A GENERALIZATION OF POST-WIDDER OPERATORS  
BASED ON $q$-INTEGERS 

BY 

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Abstract. In this paper we introduce a $q$-generalization of Post-Widder operators $P_{n,q}$. We give a Voronovskaja-type approximation result and rate of that convergence. We also study approximation properties of $P_{n,q}$ in a weighted space. We also show that the rates of convergence of these generalized operators to approximating function $f$ as weight are at least so faster than that of the classical Post-Widder operators.

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1. Introduction

The Post-Widder operator is defined for all $n \in \mathbb{N}$, $x > 0$, by

$$P_n (f) (x) = \frac{1}{n!} \left( \frac{n}{x} \right)^{n+1} \int_0^\infty t^n e^{-nt} f(t) \, dt,$$

for $f \in C(0,\infty)$. The convergence of Post-Widder operators was investigated deeply in (Ch. VII of [23]) by Widder. The saturation and inverse problems for Post-Widder operators of the form

$$P_n (f) (x) = \frac{1}{(n-1)!} \left( \frac{n}{x} \right)^n \int_0^\infty t^{n-1} e^{-t} f(t) \, dt$$

were presented by May in [16]. Simultaneous approximation with an asymptotic formula of these operators were carried out by Rathore and Singh [18]. A modification of Post-Widder operators in polynomial weighted spaces of
differentiable functions and strong approximation was studied by Rempulska and Skorupka in [19]. On the other hand, Voronovskaja-type results were discussed by the same authors in [20]. A deep analysis related to the characterization of the weighted approximation errors of the Post-Widder and the Gamma operators for functions belonging $L_p(0, \infty), 1 \leq p \leq \infty$ with a suitable weight was achieved by Draganov and Ivanov in [10]. Uniform approximation in a weighted space by the Post-Widder operators was obtained by Holhos in [14]. Approximation and geometric properties of complex Post-Widder operators were studied in [2].

As it is known, in 1997, an interesting contribution to the theory of approximation by linear positive operators was done by Phillips by expressing Bernstein polynomials in terms of $q$-calculus. [17]. After, linear positive discrete and integral operators were designed by using machinery of $q$-calculus and investigated their approximation properties by many authors. Some are in [1], [3], [4], [5], [6], [7], [17].

In this work, we introduce a $q$-generalization of the Post-Widder operator denoted by $P_{n,q}, n \in \mathbb{N}$, and give a Voronovskaja-type result for $P_{n,q}$. Next, we study the rate of this convergence. On the other hand, we deal with the approximation by $P_{n,q}$ in polynomial weighted space using the theorem proved by Gadziev in [12]. For the rate of the approximation, we use the weighted modulus of continuity introduced by Ispir in [15].

Throughout the paper, related to $q$-special functions, we shall use the following standard notations of $q$-numbers (see [13]):

Let $q > 0$, then $q$-integer is defined as

$$[n]_q = \begin{cases} \frac{1-q^n}{1-q}, & q \in \mathbb{R}^+ \setminus \{1\} \\ n, & q = 1 \end{cases},$$

for $n \in \mathbb{N} = \{1, 2, \ldots\}$ and $[0]_q = 0$. For $0 < |q| < 1$, the $q$-extension of exponential function $e^x$ is defined by

$$E_q(x) := \sum_{n=0}^{\infty} q^{\frac{n(n-1)}{2}} \frac{x^n}{(q; q)_n} = (-x; q)_\infty, \quad x \in \mathbb{R},$$

where $(a; q)_n = \prod_{k=0}^{n-1}(1 - aq^k)$ and $(-x; q)_\infty = \prod_{k=0}^{\infty}(1 + xq^k)$.

Also, $q$-factorial is defined by

$$[n]_q! = \begin{cases} \frac{[n]_q [n-1]_q \cdots [1]_q}{[1]_q}, & n \in \mathbb{N} \\ 1, & n = 0 \end{cases}. $$
In [8], Atakishiyev and Atakishiyeva introduced the \( q \)-extension of the Euler Gamma integral as

\[
(1.2) \quad c_q(x) \Gamma_q(x) = \frac{1 - q}{\ln q} q^{\frac{x(x-1)}{2}} \int_0^\infty t^{x-1} E_q((1-q)t) dt, \quad \Re x > 0
\]

for \( 0 < q < 1 \), where \( \Gamma_q(x) \) is the \( q \)-extension of Gamma function defined by

\[
(1.3) \quad \Gamma_q(x) = (q; q)_\infty \frac{(q^x; q)_\infty}{(1-q)_\infty (1-q)^{x-1}} \]

and satisfies the analogous property \( \Gamma_q(n+1) = [n]_q! \), for \( n \in \mathbb{N} \cup \{0\} \).

Moreover, the factor \( c_q(x) \) in (1.2) satisfies the following conditions:

a. \( c_q(x + 1) = c_q(x) \)

b. \( c_q(n) = 1, n = 0, 1, 2, ... \)

c. \( \lim_{q \to 1^-} c_q(x) = 1 \).

Below, taking (1.2) into consideration, we introduce a \( q \)-extension of the Post-Widder operator.

**Definition 1.** Let \( 0 < q < 1, n \in \mathbb{N} \) and \( x > 0 \), the \( q \)-Post-Widder operator will be defined by

\[
(1.4) \quad P_{n;q}(f)(x) = A_{n;q} \frac{[n]_q}{x} \int_0^\infty \frac{t^{n-1}}{E_q((1-q)t)} f(q^n t) dt,
\]

for all real valued measurable functions defined on \((0, \infty)\) such that the integral in (1.4) is convergent for almost all \( x \), where \( A_{n;q} := \frac{(1-q)^n n!}{\ln q} \frac{n(n-1)}{2 \ln q} \), and \( \Gamma_q(n) \) is the \( q \)-gamma function given in (1.3).

Let \( e_\alpha(x) := x^\alpha, \alpha \in [0, \infty), x > 0 \). Note that the Post-Widder operator \( P_n \) satisfies \( P_n(e_\alpha)(x) = \frac{\Gamma(n+\alpha)}{n! \Gamma(n)} x^\alpha \), where \( \Gamma \) is the familiar gamma function.

Moreover, we have \( \lim_{n \to \infty} \frac{\Gamma(n+\alpha)}{n! \Gamma(n)} = 1 \) (see, for example, in the proof of Lemma 2.1 of [14]).

Analogously, in the following lemma, we give the behavior of \( q \)-Post-Widder operator on the power function \( e_\alpha, \alpha \geq 0 \).
Lemma 1. For all $e_\alpha(x) = x^\alpha$, $\alpha \in [0, \infty)$, $x > 0$, we have $P_{n,q}(e_0)(x) = 1$ and

\[(1.5) \quad P_{n,q}(e_\alpha)(x) = c_q(n + \alpha)q^{-\frac{\alpha}{2}(\alpha - 1)}\frac{\Gamma_q(n + \alpha)}{\Gamma_q(n)}x^\alpha,\]

where $\Gamma_q$ is the $q$-gamma function given by (1.3).

Proof. By the definition (1.2), it is clear that $P_{n,q}(e_0)(x) = 1$. For $\alpha \in [0, \infty)$ we have

\[\begin{align*}
P_{n,q}(e_\alpha)(x) & = \frac{(1-q)q^{n(n-1)}}{\ln q^{-1}\Gamma_q(n)} \left(\frac{[n]_q}{x}\right)^n \int_0^\infty \frac{q^{n\alpha + n + \alpha - 1}t^n}{E_q((1-q)\frac{[n]_q}{x})} dt. \\
\end{align*}\]

Making use of substitution $\frac{[n]_q}{x} = u$ gives that

\[\begin{align*}
P_{n,q}(e_\alpha)(x) & = \frac{(1-q)q^{n(n-1)}n\alpha}{\ln q^{-1}\Gamma_q(n)} \left(\frac{x}{[n]_q}\right)^\alpha \int_0^\infty \frac{u^{n+\alpha-1}}{E_q((1-q)u)} du, \\
\end{align*}\]

from which we can reach to the result by taking (1.2) into consideration. \(\square\)

2. Voronovskaja-type results

In this section, we first obtain a Voronovskaja-type result for $P_{n,q}$. Subsequently, we study the rate of this convergence in a similar way to that for the classical Post-Widder operator obtained by Rempulska and Scorupka in [20]. So, by considering the weight function $w_r(x) := \frac{1}{1+x^r}$ for $r \geq 1$, and $w_0(x) := 1$, we similarly take $C_r$, $r \in \mathbb{N}$, as the space of all real valued functions on $(0, \infty)$ for which $w_rf$ is continuous and bounded on $(0, \infty)$ with norm $\|f\|_r = \sup_{x>0} w_r(x)|f(x)|$. Moreover, we simply take $C_0$ as the space of all functions continuous and bounded on $(0, \infty)$. We also consider $C_r^2$, $r \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, as the set of all functions $f \in C_r$ which are twice differentiable on $(0, \infty)$ and $f^{(k)} \in C_{r-k}$ for $k = 1, 2$ and $C_{r-k} \equiv C_0$ if $r - k \leq 0$.

The modulus of continuity of a function $f \in C_r$ is defined as

\[(2.1) \quad \omega(f; C_r; \delta) := \sup_{0 \leq h \leq \delta} \|f(x + h) - f(x)\|_r.\]

Let $\varphi_x := t - x$. In the following, we give $P_{n,q}(\varphi_x^m)(x)$ for $m = 1, 2, 4$. 
Lemma 2. Let $P_{n,q}$ be the operator defined by (1.4). Then we have:

$$P_{n,q}(\varphi_x)(x) = 0, \quad P_{n,q}(\varphi_x^2)(x) = \frac{x^2}{q[n]_q},$$

$$P_{n,q}(\varphi_x^4)(x) = \left\{ \frac{[3]_q [2]_q}{q^6 [n]_q^3} + \frac{[3]_q}{q^5 [n]_q^2} + 2 - q [2]_q [n]_q + \left(\frac{1}{q} - 1\right)^2 \right\} x^4. \tag{2.2}$$

Proof. From (1.5), $P_{n,q}(\varphi_x)(x) = 0$ is clear. For each $k = 1, 2, 3$, taking the fact $[n + k]_q - q^k [n]_q = [k]_q$ and the property b. of $c_q$ into consideration, we obtain that

$$P_{n,q}(\varphi_x^2)(x) = \left(\frac{\Gamma_q(n + 2)}{q[n]_q^2 \Gamma_q(n)} - 1 \right) x^2 = \left(\frac{[n + 1]_q}{q[n]_q^2} - 1 \right) x^2 = \frac{x^2}{q[n]_q^2},$$

and

$$P_{n,q}(\varphi_x^4)(x) = \left\{ \frac{\Gamma_q(n + 4)}{q^6 [n]_q^3 \Gamma_q(n)} - 4 \frac{\Gamma_q(n + 3)}{q^5 [n]_q^2 \Gamma_q(n)} + \frac{\Gamma_q(n + 2)}{q^4 [n]_q \Gamma_q(n)} - 3 \right\} x^4 = \frac{1}{q^6 [n]_q^3} \left\{ [3]_q [2]_q [n + 2]_q [n + 1]_q - 3q^3 [n]_q [2]_q + q [n]_q \right\} x^4 = \frac{1}{q^6 [n]_q^3} \left\{ [3]_q [2]_q [n + 2]_q + q^2 [n]_q (2 - q [2]_q) + q^3 [n]_q^2 (1 - q)^2 \right\} x^4 = \frac{1}{q^6 [n]_q^3} \left\{ \frac{[3]_q [2]_q}{q^6 [n]_q^3} + \frac{[3]_q}{q^5 [n]_q^2} + \frac{2 - q [2]_q}{q^4 [n]_q} + \left(\frac{1}{q} - 1\right)^2 \right\} x^4,$$

which completes the proof. \qed

Remark 1. We point out that our $q$-generalization $P_{n,q}(f), n \in \mathbb{N}$, is different by $q$-analogue of the Post-Widder operators, recently introduced in [22] as follows.

$$P_{n,q}^*(f)(x) = \int_0^{1/(1-q)} f \left( \frac{xt}{[n]_q} \right) t^n E_q(-qt) d_qt,$$

where $x \in [0, \infty)$ and $q \in (0, 1)$. Since $P_{n,q}(\varphi_x)(x) = 0$, $P_{n,q}(\varphi_x^2)(x) = \frac{x^2}{q[n]_q}$ and $P_{n,q}^*(\varphi_x)(x) = (\frac{[n + 1]_q}{[n]_q} - 1)x$, $P_{n,q}^*(\varphi_x^2)(x) = (\frac{[n + 1]_q [n + 2]_q}{[n]_q^2} - \frac{2[n + 1]_q}{[n]_q} + 1)x^2$, the behavior of $P_{n,q}(f)$ on $\varphi_x$ and $\varphi_x^2$ are better than $P_{n,q}^*(f)$ on $\varphi_x$ and $\varphi_x^2$. 
Using Lemma 2 we reach to the following result easily.

**Lemma 3.** Let \((q_n)\) be a sequence in \((0,1)\) such that \(\lim_{n \to \infty} q_n = 1\). Then for every \(x > 0\) we have

\[
\lim_{n \to \infty} [n]_{q_n} P_{n,q_n} \left( \varphi_x \right) (x) = 0
\]

\[
\lim_{n \to \infty} [n]_{q_n} P_{n,q_n} \left( \varphi_x^2 \right) (x) = x^2
\]

\[
\lim_{n \to \infty} [n]_{q_n} P_{n,q_n} \left( \varphi_x^4 \right) (x) = 0.
\]

Below, we present a Voronovskaja-type result for the \(q\)-Post-Widder operator \(P_{n,q}\).

**Theorem 1.** Let \((q_n)\) be a sequence in \((0,1)\) such that \(\lim_{n \to \infty} q_n = 1\) and \(f \in C^2_0\). Then we have \(\lim_{n \to \infty} [n]_{q_n} \{P_{n,q_n}(f)(x) - f(x)\} = \frac{x^2}{2} f''(x)\) at every \(x > 0\).

**Proof.** Proof follows from Lemma 3 and Theorem 1 of [21].

Now, we study the rate of this convergence by extending the result of Rempulska and Scorupka for the Post-Widder operator (Theorem 2 of section 3.2 of [20]) to the \(q\)-Post-Widder operator \(P_{n,q}\).

**Theorem 2.** Let \(f \in C^2_r\) with a fixed \(r = 0, 1, 2\). Then

\[
w_2(x) \left| [n]_{q} \{P_{n,q}(f)(x) - f(x)\} - \frac{x^2}{2q} f''(x) \right|
\]

\[
\leq \frac{1}{2} \omega \left( f''; C_0; \frac{x}{\sqrt{q[n]}} \right) \left\{ \frac{[3]_q ([2]_q + q)}{q^3} + \frac{2 - q [2]_q}{q^3} + \frac{1}{q^2} \right\},
\]

where \(\omega (f'', C_0)\) is the modulus of continuity of \(f''\) defined in (2.1) and \(w_2(x) = \frac{1}{1 + x^2}\).

**Proof.** Let \(f \in C^2_r\) and \(x > 0\) be fixed. From Taylor’s formula with integral form of the remainder we have

\[
f(t) = f(x) + f'(x)(t-x) + \frac{1}{2} f''(x)(t-x)^2 + (t-x)^2 F(t,x)
\]

for \(t > 0\), where

\[
F(t,x) = \int_0^1 (1-u) [f''(x + u(t-x)) - f''(x)] du.
\]
Taking \( \varphi_x = t - x \) and using Lemma 2, we obtain from (2.3) that

\[
\begin{align*}
    w_2(x) \left[ n \right]_q \{ P_{n,q}(f)(x) - f(x) \} - \frac{x^2}{2q} f''(x) \\
\leq w_2(x) \left[ n \right]_q P_{n,q}(\varphi_x^2 | F(., x)|)(x),
\end{align*}
\]

(2.5)

where we have used the fact that \( P_{n,q} \) is linear and positive. Recall the following property of the modulus of continuity \( \omega(f; C_0; t) \leq (1 + \frac{t^2}{2}) \omega(f; C_0; \delta) \), for \( f \in C_0 \). Using this, we obtain the following estimate for \(| F(t, x) |\):

\[
| F(t, x) | \leq \int_0^1 (1 - u) \omega \left( f''; C_0; u | t - x | \right) du \\
\leq \omega \left( f''; C_0; | t - x | \right) \int_0^1 (1 - u) du \\
\leq \frac{1}{2} \omega \left( f''; C_0; \frac{x}{\sqrt{q \left[ n \right]_q}} \right) \left( 1 + \frac{q \left[ n \right]_q}{x^2} (t - x)^2 \right).
\]

Taking the last inequality into account we deduce

\[
\begin{align*}
    \left[ n \right]_q P_{n,q}(\varphi_x^2 | F(., x)|)(x) \\
\leq \frac{1}{2} \omega \left( f''; C_0; \frac{x}{\sqrt{q \left[ n \right]_q}} \right) \left\{ \left[ n \right]_q P_{n,q}(\varphi_x^2)(x) + \frac{q \left[ n \right]_q^2}{x^2} P_{n,q}(\varphi_x^4)(x) \right\}.
\end{align*}
\]

(2.6)

Here, from Lemma 2 we find that

\[
\begin{align*}
    \left[ n \right]_q P_{n,q}(\varphi_x^2)(x) = \frac{x^2}{q},
\end{align*}
\]

(2.7)

on the other hand, since \( \left[ n \right]_q < \frac{1}{1-q} \), we obtain that

\[
\begin{align*}
    \frac{q \left[ n \right]_q^2}{x^2} P_{n,q}(\varphi_x^4)(x) \\
= \frac{x^2}{q^3 \left[ n \right]_q} \left\{ 3q \left[ 2 \right]_q + q \left[ n \right]_q \right\} + \frac{2 - q \left[ 2 \right]_q}{q^3} + \frac{\left[ n \right]_q (1 - q)^2}{q^2} \\
= \frac{q \left[ n \right]_q^2}{x^2} \left\{ \left[ 3 \right]_q \right\} + \frac{3q}{q^4} + \frac{2 - q \left[ 2 \right]_q}{q^3} + \frac{\left[ n \right]_q (1 - q)^2}{q^2} \\
< \frac{x^2}{q^3} \left\{ \left[ 3 \right]_q \right\} + \frac{3q}{q^4} + \frac{2 - q \left[ 2 \right]_q}{q^3} + \frac{1 - q}{q^2}.
\end{align*}
\]
which together with (2.6) and (2.7), we reach to

\[
\begin{align*}
\frac{w_2(x)}{[n]_q} P_{n,q}(\varphi_{\varphi}^2(x)) (x) \\
< \frac{1}{2} \omega \left( f''; C_0; \frac{x}{\sqrt{q [n]_q}} \right) \left\{ \frac{[3]_q [2]_q}{q^4} + \frac{[3]_q}{q^3} + \frac{2 - q [2]_q}{q^2} + \frac{1 - q}{q} \right\} \\
= \frac{1}{2} \omega \left( f''; C_0; \frac{x}{\sqrt{q [n]_q}} \right) \left\{ \frac{[3]_q ([2]_q + 1)}{q^4} + \frac{2 - q [2]_q}{q^3} + \frac{1}{q^2} \right\}.
\end{align*}
\]

Using the last inequality in (2.5), the result follows.

Recall that a continuous function \( f \) from \( D \subseteq \mathbb{R} \) into \( \mathbb{R} \) is said to be \( \text{Lipschitz continuous of order} \ \alpha, \ \alpha \in (0, 1) \), if there exists a constant \( M > 0 \) such that for every \( x, y \in D \), \( f \) satisfies \( |f(x) - f(y)| \leq M |x - y|^{\alpha} \). The set of Lipschitz continuous functions is denoted by \( \text{Lip}_M^\alpha (D) \).

**Remark 2.** Let \( f \in C^2_r \) for \( r = 0, 1, 2 \), such that \( f'' \in \text{Lip}_M^\alpha (0, \infty) \). Then we have

\[
\begin{align*}
\frac{w_2(x)}{[n]_q} \left\{ P_{n,q}(f)(x) - f(x) \right\} - \frac{x^2}{2q} f''(x) \\
< \frac{1}{q(\alpha + 2)} B(\alpha, 2) \left\{ \frac{[3]_q [2]_q}{q^4} + \frac{[3]_q}{q^3} + \frac{2 - q [2]_q}{q^2} + \frac{1 - q}{q} \right\} \left( \frac{x}{\sqrt{q [n]_q}} \right)^{2 + \alpha},
\end{align*}
\]

where \( B(., .) \) is the usual beta function.

**3. Approximation properties in a weighted space**

In this section, by using a Bohman-Korovkin type theorem proved in [12] by Gadzhiev, we present a direct approximation property of the operator \( P_{n,q} \).

Let \( B_2 \) denote the space of real-valued functions \( f \) defined on \( (0, \infty) \) with \( w_2(x) |f(x)| \leq M_f \) for all \( x > 0 \), where \( M_f \) is a constant depending on the function \( f \). Therefore \( C_2 \) can be regarded as a subspace of \( B_2 \), namely \( C_2 = \{ f \in B_2 : w_2 f \text{ is continuous on } (0, \infty) \} \). We also consider the space of functions \( C_{2,k} = \{ f \in C_2 : \lim_{x \to \infty} w_2(x) f(x) = k \in \mathbb{R}, \ x > 0 \} \), with the norm \( \|f\|_2 = \sup_{x > 0} w_2(x) |f(x)| \). Related to the spaces given above, we recall the following result due to Gadzhiev in [12].
Theorem 3. Let $T_n$ be a sequence of linear positive operators mapping $C_2$ into $B_2$ and satisfying the conditions $\lim_{n \to \infty} \|T_n (e_i) - e_i\|_2 = 0$, for $i = 0, 1, 2$. Then for any $f \in C_{2,k}$, we have $\lim_{n \to \infty} \|T_n (f) - f\|_2 = 0$ and there exists a function $f^* \in C_2 \setminus C_{2,k}$ such that $\lim_{n \to \infty} \|T_n (f^*) - f^*\|_2 \geq 1$.

Now, for $f \in C_{2,k}$, we will consider the weighted modulus of continuity defined by $\Omega_2 (f, \delta) = \sup_{x>0, |h| \leq \delta} \frac{|f(x+h) - f(x)|}{(1+x^2)^{\frac{1}{2}+\delta^2}}$ in [15]. This function has the following properties:
1. $\Omega_2 (f, \delta) \leq 2\|f\|_2$,
2. $\Omega_2 (f, m\delta) \leq 2m (1 + \delta^2) \Omega_2 (f, \delta)$, $m \in \mathbb{N}$,
3. $\lim_{\delta \to 0} \Omega_2 (f, \delta) = 0$.

Note that, we cannot find a rate of convergence in terms of usual first modulus of continuity $\omega (f; C_0; \delta)$ of function $f$, because, the modulus of continuity $\omega (f; C_0; \delta)$ on the infinite interval does not tend to zero as $\delta \to 0$. For this reason, we consider the weighted modulus of continuity $\Omega_2 (f, \delta)$.

Remark 3. Since any linear and positive operator is monotone, the relations (1.5) guarantee that $P_{n,q} (f) \in C_2$ for each $f \in C_2$.

Theorem 4. Let $\{q_n\}$ be a sequence in $(0, 1)$ such that $\lim_{n \to \infty} q_n = 1$. Then for each $f \in C_{2,k}$ we have $\lim_{n \to \infty} \|P_{n,q_n} (f) - f\|_2 = 0$.

Proof. From Lemma 2 it is clear that $\lim_{n \to \infty} \|P_{n,q_n} (e_i) - e_i\|_2 = 0$, for $i = 0, 1$. For $i = 2$, we get
\[
\|P_{n,q_n} (e_2) - e_2\|_2 = \sup_{x>0} \frac{|P_{n,q_n} (e_2) (x) - e_2 (x)|}{1 + x^2} \\
\leq \sup_{x>0} \frac{x^2}{1 + x^2} \left| \frac{[n+1]_{q_n} - q [n]_{q_n}}{q_n [n]_{q_n}} \right| = \frac{1}{q_n [n]_{q_n}},
\]
which implies that $\lim_{n \to \infty} \|P_{n,q_n} (e_2) - e_2\|_2 = 0$. Since the conditions of Theorem 3 are satisfied, then we obtain for any $f \in C_{2,k}$, $\lim_{n \to \infty} \|P_{n,q_n} (f) - f\|_2 = 0$.

Theorem 5. For $f \in C_{2,k}$, $n \in \mathbb{N}$ we have
\[
\sup_{x>0} \frac{|P_{n,q} (f) (x) - f(x)|}{(1 + x^2)^{\frac{1}{2}}} \leq K_q \Omega_2 \left( f, [n]_{q}^{-1/4} \right),
\]
where
\[
K_q = 16 \left( 1 + \frac{1}{q^6} ([3]_q ([2]_q + q) + q^2 (2 - q[2]_q) + q^3 (1 - q)^2) \right)
\]
is independent of \( f \) and \( n \).

**Proof.** From the properties of \( \Omega_2 \) it is obvious that for any \( \lambda > 0 \), \( \Omega_2 (f, \lambda \delta) \leq 2 (\lambda + 1) (1 + \delta^2) \Omega_2 (f, \delta) \). For \( 0 < \delta \leq 1 \), if we use the definition of \( \Omega_2 \) and the last inequality with \( \lambda = \frac{|t-x|}{\delta} \) we have

\[
|f(t) - f(x)| \leq (1 + x^2)(1 + |t-x|^2)\Omega_2 (f, |t-x|) \\
\leq 2(1 + x^2)(1 + |t-x|^2) \left( 1 + \frac{|t-x|}{\delta} \right) \Omega_2 (f, \delta).
\]

Applying \( P_{n,q} \) to the last inequality we obtain

\[
|P_{n,q}(f)(x) - f(x)| \leq 16 (1 + x^2) \Omega_2 (f, \delta) \left( 1 + \frac{1}{\delta^4} P_{n,q} (\varphi^2) (x) \right).
\]

From (2.2) we can write

\[
\left| [n]_q P_{n,q} (\varphi^4) (x) \right| \leq x^4 \left\{ \frac{1}{q^6[n]_q} \left[ 3[2]_q + q[n]_q \right] + \frac{(2 - q[2]_q)}{q^4[n]_q} + \frac{(1 - q^2)}{q^4} \right\}
\]

\[
\leq x^4 \left\{ \frac{[3]_q [2]_q + q}{q^6} + \frac{(2 - q[2]_q)}{q^4} + \frac{(1 - q^2)}{q^4} \right\},
\]

for all \( x > 0 \) and \( n \in \mathbb{N} \). Taking (3.1) and (3.2) into account and by choosing \( \delta = [n]^{-1/4}_q \) we obtain that

\[
\sup_{x>0} \frac{|P_{n,q_n}(f)(x) - f(x)|}{(1 + x^2)^3} \leq K_q \Omega_2 \left( f, [n]^{-1/4}_q \right),
\]

where \( K_q \) is the given in the hypothesis, which is independent of \( f \) and \( n \). \( \square \)

**Remark 4.** Let \( q \) be fixed. Since \( [n]_q \to \frac{1}{1-q} \) as \( n \to \infty \), we cannot obtain that \( \lim_{n \to \infty} \sup_{x>0} \frac{|P_{n,q_n}(f)(x) - f(x)|}{(1 + x^2)^3} = 0 \) in Theorem 5. But we can improve this, by choosing \( q = q_n \), where \( 0 < q_n < 1 \) with \( q_n \to 1 \) as \( n \to \infty \). Actually, in this case we obtain that \( \lim_{n \to \infty} [n]^{-1/4}_q = \infty \). Therefore we have \( \lim_{n \to \infty} \Omega_2 (f, [n]^{-1/4}_q) = 0 \) and \( \lim_{n \to \infty} K_{q_n} \) is finite, which show that we obtain a weighted approximation for \( P_{n,q_n}(f)(x) \) to \( f(x) \) as \( n \to \infty \) in Theorem 5. This theorem tells us that depending on the selection of \( q_n \), the rate of convergence of \( P_{n,q_n}(f)(x) \) to \( f(x) \) as weight is \( [n]^{-1/4}_q \) that is at least so faster than \( n^{-1/4} \) which is the rate of convergence for the classical Post-Widder operators. On the other hand, the saturation rate of the Post-Widder operator in \( L_p \)-norm \( 1 \leq p \leq \infty \), with power-type weight is \( \frac{1}{n} \) (see chapter 10 of [9] or [11]).
REFERENCES


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