sqrt{2(S+\sqrt{S^2-4(n-1)})}}
\]

It may be easily checked that \( H_1(t_1) \leq H_1(t_2) \), the mean curvature of \( M^n \) is \( H = H_1(t) \) and the mean curvature of the Riemannian product \( S^{n-1}(a) \times S^1(\sqrt{1-a^2}) \) is \( H_1(t_1) \) or \( H_1(t_2) \).

If \( c = -1 \), then \( P_2(t) \) has two positive real roots \( t_1 = \frac{s-\sqrt{s^2-4(n-1)}}{2(n-1)} \) and \( t_2 = \frac{s+\sqrt{s^2-4(n-1)}}{2(n-1)} \) and

\[
H_2(t_1) = \frac{\sqrt{n-1}(S-\sqrt{S^2-4(n-1)}+2)}{n\sqrt{2(S-\sqrt{S^2-4(n-1)})}}
\]

\[
H_2(t_2) = \frac{\sqrt{n-1}(S+\sqrt{S^2-4(n-1)}+2)}{n\sqrt{2(S+\sqrt{S^2-4(n-1)})}}
\]

It may be easily checked that \( H_2(t_1) \leq H_2(t_2) \), the mean curvature of \( M^n \) is \( H = H_2(t) \) and the mean curvature of the Riemannian product \( S^{n-1}(a) \times H^1(\sqrt{1-a^2}) \) is \( H_2(t_1) \) or \( H_2(t_2) \).

We also have the following:

**Theorem 1.3.** Let \( M^n \) be an \( n(n \geq 3) \)-dimensional complete connected and oriented hypersurface in \( M^{n+1}(c) \) (\( c = \pm 1 \)) with nonzero constant squared norm of the second fundamental form and two distinct nonzero principal curvatures \( \lambda \) and \( \mu \) of multiplicities \( n-1 \) and 1.

(1) If \( \lambda \) is bounded from below by a positive constant \( \kappa > \frac{\sqrt{n}}{n} \), then

(i) for \( c = 1 \), if \( H \geq H_1(t_2) \), then \( M^n \) is isometric to the Riemannian product \( S^{n-1}(a) \times S^1(\sqrt{1-a^2}) \), \( a^2 = \frac{\sqrt{S^2-4(n-1)+2(n-1)}}{2(n-1)} \), \( \alpha(k) < a^2 \leq \frac{1}{1+\kappa^2} \), \( \kappa \neq 1 \) and \( \kappa^2 > \frac{2\sqrt{n-1}}{n} \), where \( H_1(t_2) \) is denoted by (1.12) and \( \alpha(k) \) is the value of (1.1) at \( k \),
(ii) for \( c = -1 \), if \( H \geq H_2(t_2) \), then \( M^n \) is isometric to the Riemannian product \( S^{n-1}(a) \times H^{1}(-\sqrt{1 + a^2}) \), \( a^2 = \frac{2(2n-1)}{s + \sqrt{s^2 - 4(n-1) - 2(n-1)}} \), \( a^2 < \gamma(\kappa) \) and \( \kappa > 1 \), where \( H_2(t_2) \) is denoted by (1.14) and \( \gamma(\kappa) \) is the value of (1.5) at \( \kappa \);

(2) If \( \lambda \) is bounded from above by a positive constant \( \kappa' < \sqrt{n} \), then

(i) for \( c = 1 \), if \( H \leq H_1(t_1) \), then \( M^n \) is isometric to the Riemannian product \( S^{n-1}(a) \times S^1(\sqrt{1 - a^2}) \), \( a^2 = \frac{2(n-1)}{n - \sqrt{n^2 - 4(n-1) + 2(n-1)}} \), \( a^2 > \beta(\kappa') \), \( \kappa^2 \geq \frac{2(n-1)}{n} \), or \( \frac{1}{1 + n^2} \leq a^2 < \alpha(\kappa') \) and \( \frac{2(n-1)}{n} \leq \kappa^2 < 1 \), where \( H_1(t_1) \) is denoted by (1.11) and \( \alpha(\kappa'), \beta(\kappa') \) are the values of (1.1), (1.2) at \( \kappa' \).

(ii) for \( c = -1 \), if \( H < H_2(t_1) \), then \( M^n \) is isometric to the Riemannian product \( S^{n-1}(a) \times H^1(-\sqrt{1 + a^2}) \), \( a^2 = \frac{2(n-1)}{s - \sqrt{s^2 - 4(n-1) - 2(n-1)}} \), \( \kappa^2 < a^2 < \delta(\kappa') \) and \( \kappa' > 1 \), where \( H_2(t_1) \) is denoted by (1.13) and \( \delta(\kappa') \) is the values of (1.6) at \( \kappa' \).

2. Preliminaries

Let \( M^{n+1}(c) \) be an \( (n + 1) \)-dimensional connected Riemannian manifold with constant sectional curvature \( c \). Let \( M^n \) be an \( n \)-dimensional hypersurface in \( M^{n+1}(c) \). We choose a local orthonormal frame \( e_1, \ldots, e_{n+1} \) in \( M^{n+1}(c) \) such that \( e_1, \ldots, e_{n+1} \) are tangent to \( M^n \). Let \( \omega_1, \ldots, \omega_{n+1} \) be the dual coframe. We use the following convention on the range of indices \( 1 \leq A, B, C, \ldots \leq n + 1; 1 \leq i, j, k, \ldots \leq n \). The structure equations of \( M^{n+1}(c) \) are given by

\[
(2.1) \quad d\omega_A = \sum_B \omega_{AB} \wedge \omega_B, \quad \omega_{AB} + \omega_{BA} = 0,
\]

\[
(2.2) \quad d\omega_{AB} = \sum_C \omega_{AC} \wedge \omega_{CB} - \frac{1}{2} \sum_{C,D} K_{ABCD} \omega_C \wedge \omega_D,
\]

\[
(2.3) \quad K_{ABCD} = c(\delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC}).
\]

Restricting to \( M^n \),

\[
(2.4) \quad \omega_{n+1} = 0,
\]

\[
(2.5) \quad \omega_{n+1} = \sum_j h_{ij} \omega_j, \quad h_{ij} = h_{ji}.
\]
The structure equations of $M^n$ are

\begin{align}
(2.6) & \quad d\omega_i = \sum_j \omega_{ij} \wedge \omega_j, \quad \omega_{ij} + \omega_{ji} = 0, \\
(2.7) & \quad d\omega_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj} - \frac{1}{2} \sum_{k,l} R_{ijkl} \omega_k \wedge \omega_l,
\end{align}

\begin{align}
R_{ijkl} &= c(\delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk}) + (h_{ik} h_{jl} - h_{il} h_{jk}), \\
(2.8) & \quad n(n-1)(r-c) = n^2 H^2 - S,
\end{align}

where $n(n-1)r = R$ is the scalar curvature, $H$ is the mean curvature and $S$ is the squared norm of the second fundamental form of $M^n$.

From (2.5) we have

\begin{align}
(2.10) & \quad \omega_{n+1i} = \lambda_i \omega_i, \quad i = 1, 2, \ldots, n.
\end{align}

Thus, we have from the structure equations of $M^n$

\begin{align}
(2.11) & \quad d\omega_{n+1i} = d\lambda_i \wedge \omega_i + \lambda_i \omega_i = d\lambda_i \wedge \omega_i + \lambda_i \sum_j \omega_{ij} \wedge \omega_j.
\end{align}

On the other hand, we have on the curvature forms of $M^{n+1}(c)$,

\begin{align}
\Omega_{n+1i} &= -\frac{1}{2} \sum_{C,D} K_{n+1iCD} \omega_C \wedge \omega_D \\
(2.12) & \quad = -\frac{1}{2} \sum_{C,D} c(\delta_{n+1C} \delta_{iD} - \delta_{n+1D} \delta_{iC}) \omega_C \wedge \omega_D \\
& \quad = -c \omega_{n+1} \wedge \omega_i = 0.
\end{align}

Therefore, from the structure equations of $M^{n+1}(c)$, we have

\begin{align}
(2.13) & \quad d\omega_{n+1i} = \sum_j \omega_{n+1j} \wedge \omega_{ji} + \omega_{n+1n+1} \wedge \omega_{n+1i} + \Omega_{n+1i} \\
& \quad = \sum_j \lambda_j \omega_{ij} \wedge \omega_j.
\end{align}

From (2.11) and (2.13), we obtain

\begin{align}
(2.14) & \quad d\lambda_i \wedge \omega_i + \sum_j (\lambda_i - \lambda_j) \omega_{ij} \wedge \omega_j = 0.
\end{align}
Putting
\[ \psi_{ij} = (\lambda_i - \lambda_j)\omega_{ij}, \]
we have \( \psi_{ij} = \psi_{ji} \). Thus (2.14) can be written as
\[ \sum_j (\psi_{ij} + \delta_{ij}d\lambda_j) \wedge \omega_j = 0. \]

By E. Cartan’s Lemma, we get
\[ \psi_{ij} + \delta_{ij}d\lambda_j = \sum_k Q_{ijk}\omega_k, \]
where \( Q_{ijk} \) are uniquely determined functions such that
\[ Q_{ijk} = Q_{ikj}. \]

3. Proofs of theorems
We have the following Proposition 3.1 original due to OTSUJI [4]:

**Proposition 3.1.** Let \( M^n \) be a hypersurface in a real space form \( M^{n+1}(c) \)
such that the multiplicities of the principal curvatures are constant. Then the distribution of the space of the principal vectors corresponding to each principal curvature is completely integrable. In particular, if the multiplicity of a principal curvature is greater than 1, then this principal curvature is constant on each integral submanifold of the corresponding distribution of the space of the principal vectors.

**Proof** (Proof of Theorem 1.1). (1) If \( M^n \) has two distinct principal curvatures \( \lambda \) and \( \mu \) of multiplicities \( k \) and \( n-k \) and one of which is zero, without loss of generality, we may assume that \( \lambda = 0 \) and \( \mu \neq 0 \). Thus \( S = (n-k)\mu^2 \) and we see that \( \lambda \) and \( \mu \) are all constant. By CARTAN [3], we know that \( M^n \) is isometric to one of the Riemannian products: \( \mathbb{R}^k \times S^{n-k}(a), a^2 = \frac{n-k}{2} \) for \( c = 0 \), or \( S^k(a) \times S^{n-k}(\sqrt{1-a^2}) \) for \( c = 1 \), or \( S^k(a) \times H^{n-k}(\sqrt{1+a^2}) \) for \( c = -1 \). But, it is obvious that this is a contradiction for \( c = \pm 1 \).

(2) If \( M^n \) has two distinct nonzero principal curvatures \( \lambda \) and \( \mu \) of multiplicities \( k \) and \( n-k \), where \( 1 < k < n-1 \), we have
\[ 0 \neq S = k\lambda^2 + (n-k)\mu^2. \]
Denote by $D_{\lambda}$ and $D_{\mu}$ the integral submanifolds of the corresponding distribution of the space of principal vectors corresponding to the principal curvature $\lambda$ and $\mu$, respectively. From Proposition 3.1, we know that $\lambda$ is constant on $D_{\lambda}$. From (3.1), we infer that $\mu$ is constant on $D_{\lambda}$. By Proposition 3.1 again, we have that $\mu$ is constant on $M^n$. By the same assertion we know that $\lambda$ is constant on $M^n$. Thus $M^n$ is isoparametric. By Cartan [3], we know that $M^n$ is isometric to one of the Riemannian products: $R^k \times S^{n-k}(a)$, $a^2 = \frac{n-k}{S}$ for $c = 0$, obviously this is a contradiction, or $S^k(a) \times S^{n-k}(\sqrt{1-a^2})$, $a^2 = \frac{2k}{S\pm\sqrt{S^2-4k(n-k)+2k}}$ for $c = 1$, or $S^k(a) \times \mathbb{H}^{n-k}(-\sqrt{1+a^2})$, $a^2 = \frac{2k}{S\pm\sqrt{S^2-4k(n-k)-2k}}$ for $c = -1$, where $1 < k < n - 1$. We complete the proof of Theorem 1.1. □

Let $M^n$ be an $n$-dimensional complete hypersurface with nonzero constant squared norm of the second fundamental form and two distinct nonzero principal curvatures $\lambda$ and $\mu$ of multiplicities $n - 1$ and 1. By changing the orientation for $M^n$ and renumbering $e_1, \ldots, e_n$ if necessary, we may assume that $\lambda > 0$. Thus, we have that

\begin{align}
S &= (n-1)\lambda^2 + \mu^2, \\
\mu &= \pm \sqrt{S - (n-1)\lambda^2}, \\
0 &\neq \lambda - \mu = \lambda \mp \sqrt{S - (n-1)\lambda^2}.
\end{align}

We denote the integral submanifold through $x \in M^n$ corresponding to $\lambda$ by $M^{n-1}_1(x)$. Putting

\begin{equation}
d\lambda = \sum_{k=1}^{n} \lambda_k \omega_k, \quad d\mu = \sum_{k=1}^{n} \mu_k \omega_k.
\end{equation}

From Proposition 3.1, we have

\begin{equation}
\lambda_1 = \lambda_2 = \cdots = \lambda_{n-1} = 0 \quad \text{on} \quad M^{n-1}_1(x).
\end{equation}

From (3.3), we have

\begin{equation}
d\mu = \mp \frac{(n-1)\lambda}{\sqrt{S - (n-1)\lambda^2}} d\lambda.
\end{equation}

Thus, we also have

\begin{equation}
\mu_1 = \mu_2 = \cdots = \mu_{n-1} = 0 \quad \text{on} \quad M^{n-1}_1(x).
\end{equation}
In this case, we may consider locally $\lambda$ as a function of the arc length $s$ of the integral curve of the principal vector field $e_n$ corresponding to the principal curvature $\mu$. From (2.17) and (3.6), we have for $1 \leq j \leq n - 1$,

\begin{equation}
\lambda_n \omega_n = \sum_{i=1}^{n} \lambda_i \omega_i = d\lambda = d\lambda_j = \sum_{k=1}^{n-1} Q_{jjk}\omega_k = \sum_{k=1}^{n-1} Q_{jjk}\omega_k + Q_{jjn}\omega_n.
\end{equation}

(3.9)

Therefore, we have

\begin{equation}
Q_{jjk} = 0, \quad 1 \leq k \leq n - 1, \quad \text{and} \quad Q_{jjn} = \lambda_n.
\end{equation}

(3.10)

By (2.17) and (3.8), we have

\begin{equation}
\mu_n \omega_n = \sum_{i=1}^{n} \mu_i \omega_i = d\mu = d\lambda_n = \sum_{k=1}^{n-1} Q_{nnk}\omega_k = \sum_{k=1}^{n-1} Q_{nnk}\omega_k + Q_{nnn}\omega_n.
\end{equation}

(3.11)

Hence, we obtain

\begin{equation}
Q_{nnk} = 0, \quad 1 \leq k \leq n - 1, \quad \text{and} \quad Q_{nnn} = \mu_n.
\end{equation}

(3.12)

From (3.7), we get

\begin{equation}
Q_{nnn} = \mu_n = \mp \frac{(n-1)\lambda}{\sqrt{S - (n-1)\lambda^2}} \lambda_n.
\end{equation}

(3.13)

From the definition of $\psi_{ij}$, if $i \neq j$, we have $\psi_{ij} = 0$ for $1 \leq i \leq n - 1$ and $1 \leq j \leq n - 1$. Therefore, from (2.17), if $i \neq j$ and $1 \leq i \leq n - 1$ and $1 \leq j \leq n - 1$ we have

\begin{equation}
Q_{ijk} = 0, \quad \text{for any} \quad k.
\end{equation}

(3.14)

By (2.17), (3.10), (3.12) and (3.14), for $j < n$, we get

\begin{equation}
\psi_{jn} = \sum_{k=1}^{n} Q_{jnk}\omega_k = Q_{jjn}\omega_j + Q_{jnn}\omega_n = \lambda_n \omega_j.
\end{equation}

(3.15)

From (2.15), (3.4) and (3.15), for $j < n$, we have

\begin{equation}
\omega_{jn} = \frac{\psi_{jn}}{\lambda - \mu} = \frac{\lambda_n}{\lambda - \mu} \omega_j = \frac{\lambda_n}{\lambda \mp \sqrt{S - (n-1)\lambda^2}} \omega_j.
\end{equation}

(3.16)
Thus, from the structure equations of $M^n$ we have $d\omega_n = \sum_{k=1}^{n-1} \omega_k \wedge \omega_{kn} + \omega_{nn} \wedge \omega_n = 0$. Therefore, we may put $\omega_n = ds$. By (3.6), we get $d\lambda = \lambda_n ds$, $\lambda_n = \frac{d\lambda}{ds}$. Thus, we have

$$\omega_n = \frac{d\lambda}{\lambda \mp \sqrt{S - (n - 1)\lambda^2}} \omega_j$$

(3.17)

$$d\{\ln \left| \sqrt{n-1}S(\lambda \mp \sqrt{S - (n - 1)\lambda^2})e^{\mp \sqrt{n-1} \arcsin \left(\frac{n-1}{2}\right)} \right| \frac{\lambda}{n} \lambda \mp \sqrt{S - (n - 1)\lambda^2}} \wedge \omega_j. $$

From (3.17) and the structure equations of $M^{n+1}(c)$, for $j < n$, we have

$$d\omega_j = \sum_{k=1}^{n-1} \omega_{jk} \wedge \omega_{kn} + \omega_{jn} \wedge \omega_{nn} + \omega_{jn+1} \wedge \omega_{n+1n} + \Omega_{jn}$$

$$= \sum_{k=1}^{n-1} \omega_{jk} \wedge \omega_{kn} + \omega_{jn+1} \wedge \omega_{n+1n} - c\omega_j \wedge \omega_n$$

$$= d\{\ln \left| \sqrt{n-1}S(\lambda \mp \sqrt{S - (n - 1)\lambda^2})e^{\mp \sqrt{n-1} \arcsin \left(\frac{n-1}{2}\right)} \right| \frac{\lambda}{n} \lambda \mp \sqrt{S - (n - 1)\lambda^2}} \wedge \omega_j$$

$$\cdot \sum_{k=1}^{n-1} \omega_{jk} \wedge \omega_{k} - (\lambda c + c) \omega_j \wedge ds.$$

Differentiating (3.17), we have

$$d\omega_j = \frac{d^2\{\ln \left| \sqrt{n-1}S(\lambda \mp \sqrt{S - (n - 1)\lambda^2})e^{\mp \sqrt{n-1} \arcsin \left(\frac{n-1}{2}\right)} \right| \frac{\lambda}{n} \lambda \mp \sqrt{S - (n - 1)\lambda^2}} \wedge \omega_j}{ds^2} ds \wedge \omega_j$$

$$+ \frac{d\{\ln \left| \sqrt{n-1}S(\lambda \mp \sqrt{S - (n - 1)\lambda^2})e^{\mp \sqrt{n-1} \arcsin \left(\frac{n-1}{2}\right)} \right| \frac{\lambda}{n} \lambda \mp \sqrt{S - (n - 1)\lambda^2}} \wedge \omega_j}{ds} d\omega_j$$

$$= \frac{d^2\{\ln \left| \sqrt{n-1}S(\lambda \mp \sqrt{S - (n - 1)\lambda^2})e^{\mp \sqrt{n-1} \arcsin \left(\frac{n-1}{2}\right)} \right| \frac{\lambda}{n} \lambda \mp \sqrt{S - (n - 1)\lambda^2}} \wedge \omega_j}{ds^2} ds \wedge \omega_j$$

$$+ \frac{d\{\ln \left| \sqrt{n-1}S(\lambda \mp \sqrt{S - (n - 1)\lambda^2})e^{\mp \sqrt{n-1} \arcsin \left(\frac{n-1}{2}\right)} \right| \frac{\lambda}{n} \lambda \mp \sqrt{S - (n - 1)\lambda^2}} \wedge \omega_j}{ds} \sum_{k=1}^{n-1} \omega_{jk} \wedge \omega_{k}.$$
\[
\begin{align*}
&= -\frac{d^2\{\ln |\sqrt{\frac{n-1}{S}}(\lambda \mp \sqrt{S - (n-1)\lambda^2})e^{\pm\sqrt{n-1}\arcsin \frac{n-1}{S}\lambda |\pi}}\} ds^2 \\
+ \frac{d\{\ln |\sqrt{\frac{n-1}{S}}(\lambda \mp \sqrt{S - (n-1)\lambda^2})e^{\mp\sqrt{n-1}\arcsin \frac{n-1}{S}\lambda |\pi}}\} ds \\
+ \frac{d\{\ln |\sqrt{\frac{n-1}{S}}(\lambda \mp \sqrt{S - (n-1)\lambda^2})e^{\mp\sqrt{n-1}\arcsin \frac{n-1}{S}\lambda |\pi}}\} \sum_{k=1}^{n-1} \omega_{jk} \wedge \omega_k. \\
\end{align*}
\]

From the previous two equalities, we have
\[
\begin{align*}
&d^2\{\ln |\sqrt{\frac{n-1}{S}}(\lambda \mp \sqrt{S - (n-1)\lambda^2})e^{\mp\sqrt{n-1}\arcsin \frac{n-1}{S}\lambda |\pi}}\} ds^2 \\
- \{\frac{d\{\ln |\sqrt{\frac{n-1}{S}}(\lambda \mp \sqrt{S - (n-1)\lambda^2})e^{\mp\sqrt{n-1}\arcsin \frac{n-1}{S}\lambda |\pi}}\} ds \\
- (\lambda\mu + c) = 0.
\end{align*}
\]

Putting
\[
\varpi = |\sqrt{\frac{n-1}{S}}(\lambda \mp \sqrt{S - (n-1)\lambda^2})e^{\mp\sqrt{n-1}\arcsin \frac{n-1}{S}\lambda |\pi}},
\]
from (3.18), we obtain
\[
\frac{d^2\varpi}{ds^2} + \varpi(\lambda\mu + c) = 0. 
\]

By (3.3), we have
\[
\frac{d^2\varpi}{ds^2} + \varpi(c \pm \lambda \sqrt{S - (n-1)\lambda^2}) = 0.
\]

On the other hand, from (3.17), we have \(\nabla e_n e_n = \sum_{i=1}^{n} \omega_{ni} e_i = 0\). By the definition of geodesic, we know that any integral curve of the principal vector field corresponding to the principal curvature \(\mu\) is a geodesic. Thus, we see that \(\varpi(s)\) is a function defined in \((-\infty, +\infty)\) since \(M^n\) is complete and any integral curve of the principal vector field corresponding to \(\mu\) is a geodesic.

We can prove the following Lemmas:
Lemma 3.2. (1) If $\lambda$ is bounded from below by a positive constant $\kappa > \sqrt{\frac{S}{n}}$, then $\varpi$ is bounded.

(2) If $\lambda$ is bounded from above by a positive constant $\kappa' < \sqrt{\frac{S}{n}}$, then $\varpi$ is also bounded.

Proof. Since $S - (n - 1)\lambda^2 = \mu^2 > 0$, we have $\lambda < \sqrt{\frac{S}{n-1}}$. Putting $\theta = \arcsin \sqrt{\frac{n-1}{S}} \lambda$, we have $|\sin \theta| = \sqrt{\frac{n-1}{S}} |\lambda| < 1$. Thus $|\theta| = |\arcsin \sqrt{\frac{n-1}{S}} \lambda| < \frac{\pi}{2}$.

(1) If $\lambda$ is bounded from below by a positive constant $\kappa > \sqrt{\frac{S}{n}}$, we have

$$\lambda - \sqrt{S - (n - 1)\lambda^2} \geq \kappa' - \sqrt{S - (n - 1)\kappa'^2} > \sqrt{\frac{S}{n}} - \sqrt{S - (n - 1)\frac{S}{n}} = 0.$$ 

Thus, we see that

$$|\sqrt{\frac{n-1}{S}} (\lambda - \sqrt{S - (n - 1)\lambda^2}) e^{-\sqrt{n-1} \arcsin \sqrt{\frac{n-1}{S}} \lambda}|$$

$$> \sqrt{\frac{n-1}{S}} |\lambda - \sqrt{S - (n - 1)\lambda^2}| e^{-\frac{\pi}{2} \sqrt{n-1}}$$

$$\geq \sqrt{\frac{n-1}{S}} |\kappa - \sqrt{S - (n - 1)\kappa^2}| e^{-\frac{\pi}{2} \sqrt{n-1}} > 0,$$ 

and $0 < \varpi < \left(\sqrt{\frac{n-1}{S}} |\kappa - \sqrt{S - (n - 1)\kappa^2}| e^{-\frac{\pi}{2} \sqrt{n-1}}\right)^{-\frac{1}{n}}$.

On the other hand

$$|\sqrt{\frac{n-1}{S}} (\lambda + \sqrt{S - (n - 1)\lambda^2}) e^{\sqrt{n-1} \arcsin \sqrt{\frac{n-1}{S}} \lambda}|$$

$$> \sqrt{\frac{n-1}{S}} |\lambda + \sqrt{S - (n - 1)\lambda^2}| e^{\frac{\pi}{2} \sqrt{n-1}}$$

$$\geq \sqrt{\frac{n-1}{S}} \kappa e^{\frac{\pi}{2} \sqrt{n-1}} > \sqrt{\frac{n-1}{n}} e^{\frac{\pi}{2} \sqrt{n-1}} > 0.$$ 

Thus $0 < \varpi < \left(\sqrt{\frac{n-1}{n}} e^{-\frac{\pi}{2} \sqrt{n-1}}\right)^{-\frac{1}{n}}$.

(2) If $\lambda$ is bounded from above by a positive constant $\kappa' < \sqrt{\frac{S}{n}}$, we have

$$\lambda - \sqrt{S - (n - 1)\lambda^2} \leq \kappa' - \sqrt{S - (n - 1)\kappa'^2} < \sqrt{\frac{S}{n}} - \sqrt{S - (n - 1)\frac{S}{n}} = 0.$$
Thus, we see that

\[
\left| \sqrt{\frac{n-1}{S}} (\lambda - \sqrt{S - (n-1)\lambda^2}) e^{-\sqrt{n-1} \arcsin \sqrt{\frac{1}{n}}} \right| > \sqrt{\frac{n-1}{S}} |\kappa' - \sqrt{S - (n-1)\kappa'^2}| e^{-\frac{2}{n} \sqrt{n-1}} > 0,
\]

and \(0 < \omega < \left( \sqrt{\frac{n-1}{S}} |\kappa' - \sqrt{S - (n-1)\kappa'^2}| e^{-\frac{2}{n} \sqrt{n-1}} \right)^{-\frac{1}{n}}\).

On the other hand

\[
\left| \sqrt{\frac{n-1}{S}} (\lambda + \sqrt{S - (n-1)\lambda^2}) e^{\sqrt{n-1} \arcsin \sqrt{\frac{1}{n}}} \right| > \sqrt{\frac{n-1}{S}} \sqrt{S - (n-1)\kappa'^2} e^{-\frac{2}{n} \sqrt{n-1}} > \sqrt{\frac{n-1}{n}} e^{-\frac{2}{n} \sqrt{n-1}} > 0.
\]

Thus \(0 < \omega < \left( \frac{1}{\sqrt{n}} e^{\frac{2}{n} \sqrt{n-1}} \right)^{-\frac{1}{n}}\). We complete our proof.

\[\square\]

**Lemma 3.3.** (1) Let \(P_1(t) = c - \sqrt{1} \sqrt{S - (n-1)t}\), and \(t_0 = \frac{S}{2(n-1)}\).

For \(c = 1\), if \(S < 2\sqrt{n-1}\), then \(P_1(t) > 0\); if \(S \geq 2\sqrt{n-1}\), then

\(P_1(t)\) has two positive real roots \(t_1 = \frac{S - \sqrt{S^2 - 4(n-1)}}{2(n-1)}\), \(t_2 = \frac{S + \sqrt{S^2 - 4(n-1)}}{2(n-1)}\)

and \(t_1 \leq t_0 \leq t_2\).

(i) if \(t \geq t_0\), then \(t \geq t_2\) holds if and only if \(P_1(t) \geq 0\) and \(t \leq t_2\) holds if and only if \(P_1(t) \leq 0\);

(ii) if \(t \leq t_0\), then \(t \leq t_1\) holds if and only if \(P_1(t) \leq 0\) and \(t \geq t_1\) holds if and only if \(P_1(t) \geq 0\).

(2) Let \(P_2(t) = c + \sqrt{1} \sqrt{S - (n-1)t}\), and \(t_0 = \frac{S}{2(n-1)}\).

For \(c = -1\), if \(S < 2\sqrt{n-1}\), then \(P_2(t) < 0\); if \(S \geq 2\sqrt{n-1}\), then

\(P_2(t)\) has two positive real roots \(t_1 = \frac{S - \sqrt{S^2 - 4(n-1)}}{2(n-1)}\), \(t_2 = \frac{S + \sqrt{S^2 - 4(n-1)}}{2(n-1)}\)

and \(t_1 \leq t_0 \leq t_2\).

(i) if \(t \geq t_0\), then \(t \geq t_2\) holds if and only if \(P_2(t) \leq 0\) and \(t \leq t_2\) holds if and only if \(P_2(t) \geq 0\);

(ii) if \(t \leq t_0\), then \(t \leq t_1\) holds if and only if \(P_2(t) \leq 0\) and \(t \geq t_1\) holds if and only if \(P_2(t) \geq 0\).

**Proof.** (1) We have \(\frac{dP_1(t)}{dt} = \frac{2(n-1) - S}{2\sqrt{1} \sqrt{S - (n-1)t}}\), it follows that the solution of \(\frac{dP_1(t)}{dt} = 0\) is \(t_0 = \frac{S}{2(n-1)}\). Therefore, we know that \(t \leq t_0\) if and only if \(P_1(t)\)
is a decreasing function, \( t \geq t_0 \) if and only if \( P_1(t) \) is an increasing function and \( P_1(t) \) obtain its minimum at \( t_0 = \frac{S}{2(n-1)} \) and \( P_1(t_0) = 1 - \frac{S}{2\sqrt{n-1}} \).

If \( S < 2\sqrt{n-1} \), we have \( P_1(t) \geq P_1(t_0) > 0 \);

If \( S \geq 2\sqrt{n-1} \), then \( P_1(t) \) has two positive real roots \( t_1 = \frac{S - \sqrt{S^2 - 4(n-1)}}{2(n-1)} \), \( t_2 = \frac{S + \sqrt{S^2 - 4(n-1)}}{2(n-1)} \). We easily see that \( t_1 \leq t_0 \leq t_2 \).

(i) if \( t \geq t_0 \), since \( P_1(t) \) is an increasing function, we have \( t \geq t_2 \) holds if and only if \( P_1(t) \geq P_1(t_2) = 0 \) and \( t \leq t_2 \) holds if and only if \( P_1(t) \leq P_1(t_2) = 0 \).

(ii) if \( t \leq t_0 \), since \( P_1(t) \) is a decreasing function, we have \( t \leq t_1 \) holds if and only if \( P_1(t) \geq P_1(t_1) = 0 \) and \( t \geq t_1 \) holds if and only if \( P_1(t) \leq P_1(t_1) = 0 \).

(2) By the same method, the rest of Lemma 3.3 follows. \( \square \)

**Lemma 3.4.** (1) Let \( H_1(t) = (n-1)\sqrt{t} - \sqrt{S} - (n-1)t \). Then:

(i) \( t \geq t_2 \) holds if and only if \( H_1(t) \geq H_1(t_2) \) and \( t \leq t_2 \) holds if and only if \( H_1(t) \leq H_1(t_2) \);

(ii) \( t \geq t_1 \) holds if and only if \( H_1(t) \geq H_1(t_1) \) and \( t \leq t_1 \) holds if and only if \( H_1(t) \leq H_1(t_1) \).

(2) Let \( H_2(t) = (n-1)\sqrt{t} + \sqrt{S} - (n-1)t \) and \( t'_0 = \frac{S}{n} \). Then

(i) if \( t \geq t'_0 \), when \( t_2 \geq t'_0 \), then \( t \geq t_2 \) holds if and only if \( H_2(t) \leq H_2(t_2) \) and \( t \leq t_2 \) holds if and only if \( H_2(t) \geq H_2(t_2) \);

(ii) if \( t \leq t'_0 \), when \( t_2 \leq t'_0 \), then \( t \geq t_2 \) holds if and only if \( H_2(t) \geq H_2(t_2) \) and \( t \leq t_2 \) holds if and only if \( H_2(t) \leq H_2(t_2) \). In addition, \( t \geq t_1 \) holds if and only if \( H_2(t) \geq H_2(t_1) \) and \( t \leq t_1 \) holds if and only if \( H_2(t) \leq H_2(t_1) \).

**Proof.** (1) We have \( \frac{dH_1(t)}{dt} = \frac{n-1}{2\sqrt{t}} + \frac{1}{\sqrt{S} - (n-1)t} > 0 \), it follows that \( H_1(t) \) is an increasing function, we conclude.

(2) We have \( \frac{dH_2(t)}{dt} = \frac{n-1}{2\sqrt{t}} - \frac{1}{\sqrt{S} - (n-1)t} \), it follows that the solution of \( \frac{dH_2(t)}{dt} = 0 \) is \( t'_0 = \frac{S}{n} \). Therefore, we know that \( t \leq t'_0 \) if and only if \( H_2(t) \) is an increasing function, \( t \geq t'_0 \) if and only if \( H_2(t) \) is a decreasing function and \( H_2(t) \) obtain its maximum at \( t'_0 = \frac{S}{n} \) and \( H_2(t'_0) = \sqrt{MS} \). We see that (i) and (ii) follows by the monotonicity of \( H_2(t) \). \( \square \)

**Proof (Proof of Theorem 1.2).** Putting \( t = \lambda^2(> 0) \), from (3.20), we
have \( \frac{d^2 \varpi}{ds^2} + \varpi (c \pm \sqrt{S - (n-1)t}) = 0 \), that is

\begin{equation}
(3.21) \quad \frac{d^2 \varpi}{ds^2} + \varpi P_1(t) = 0,
\end{equation}

or

\begin{equation}
(3.22) \quad \frac{d^2 \varpi}{ds^2} + \varpi P_2(t) = 0.
\end{equation}

(1) If \( c = 0 \), we have \( P_1(t) < 0 \) or \( P_2(t) > 0 \). If \( P_1(t) < 0 \), From (3.21), we have \( \frac{d^2 \varpi}{ds^2} > 0 \). This implies that \( \frac{d \varpi(s)}{ds} \) is a strictly monotone increasing function of \( s \) and thus it has at most one zero point for \( s \in (-\infty, +\infty) \). If \( \frac{d \varpi(s)}{ds} \) has no zero point in \((-\infty, +\infty)\), then \( \varpi(s) \) is a monotone function of \( s \) in \((-\infty, +\infty)\). If \( \frac{d \varpi(s)}{ds} \) has exactly one zero point \( s_0 \) in \((-\infty, +\infty)\), then \( \varpi(s) \) is a monotone function of \( s \) in both \((-\infty, s_0]\) and \([s_0, +\infty)\).

On the other hand, from Lemma 3.2, we know that \( \varpi(s) \) is bounded. Since \( \varpi(s) \) is bounded and monotonic when \( s \) tends to infinity, we know that both \( \lim_{s \to -\infty} \varpi(s) \) and \( \lim_{s \to +\infty} \varpi(s) \) exist and then we get

\begin{equation}
(3.23) \quad \lim_{s \to -\infty} \frac{d \varpi(s)}{ds} = \lim_{s \to +\infty} \frac{d \varpi(s)}{ds} = 0.
\end{equation}

This is impossible because \( \frac{d \varpi(s)}{ds} \) is a strictly monotone increasing function of \( s \). Therefore, we know that there are no hypersurfaces in \( M^{n+1}(c)(c = 0) \). If \( P_2(t) > 0 \), it follows by the same method.

(2) for \( c = \pm 1 \), if the sectional curvature of \( M^n \) is nonnegative, that is, for \( i \neq j \), \( \lambda_i \lambda_j + c = R_{ijij} \geq 0 \), we see that \( \lambda \mu + c \geq 0 \). From (3.19), we have \( \frac{d \varpi}{ds} \leq 0 \). Thus, \( \frac{d \varpi}{ds} \) is a monotone function of \( s \in (-\infty, +\infty) \). Therefore, as observed by Wei [5], \( \varpi(s) \) must be monotone when \( s \) tends to infinity. From Lemma 3.2, we know that the positive function \( \varpi(s) \) is bounded. Since \( \varpi(s) \) is bounded and monotonic when \( s \) tends to infinity, we know that both \( \lim_{s \to -\infty} \varpi(s) \) and \( \lim_{s \to +\infty} \varpi(s) \) exist and (3.23) holds.

From the monotonicity of \( \frac{d \varpi(s)}{ds} \), we have \( \frac{d \varpi(s)}{ds} \equiv 0 \) and \( \varpi(s) = \text{constant} \).

Combining \( \varpi = \left| \sqrt{\frac{n-1}{n}} \lambda \mp \sqrt{S - (n-1)\lambda^2} \right| e^{\pm \sqrt{n-1} \arcsin \sqrt{\frac{2n-1}{2n}}} - \frac{1}{n} \) and (3.3), we conclude that \( \lambda \) and \( \mu \) are constant, that is, \( M^n \) is isoparametric.

According to CARTAN [3], Example 1.2 and Example 1.3, we know that \( M^n \) is isometric to one of the Riemannian products: \( S^{n-1}(a) \times S^1(\sqrt{1-a^2}) \), \( \alpha(\kappa) < a^2 \leq \frac{1}{1+\kappa^2} \), \( \kappa \neq 1 \) and \( \kappa^2 > \frac{2\sqrt{n-1}}{n} \), or \( a^2 > \beta(\kappa′) \) and \( \kappa^2 \geq \frac{2\sqrt{n-1}}{n} \).
or \( \frac{1}{1 + \kappa'^2} \leq a^2 < \alpha(\kappa') \) and \( \frac{2\sqrt{n-1}}{n} \leq \kappa'^2 < 1 \); or \( S^{n-1}(a) \times H^1(-\sqrt{1 + a^2}) \), we have \( a^2 = \frac{2(n-1)}{S + 2(n-1) - 2(n-1)} \), \( \gamma(\kappa) < a^2 < \delta(\kappa) \), \( \frac{2\sqrt{n-1}}{n} < \kappa'^2 < 1 \), or \( a^2 < \gamma(\kappa), \kappa > 1 \), or \( \frac{1}{n^2 - 1} < a^2 < \delta(\kappa') \) and \( \kappa' > 1 \). This completes our proof. □

**Proof (Proof of Theorem 1.3).** (1) If \( \lambda \) is bounded from below by a positive constant \( \kappa > \sqrt{\frac{n}{n}} \), we have \( t > \frac{s}{n} > t_0 \).

(i) If \( c = 1 \), we may easily see that (3.22) does not hold. In fact, since \( P_2(t) > 0 \), from (3.22), we have \( \frac{d\varphi}{ds} < 0 \). This implies that \( \frac{d\varphi(s)}{ds} \) is a strictly monotone decreasing function of \( s \) and thus it has at most one zero point for \( s \in (-\infty, +\infty) \). By the same arguments as in the proof of (1) of Theorem 1.2, we know that this is impossible. We conclude that only (3.21) holds.

Case i. If \( S < 2\sqrt{n-1} \), by Lemma 3.3, we have \( P_1(t) > 0 \). From (3.21), we have \( \frac{d^2\varphi}{ds^2} < 0 \). This implies that \( \frac{d\varphi(s)}{ds} \) is a strictly monotone decreasing function of \( s \). By the same arguments as in the proof of (1) of Theorem 1.2, we know that this is impossible.

Case ii. If \( S \geq 2\sqrt{n-1} \), since \( t > t_0 \), from Lemma 3.4, Lemma 3.3 and (3.21), we have that \( H = H_1(t_1) \geq H_1(t_2) \) holds if and only if \( t \geq t_2 \) if and only if \( P_1(t) \geq 0 \) and if and only if \( \frac{d\varphi(t)}{ds} \leq 0 \). Thus \( \frac{d\varphi}{ds} \) is a monotonic function of \( s \in (-\infty, +\infty) \). By the same arguments as in the proof of (2) of Theorem 1.2 and Example 1.2, we conclude that \( M^n \) is isometric to the Riemannian product \( S^{n-1}(a) \times S^1(\sqrt{1 - a^2}) \), \( a^2 = \frac{2(n-1)}{S + 2(n-1) - 2(n-1)} \) and \( \alpha(\kappa) < a^2 \leq \frac{1}{1 + \kappa'^2} \), \( \kappa \neq 1 \) and \( \kappa'^2 > \frac{2\sqrt{n-1}}{n} \).

(ii) If \( c = -1 \), we may easily see that (3.21) does not hold. In fact, since \( P_1(t) < 0 \), from (3.21), we have \( \frac{d^2\varphi}{ds^2} > 0 \). This implies that \( \frac{d\varphi(s)}{ds} \) is a strictly monotone increasing function of \( s \). By the same arguments as in the proof of (1) of Theorem 1.2, we know that this is impossible. We conclude that only (3.22) holds.

Case i. If \( S < 2\sqrt{n-1} \), by Lemma 3.3, we have \( P_2(t) < 0 \). From (3.21), we have \( \frac{d^2\varphi}{ds^2} > 0 \). This implies that \( \frac{d\varphi(s)}{ds} \) is a strictly monotone increasing function of \( s \). By the same arguments as in the proof of (1) of Theorem 1.2, we know that this is impossible.

Case ii. If \( S \geq 2\sqrt{n-1} \), we consider two cases \( S \geq n \) and \( 2\sqrt{n-1} \leq S < n \).

If \( S \geq n \), we easily check that \( t_2 \geq t_0' \). Since \( t_0' = \frac{S}{n} > t_0 \), we have \( t_2 \geq t_0' > t_0 \). Since \( t > t_0' > t_0 \), from Lemma 3.4, Lemma 3.3 and (3.22), we have
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that \( H_2(t) \geq H_3(t_2) \) holds if and only if \( t \leq t_2 \) if and only if \( P_2(t) \geq 0 \) and if and only if \( \frac{\partial P_2}{\partial s} \leq 0 \). Thus \( \frac{\partial P_2}{\partial s} \) is a monotonic function of \( s \in (-\infty, +\infty) \).

By the same arguments as in the proof of (2) of Theorem 1.2, \( n \kappa^2 > S \geq n \), that is \( \kappa > 1 \), and Example 1.3, we conclude that \( M \) is isometric to the Riemannian product \( S^{n-1}(a) \times H^1\left(-\sqrt{1 + a^2}\right) \), \( a^2 = \frac{2(n-1)}{S + \sqrt{S^2 - 4(n-1) - 2(n-1)}} \), \( a^2 < c(n) \) and \( \kappa > 1 \).

If \( 2\sqrt{n-1} \leq S < n \), we easily check that \( t_2 > t_0' \). Since \( t_2 > t_0 \), we have \( t_0' > t_2 \geq t_0 \). Thus \( t > t_2 \geq t_0 \). From Lemma 3.3 and (3.22), we have \( P_2(t) < 0 \) and \( \frac{\partial P_2}{\partial s} > 0 \). This implies that \( \frac{\partial P_2}{\partial s} \) is a strictly monotone increasing function of \( s \). By the same arguments as in the proof of (1) of Theorem 1.2, we know that this is impossible. Thus, the case \( 2\sqrt{n-1} \leq S < n \) does not occur.

(2) If \( \lambda \) is bounded from above by a positive constant \( \kappa' < \sqrt{\frac{2}{n}} \), we have \( t < \frac{S}{n} = t_0' \).

(i) If \( c = 1 \), from the arguments in (1), we know that (3.22) does not hold. We conclude that only (3.21) holds.

Case i. If \( S < 2\sqrt{n-1} \), by the arguments in (1), we know that this is impossible.

Case ii. If \( S \geq 2\sqrt{n-1} \), from Lemma 3.4, we know that \( H = H_1(t) \leq H_1(t_1) \) holds if and only if \( t \leq t_1 \). Since \( t_1 \leq t_0 \), from Lemma 3.3 and (3.21), we have that \( t \leq t_1 \) if and only if \( P_1(t) \geq 0 \) and if and only if \( \frac{\partial P_1}{\partial s} \leq 0 \). Thus \( \frac{\partial P_1}{\partial s} \) is a monotonic function of \( s \in (-\infty, +\infty) \). By the same arguments as in the proof of (2) of Theorem 1.2 and Example 1.2, we conclude that \( M \) is isometric to the Riemannian product \( S^{n-1}(a) \times S^1\left(\sqrt{1 - a^2}\right) \), \( a^2 = \frac{2(n-1)}{S - \sqrt{S^2 - 2(n-1) + 2(n-1)}} \), \( a^2 > \beta(n') \), \( \kappa'^2 \geq 2\sqrt{n-1} \), or \( \frac{1}{1 + \kappa'^2} \leq a^2 < \alpha(n') \) and \( \frac{2\sqrt{n-1}}{n} \leq \kappa' < 1 \).

(ii) If \( c = -1 \), we similarly know that (3.21) does not hold, then only (3.22) holds.

Case i. If \( S < 2\sqrt{n-1} \), by the arguments in (1), we know that this is impossible.

Case ii. If \( S \geq 2\sqrt{n-1} \), we may easily check that \( t_1 \leq t_0' \). Since \( t < t_0' \), from Lemma 3.4, we have that \( H_2(t) \leq H_2(t_1) \) holds if and only if \( t \leq t_1 \). Since \( t_1 \leq t_0 \), by Lemma 3.3 and (3.22), we have that \( t \leq t_1 \) if and only if \( P_2(t) \leq 0 \) and if and only if \( \frac{\partial P_2}{\partial s} \geq 0 \). Thus \( \frac{\partial P_2}{\partial s} \) is a monotonic function of \( s \in (-\infty, +\infty) \). By the same arguments as in the proof of (2)
of Theorem 1.2 and Example 1.3, we conclude that $M^n$ is isometric to the Riemannian product $S^{n-1}(a) \times H^1(-\sqrt{1+a^2})$, 
\[ a^2 = \frac{2(n-1)}{S-\sqrt{S^2-4(n-1)^2}}. \]
\[ \frac{1}{\kappa'^2-1} < a^2 < \delta(\kappa') \quad \text{and} \quad \kappa' > 1. \]
This completes our proof. \qed

Acknowledgment. The authors would like to thank the referee for his/her careful reading of the original manuscript and many valuable suggestions that really improve the paper.

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