CLASSICAL TAUBERIAN THEOREMS FOR THE \((C, 1, 1)\) SUMMABILITY METHOD

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Abstract. In this paper we generalize some classical Tauberian theorems for single sequences to double sequences. One-sided Tauberian theorem and generalized Littlewood theorem for \((C, 1, 1)\) summability method are given as corollaries of the main results.

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1. Introduction and preliminaries

A double sequence \(u = (u_{mn})\) is called Pringsheim convergent (or \(P\)-convergent), see [13], if for a given \(\varepsilon > 0\) there exists a positive integer \(N_0\) such that \(|u_{mn} - L| < \varepsilon\) for all nonnegative integer \(m, n \geq N_0\). The \((C, 1, 1)\) means of \((u_{mn})\) are defined by

\[
\sigma_{mn}^{(11)}(u) = \frac{1}{(m + 1)(n + 1)} \sum_{i=0}^{m} \sum_{j=0}^{n} u_{ij}
\]

for nonnegative integers \(m, n\) (see [12]). The sequence \((u_{mn})\) is said to be \((C, 1, 1)\) summable to a finite number \(s\) if

\[
\lim_{m,n \to \infty} \sigma_{mn}^{(11)}(u) = s.
\]

The \((C, 1, 0)\) and \((C, 0, 1)\) means of \((u_{mn})\) are defined respectively by

\[
\sigma_{mn}^{(10)}(u) = \frac{1}{m + 1} \sum_{i=0}^{m} u_{in} \quad \text{and} \quad \sigma_{mn}^{(01)}(u) = \frac{1}{n + 1} \sum_{j=0}^{n} u_{mj},
\]
for nonnegative integers \(m, n\). The sequence \((u_{mn})\) is said to be \((C, 1, 0)\) summable to a finite number \(s\) if \(\lim_{m \to \infty} \sigma_{mn}^{(10)}(u) = s\). In the light of above discussion, the \((C, 0, 1)\) summability is defined analogously.

A double sequence \((u_{mn})\) is bounded if there exists a real number \(C > 0\) such that \(|u_{mn}| \leq C\) for all nonnegative \(m, n\). Moreover, a double sequence \((u_{mn})\) is said to be one-sided bounded if there exists a real number \(C > 0\) such that \(u_{mn} \leq C\) for all nonnegative integer \(m, n\).

For a double sequence \((u_{mn})\), we define 
\[
\Delta_{m} u_{mn} = u_{mn} - u_{m-1,n}, \quad \Delta_{n} u_{mn} = u_{m,n} - u_{m,n-1},
\]
and \(\Delta_{m,n} u_{mn} = \Delta_{m}(\Delta_{n} u_{mn}) = \Delta_{n}(\Delta_{m} u_{mn})\) for integers \(m, n \geq 1\).

The Kronecker identity for single sequences takes the following form for double sequences. For nonnegative integers \(m, n\), we have
\[
(u_{mn}) = \mathcal{V}_{mn}^{(10)}(\Delta u),
\]
where \(\mathcal{V}_{mn}^{(10)}(\Delta u) = \sum_{i=1}^{m} \sum_{j=1}^{n} ij \Delta_{i,j} u_{ij}\) (see [9]). The sequence \(\mathcal{V}_{mn}^{(10)}(\Delta u)\) is the \((C, 1, 1)\) means of the sequence \((mn \Delta_{m,n} u_{mn})\).

Moreover, in analogy to Kronecker identity for a single sequence, we have
\[
(u_{mn}) = \mathcal{V}_{mn}^{(01)}(\Delta u),
\]
where \(\mathcal{V}_{mn}^{(01)}(\Delta u) = \sum_{i=1}^{m} \sum_{j=1}^{n} i \Delta_{i,j} u_{ij}\). The sequences \(\mathcal{V}_{mn}^{(10)}(\Delta u)\) and \(\mathcal{V}_{mn}^{(01)}(\Delta u)\) are the \((C, 1, 0)\) means of the sequence \((m \Delta_{m} u_{mn})\), and the \((C, 0, 1)\) means of the sequence \((n \Delta_{n} u_{mn})\), respectively.

We define de la Vallée Poussin means of the double sequence \((u_{mn})\) as follows: if \(\lambda > 1\)
\[
\tau_{mn}^{>}(u) = \frac{1}{([\lambda m] - m)([\lambda n] - n)} \sum_{m+1}^{[\lambda m]} \sum_{n+1}^{[\lambda n]} u_{jk},
\]
and if \(0 < \lambda < 1\)
\[
\tau_{mn}^{<}(u) = \frac{1}{(m - [\lambda m])(n - [\lambda n])} \sum_{j=[\lambda m]+1}^{m} \sum_{k=[\lambda n]+1}^{n} u_{jk},
\]
for sufficiently large nonnegative integers \(m, n\).

The symbols \(u_{mn} = o(1)\) and \(u_{n} = o(1)\) represent that \((u_{mn})\) is \(P\)-convergent to zero as \(m, n \to \infty\) and \((u_{n})\) is convergent to zero as \(n \to \infty\), respectively.
If a double sequence is $P$-convergent to $s$, then it is $(C, 1, 1)$ summable to $s$ provided that it is bounded. However the converse is not necessarily true. Namely, the double sequence which is bounded and $(C, 1, 1)$ summable may not be $P$-convergent. In the following example we give an example of a double sequence which is $(C, 1, 1)$ summable and bounded, but not $P$-convergent.

**Example 1.** The sequence $(u_{mn}) = \left( \sum_{i=0}^{m} \sum_{j=0}^{n} (-1)^{i+j} \right)$ is not $P$-convergent. But it is $(C, 1, 1)$ summable to $\frac{1}{4}$. Indeed, from the definition of $(C, 1, 1)$ means, we have

$$
\sigma_{mn}^{(11)}(u) = \begin{cases} 
\frac{1}{4}, & \text{if } m, n \text{ are odd,} \\
\frac{m+2}{4m+4}, & \text{if } m \text{ is even, } n \text{ is odd,} \\
\frac{4n+4}{n+2}, & \text{if } m \text{ is odd, } n \text{ is even,} \\
\frac{(m+2)(n+2)}{(2m+2)(2n+2)}, & \text{if } m, n \text{ are even.}
\end{cases}
$$

Hence, we get $\lim_{m,n \to \infty} \sigma_{mn}^{(11)}(u) = \frac{1}{4}$.

We can recover $P$-convergence of a double sequence from its $(C, 1, 1)$ summability under some suitable conditions. Such a condition is called a Tauberian condition and the resulting theorem is called a Tauberian theorem.

Now, let us give some well known classical type Tauberian theorems, which are called the Hardy-Landau theorem and the generalized Littlewood theorem for the $(C, 1)$ summability method of a single sequence, respectively.

**Theorem 2** ([7, 11, 8]). Let the single sequence $(u_n)$ be $(C, 1)$ summable to $s$. If $n \Delta_n u_n \geq -C$ for some $C \geq 0$, then $(u_n)$ is convergent to $s$.

Remember that a single sequence $(u_n)$ is said to be slowly oscillating if

$$\lim_{\lambda \to 1^+} \limsup_{n \to \infty} \max_{n+1 \leq j \leq |\lambda n|} \left| \sum_{k=n+1}^{j} \Delta_k u_k \right| = 0.$$

**Theorem 3** ([8, 14]). Let the single sequence $(u_n)$ be $(C, 1)$ summable to $s$. If $(u_n)$ is slowly oscillating, then $(u_n)$ is convergent to $s$. 
In the literature, several Tauberian theorems have been deeply investigated by ÇANAK [5], ÇANAK and TOTUR [4], ESTRADA and VINDAS [6], Szász [15] for the \((C, 1)\) summability method of single sequences, and studied by MÓRICZ [12], KNOPP [9] for the \((C, 1, 1)\) summability method of double sequences. Moreover, VLADIMIROV, DROZHJINOV and ZAVIALOV [17] have worked intensely on multidimensional Tauberian theorems for the Laplace transform including as particular cases those of multi-sequences.

We should mention the following novelties of the present paper. Certain conditions on the double sequence \((u_{mn})\) or the sequence \((V_{mn}(\Delta u))\) related to \((u_{mn})\) are sufficient conditions for \((C, 1, 1)\) summable to be \(P\)-convergent. Furthermore, we prove some classical type Tauberian theorems for the \((C, 1, 1)\) summability method.

2. Lemmas

In this part of the paper, we state the following assertions which will be used in the proof of our main theorems. The version for single sequences of the next lemma is due to ÇANAK [5].

**Lemma 4.** If \(u=(u_{mn})\) is \((C, 1, 1)\) summable to \(s\), then \(\lim_{m,n\to\infty} \tau_{mn}^>(u) = s\) and \(\lim_{m,n\to\infty} \tau_{mn}^<(u) = s\).

**Proof.** For \(\lambda > 1\), by the definition of de la Vallée Poussin means of \((u_{mn})\), we have that

\[
\tau_{mn}^>(u) = \frac{1}{([\lambda n] - n)} \sum_{j=0}^{[\lambda n]} \sum_{k=0}^{[\lambda n]} u_{jk} = \frac{1}{([\lambda n] - n)} \left( \sum_{j=0}^{[\lambda n]} \sum_{k=0}^{m} u_{jk} - \sum_{j=0}^{m} \sum_{k=0}^{n} u_{jk} + \sum_{j=0}^{m} \sum_{k=0}^{n} u_{jk} \right) = \frac{1}{([\lambda m] - m)([\lambda n] - n)}. 
\]
\begin{align*}
\cdot ([\lambda m] + 1)([\lambda n] + 1)\sigma^\text{(11)}_{\lambda m,[\lambda n]}(u) - ([\lambda m] + 1)(n + 1)\sigma^\text{(11)}_{\lambda m,n}(u) \\
- (m + 1)([\lambda n] + 1)\sigma^\text{(11)}_{m,[\lambda n]}(u) + (m + 1)(n + 1)\sigma^\text{(11)}_{mn}(u),
\end{align*}

and

\begin{align*}
\tau^\text{>}_{mn}(u) &= \frac{([\lambda m] + 1)([\lambda n] + 1)}{([\lambda m] - m)([\lambda n] - n)}\sigma^\text{(11)}_{\lambda m,[\lambda n]}(u) \\
&\quad - \left[ \frac{([\lambda m] + 1)([\lambda n] + 1)}{([\lambda m] - m)([\lambda n] - n)}\sigma^\text{(11)}_{\lambda m,n}(u) - \frac{[\lambda m] + 1}{[\lambda m] - m}\sigma^\text{(11)}_{\lambda m,n}(u) \right] \\
&\quad - \left[ \frac{([\lambda m] + 1)([\lambda n] + 1)}{([\lambda m] - m)([\lambda n] - n)}\sigma^\text{(11)}_{n,\lambda n}(u) - \frac{[\lambda n] + 1}{[\lambda n] - n}\sigma^\text{(11)}_{m,\lambda n}(u) \right] \\
&\quad + \left[ \frac{([\lambda m] + 1)([\lambda n] + 1)}{([\lambda m] - m)([\lambda n] - n)}\sigma^\text{(11)}_{m,n}(u) - \frac{[\lambda m] + 1}{[\lambda m] - m}\sigma^\text{(11)}_{m,n}(u) \right] \\
&\quad - \frac{[\lambda n] + 1}{[\lambda n] - n}\sigma^\text{(11)}_{m,n}(u) + \sigma^\text{(11)}_{mn}(u).
\end{align*}

The difference \(\tau^\text{>}_{mn}(u) - \sigma^\text{(11)}_{mn}(u)\) can be written as

\begin{equation}
\tau^\text{>}_{mn}(u) - \sigma^\text{(11)}_{mn}(u) = \frac{([\lambda m] + 1)([\lambda n] + 1)}{([\lambda m] - m)([\lambda n] - n)} \sigma^\text{(11)}_{\lambda m,[\lambda n]}(u) - \sigma^\text{(11)}_{\lambda m,n}(u) + \sigma^\text{(11)}_{m,\lambda n}(u)
\end{equation}

Taking the limit of both sides of (1) as \(m, n \to \infty\), we obtain

\begin{align*}
\lim_{m,n \to \infty} (\tau^\text{>}_{mn}(u) - \sigma^\text{(11)}_{mn}(u)) &= \frac{\lambda^2}{(\lambda - 1)^2} \left( \lim_{m,n \to \infty} \sigma^\text{(11)}_{\lambda m,[\lambda n]}(u) - \lim_{m,n \to \infty} \sigma^\text{(11)}_{\lambda m,n}(u) \right) \\
&\quad - \lim_{m,n \to \infty} \sigma^\text{(11)}_{m,[\lambda n]}(u) + \lim_{m,n \to \infty} \sigma^\text{(11)}_{mn}(u)) \\
&\quad + \frac{\lambda}{\lambda - 1} \left( \lim_{m,n \to \infty} \sigma^\text{(11)}_{\lambda m,n}(u) - \lim_{m,n \to \infty} \sigma^\text{(11)}_{m,n}(u) \right) \\
&\quad + \frac{\lambda}{\lambda - 1} \left( \lim_{m,n \to \infty} \sigma^\text{(11)}_{m,[\lambda n]}(u) - \lim_{m,n \to \infty} \sigma^\text{(11)}_{m,n}(u) \right) \\
&= \frac{\lambda^2}{(\lambda - 1)^2} \lim_{m,n \to \infty} \sigma^\text{(11)}_{\lambda m,[\lambda n]}(u) - \left( \frac{\lambda^2}{(\lambda - 1)^2} - \frac{\lambda}{\lambda - 1} \right) \lim_{m,n \to \infty} \sigma^\text{(11)}_{\lambda m,n}(u).
\end{align*}
\[-\left(\frac{\lambda^2}{(\lambda - 1)^2} - \frac{\lambda}{\lambda - 1}\right)\sigma_{m,\lfloor \lambda n \rfloor}^{(1)}(u) + \left(\frac{\lambda^2}{(\lambda - 1)^2} - \frac{2\lambda}{\lambda - 1}\right)\sigma_{m,n}^{(1)}(u) = 0.\]

The proof for the second part of this assertion is similar to that of the first part. Therefore, we omit it here. \(\square\)

**Lemma 5.** Let \((u_{mn})\) be a double sequence of real numbers. For sufficiently large \(m, n\):

(i) If \(\lambda > 1\)

\[u_{mn} - \sigma_{mn}^{(1)}(u) = \frac{[\lfloor \lambda m \rfloor + 1][\lfloor \lambda n \rfloor + 1]}{([\lfloor \lambda m \rfloor] - m)([\lfloor \lambda n \rfloor] - n)} \cdot (\sigma_{\lfloor \lambda m \rfloor,\lfloor \lambda n \rfloor}^{(1)}(u) - \sigma_{\lfloor \lambda m \rfloor,n}^{(1)}(u) - \sigma_{m,\lfloor \lambda n \rfloor}^{(1)}(u) + \sigma_{m,n}^{(1)}(u))\]

\[+ \frac{[\lfloor \lambda m \rfloor] + 1}{[\lfloor \lambda m \rfloor] - m} \cdot \left(\sigma_{\lfloor \lambda m \rfloor,n}^{(1)}(u) - \sigma_{m,n}^{(1)}(u)\right) + \frac{[\lfloor \lambda n \rfloor] + 1}{[\lfloor \lambda n \rfloor] - n} \cdot (\sigma_{m,\lfloor \lambda n \rfloor}^{(1)}(u) - \sigma_{m,n}^{(1)}(u))\]

\[+ \frac{1}{([\lfloor \lambda m \rfloor] - m)([\lfloor \lambda n \rfloor] - n)} \sum_{j=m+1}^{\lfloor \lambda m \rfloor} \sum_{k=n+1}^{\lfloor \lambda n \rfloor} (u_{jk} - u_{mn}),\]

(ii) if \(0 < \lambda < 1\)

\[u_{mn} - \sigma_{mn}^{(1)}(u) = \frac{[\lfloor \lambda m \rfloor + 1][\lfloor \lambda n \rfloor + 1]}{(m - [\lambda m])(n - [\lambda n])} \cdot (\sigma_{m,\lceil \lambda m \rceil}^{(1)}(u) - \sigma_{\lceil \lambda m \rceil,n}^{(1)}(u) - \sigma_{m,\lceil \lambda n \rceil}^{(1)}(u) + \sigma_{m,n}^{(1)}(u))\]

\[+ \frac{[\lambda m] + 1}{m - [\lambda m]} \cdot \left(\sigma_{m,\lceil \lambda m \rceil}^{(1)}(u) - \sigma_{\lceil \lambda m \rceil,n}^{(1)}(u)\right) + \frac{[\lambda n] + 1}{n - [\lambda n]} \cdot (\sigma_{m,\lceil \lambda n \rceil}^{(1)}(u) - \sigma_{m,n}^{(1)}(u))\]

\[+ \frac{1}{(m - [\lambda m])(n - [\lambda n])} \sum_{j=[\lambda m]+1}^{m} \sum_{k=[\lambda n]+1}^{n} (u_{mn} - u_{jk}),\]

where \([\lambda n]\) and \([\lambda m]\) denotes the integer part of \(\lambda n\) and \(\lambda m\), respectively.

**Proof.** For \(\lambda > 1\), the claimed formula from (i) follows from the equation \(u_{mn} - \sigma_{mn}^{(1)}(u) = \tau_{mn}^{>} - \sigma_{m,n}^{(1)}(u) - (\tau_{mn}^{>} - u_{mn})\) and (1).

The formula (ii) can be verified in a similar way. \(\square\)

**Remark 6.** The version of Lemma 5 for a single sequence has been frequently used in several papers to prove some Tauberian theorems [1, 2, 3, 16, 10].

**Remark 7.** A different proof of Lemma 5 is given by MÓRICZ [12].
3. One-sided Tauberian theorem for double sequences

In this section, we extend Landau’s Tauberian theorem given for single sequence to double sequence. Moreover, we establish one-sided Tauberian theorem under some conditions on the sequence \((V_{mn}(\Delta u))\) for the \((C, 1)\) summability method.

**Theorem 8.** Let the double sequence \((u_{mn})\) be bounded. If \((u_{mn})\) is \((C, 1, 1), (C, 1, 0)\) and \((C, 0, 1)\) summable to \(s\), and the conditions

\[
\begin{align*}
(2) & \quad mn\Delta_{m,n}V_{mn}^{(1)}(\Delta u) \geq -C, \quad m, n \to \infty, \\
(3) & \quad m\Delta_{m}V_{mn}^{(1)}(\Delta u) \geq -C, \quad m \to \infty, \\
(4) & \quad n\Delta_{n}V_{mn}^{(1)}(\Delta u) \geq -C, \quad n \to \infty,
\end{align*}
\]

are satisfied for some \(C \geq 0\), then \((u_{mn})\) is \(P\)-convergent to \(s\).

**Proof.** Since \((u_{mn})\) is bounded and \((C, 1, 1)\) summable to \(s\), \((\sigma_{mn}^{(1)}(u))\) is \(P\)-convergent to \(s\). We know that the \((C, 1, 1)\) summability method is regular, so \((\sigma_{mn}^{(1)}(u))\) is \((C, 1, 1)\) summable to \(s\). One can see that since \((u_{mn})\) is \((C, 1, 0)\) summable to \(s\), then we get \((\sigma_{mn}^{(1)}(u))\) is \(P\)-convergent to \(s\). Consequently, \((\sigma_{mn}^{(1)}(u))\) is \((C, 1, 1)\) summable to \(s\). Analogously, \((\sigma_{mn}^{(0)}(u))\) is \((C, 1, 1)\) summable to \(s\). It follows from the Kronecker identity that \((V_{mn}^{(1)}(\Delta u))\) is \((C, 1, 1)\) summable to 0. For \(\lambda > 1\), using the identity (1), if we replace \(u_{mn}\) by \(V_{mn}^{(1)}(\Delta u)\) in Lemma 5, we have

\[
V_{mn}^{(1)}(\Delta u) - \sigma_{mn}^{(1)}(V_{mn}^{(1)}(\Delta u)) = (\tau_{mn}^{>}(V_{mn}^{(1)}(\Delta u)) - \sigma_{mn}^{(1)}(V_{mn}^{(1)}(\Delta u)))
\]

\[
- \frac{1}{(|\lambda m| - m)(|\lambda n| - n)} \sum_{j=m+1}^{[\lambda m]} \sum_{k=n+1}^{[\lambda n]} (V_{jk}^{(1)}(\Delta u) - V_{mn}^{(1)}(\Delta u)).
\]

Taking \(\lim \sup\) of both sides of the previous equation as \(m, n \to \infty\), we get

\[
\lim \sup_{m,n \to \infty} (V_{mn}^{(1)}(\Delta u) - \sigma_{mn}^{(1)}(V_{mn}^{(1)}(\Delta u)))
\]

\[
\leq \lim \sup_{m,n \to \infty} (\tau_{mn}^{>}(V_{mn}^{(1)}(\Delta u)) - \sigma_{mn}^{(1)}(V_{mn}^{(1)}(\Delta u)))
\]

\[
+ \lim \sup_{m,n \to \infty} \left( \frac{1}{(|\lambda m| - m)(|\lambda n| - n)} \sum_{j=m+1}^{[\lambda m]} \sum_{k=n+1}^{[\lambda n]} (V_{jk}^{(1)}(\Delta u) - V_{mn}^{(1)}(\Delta u)) \right).
\]
Since the first term on the right-hand side of the last inequality vanishes by Lemma 4, we have

\[
\limsup_{m,n \to \infty} (V_{mn}^{(11)}(\Delta u) - \sigma_{mn}^{(11)}(V^{(11)}(\Delta u))) \\
\leq \limsup_{m,n \to \infty} \left( - \frac{1}{(\lfloor \lambda m \rfloor - m)(\lfloor \lambda n \rfloor - n)} \sum_{j=m+1}^{\lfloor \lambda m \rfloor} \sum_{k=n+1}^{\lfloor \lambda n \rfloor} \right) \cdot \left( \sum_{r=m+1}^{j} \Delta_r V_{rk}^{(11)}(\Delta u) + \sum_{s=n+1}^{k} \Delta_s V_{js}^{(11)}(\Delta u) \\
- \sum_{r=m+1}^{j} \sum_{s=n+1}^{k} \Delta_{rs} V_{rs}^{(11)}(\Delta u) \right).
\]

From the conditions (2), (3), and (4), we obtain

\[
\limsup_{m,n \to \infty} (V_{mn}^{(11)}(\Delta u) - \sigma_{mn}^{(11)}(V^{(11)}(\Delta u))) \\
\leq \limsup_{m,n \to \infty} \left( - \frac{1}{(\lfloor \lambda m \rfloor - m)(\lfloor \lambda n \rfloor - n)} \sum_{j=m+1}^{\lfloor \lambda m \rfloor} \sum_{k=n+1}^{\lfloor \lambda n \rfloor} \right) \cdot \left( \sum_{r=m+1}^{j} \frac{C}{r} \sum_{s=n+1}^{k} - \frac{C}{s} - \sum_{r=m+1}^{j} \sum_{s=n+1}^{k} \frac{C^2}{rs} \right) \\
\leq \limsup_{m,n \to \infty} \left( C_1 \log \left( \frac{\lfloor \lambda m \rfloor}{m} \right) + C_2 \log \left( \frac{\lfloor \lambda n \rfloor}{n} \right) + C_3 \log \left( \frac{\lfloor \lambda m \rfloor}{m} \right) \left( \frac{\lfloor \lambda n \rfloor}{n} \right) \right),
\]
for some \( C_1, C_2, C_3 \geq 0 \). Hence we get

\[
\limsup_{m,n \to \infty} (V_{mn}^{(11)}(\Delta u) - \sigma_{mn}^{(11)}(V^{(11)}(\Delta u))) \leq C_4 \log \lambda,
\]
for some \( C_4 \geq 0 \). Taking the limit of both sides as \( \lambda \to 1^+ \), we have

\[
(5) \quad \limsup_{m,n \to \infty} (V_{mn}^{(11)}(\Delta u) - \sigma_{mn}^{(11)}(V^{(11)}(\Delta u))) \leq 0.
\]

For \( 0 < \lambda < 1 \), in a similar way from Lemma 5 (ii) we have

\[
(6) \quad \liminf_{m,n \to \infty} (V_{mn}^{(11)}(\Delta u) - \sigma_{mn}^{(11)}(V^{(11)}(\Delta u))) \geq 0.
\]
By the inequalities (5) and (6), we obtain \( \lim_{m,n \to \infty} V^{(11)}_{mn}(\Delta u) = 0 \). Thus, from the Kronecker identity, we obtain that \((u_{mn})\) is \( P \)-convergent to \( s \). □

The following assertion is an extended version of the Hardy-Landau theorem for the \((C,1)\) summability method. We note that in Theorem 9 the sequence \((u_{mn})\) is not necessarily \((C,1,0)\) and \((C,0,1)\) summable.

**Corollary 9.** If \((u_{mn})\) is \((C,1,1)\) summable to \( s \), the conditions

\[
\begin{align*}
(7) & \quad mn\Delta_{m,n}u_{mn} \geq -C, \quad m, n \to \infty, \\
(8) & \quad m\Delta_{m}u_{mn} \geq -C, \quad m \to \infty, \\
(9) & \quad n\Delta_{n}u_{mn} \geq -C, \quad n \to \infty,
\end{align*}
\]

are satisfied for some \( C \geq 0 \), then \((u_{mn})\) is \( P \)-convergent to \( s \).

**Proof.** Since \((u_{mn})\) is \((C,1,1)\) summable to \( s \), \((\sigma^{(11)}_{mn}(u))\) is \( P \)-convergent to \( s \). If we apply a similar calculation for \((V^{(11)}_{mn}(\Delta u))\) as in the proof of Theorem 8 to the sequence \((u_{mn})\), one can easily obtain that \((u_{mn})\) is \( P \)-convergent to \( s \). □

If the Kronecker identity for double sequence is rewritten, then we obtain

\[
(u_{mn} - \sigma^{(10)}_{mn}(u)) + (u_{mn} - \sigma^{(11)}_{mn}(u)) - (u_{mn} - \sigma^{(11)}_{mn}(u)) = V^{(11)}_{mn}(\Delta u).
\]

It follows that \( V^{(10)}_{mn}(\Delta u) + V^{(01)}_{mn}(\Delta u) - V^{(11)}_{mn}(\Delta u) = u_{mn} - \sigma^{(11)}_{mn}(u) \). Thus, for only the \((C,1,1)\) summability of \((u_{mn})\), Theorem 8 can become the following theorem by using Theorem 9.

**Theorem 10.** Let the double sequence \((u_{mn})\) be bounded. If \((u_{mn})\) is \((C,1,1)\) summable to \( s \), and the conditions

\[
\begin{align*}
(10) & \quad mn\Delta_{m,n}V^{(11)}_{mn}(\Delta u) \geq -C, \quad m, n \to \infty, \\
(11) & \quad m\Delta_{m}V^{(11)}_{mn}(\Delta u) \geq -C, \quad m \to \infty, \\
(12) & \quad n\Delta_{n}V^{(11)}_{mn}(\Delta u) \geq -C, \quad n \to \infty
\end{align*}
\]

are satisfied for some \( C \geq 0 \), then \((u_{mn})\) is \( P \)-convergent to \( s \).

The proof of Theorem 10 is easily obtained by making use of Theorem 9. Indeed, since the sequence \((u_{mn})\) is bounded, the \((C,1,1)\) summability of \((u_{mn})\) to \( s \) implies that \((V^{(01)}_{mn}(\Delta u))\) and \((V^{(10)}_{mn}(\Delta u))\) are \((C,1,1)\) summable to 0.
4. Generalized Littlewood theorem for double sequences

In this section, we obtain some new Tauberian theorems under some conditions on the oscillatory behavior of a double sequence \((u_{mn})\) or its relation with \((V_{mn}^{(11)}(\Delta u))\).

Now, let us give the following definition of slowly oscillation for a double sequence.

**Definition 11.** A double sequence \((u_{mn})\) is said to be slowly oscillating in sense \((1,1)\) if

\[
\lim_{\lambda \to 1^+} \lim_{m,n \to \infty} \max_{m+1 \leq j \leq \lfloor \lambda m \rfloor} \left| \sum_{r=m+1}^{j} \sum_{s=n+1}^{k} \Delta_{rs} u_{rs} \right| = 0,
\]

\((u_{mn})\) is said to be slowly oscillating in sense \((1,0)\) if

\[
\lim_{\lambda \to 1^+} \lim_{m \to \infty} \max_{m+1 \leq j \leq \lfloor \lambda m \rfloor} \left| \sum_{r=m+1}^{j} \Delta_{r} u_{rn} \right| = 0,
\]

\((u_{mn})\) is said to be slowly oscillating in sense \((0,1)\) if

\[
\lim_{\lambda \to 1^+} \lim_{n \to \infty} \max_{n+1 \leq j \leq \lfloor \lambda n \rfloor} \left| \sum_{s=n+1}^{k} \Delta_{s} u_{ms} \right| = 0.
\]

**Theorem 12.** Let the double sequence \((u_{mn})\) be bounded. If \((u_{mn})\) is \((C,1,1)\), \((C,1,0)\) and \((C,0,1)\) summable to \(s\), and \((V_{mn}^{(11)}(\Delta u))\) is slowly oscillating in sense \((1,1)\), \((1,0)\), and \((0,1)\) then \((u_{mn})\) is \(P\)-convergent to \(s\).

**Proof.** Let us prove the \((C,1,1)\) summation of \((V_{mn}^{(11)}(\Delta u))\) to 0. It can be verified in exactly the same way as Theorem 8 since \((u_{mn})\) is bounded and \((C,1,1)\) summable to \(s\).

If we replace \(u_{mn}\) by \(V_{mn}^{(11)}(\Delta u)\) in Lemma 5, we obtain

\[
V_{mn}^{(11)}(\Delta u) - \sigma_{mn}^{(11)}(V^{(11)}(\Delta u)) = (\tau_{mn}^{(11)}(V^{(11)}(\Delta u)) - \sigma_{mn}^{(11)}(V^{(11)}(\Delta u)))
\]

\[
- \frac{1}{(\lfloor \lambda m \rfloor - m)(\lfloor \lambda n \rfloor - n)} \sum_{j=m+1}^{\lfloor \lambda m \rfloor} \sum_{k=n+1}^{\lfloor \lambda n \rfloor} (V_{jk}^{(11)}(\Delta u) - V_{mn}^{(11)}(\Delta u)),
\]

(10)
From the identity (10), we get
\[ |V_{mn}^{(11)}(\Delta u) - \sigma_{mn}^{(11)}(V^{(11)}(\Delta u))| \leq |\tau_{mn}^{(11)}(V^{(11)}(\Delta u)) + \sigma_{mn}^{(11)}(V^{(11)}(\Delta u))| \]
(11) \[ + \frac{1}{([\lambda m] - m)([\lambda n] - n)} \sum_{j=m+1}^{[\lambda m]} \sum_{k=n+1}^{[\lambda n]} (V_{jk}^{(11)}(\Delta u) - V_{mn}^{(11)}(\Delta u)). \]

From the second term on the right-hand side of the inequality (11), we have
\[ - \frac{1}{([\lambda m] - m)([\lambda n] - n)} \sum_{j=m+1}^{[\lambda m]} \sum_{k=n+1}^{[\lambda n]} (V_{jk}^{(11)}(\Delta u) - V_{mn}^{(11)}(\Delta u)) \]
\[ \leq \frac{1}{([\lambda m] - m)([\lambda n] - n)} \sum_{j=m+1}^{[\lambda m]} \sum_{k=n+1}^{[\lambda n]} \left( \sum_{r=m+1}^{j} \sum_{s=n+1}^{k} \Delta_{r,s} V_{rs}^{(11)}(\Delta u) \right) \]
\[ + \left| \sum_{r=m+1}^{j} \Delta_{r} V_{rn}^{(11)}(\Delta u) \right| + \left| \sum_{s=n+1}^{k} \Delta_{s} V_{ms}^{(11)}(\Delta u) \right|, \]
and then
\[ - \frac{1}{([\lambda m] - m)([\lambda n] - n)} \sum_{j=m+1}^{[\lambda m]} \sum_{k=n+1}^{[\lambda n]} (V_{jk}^{(11)}(\Delta u) - V_{mn}^{(11)}(\Delta u)) \]
\[ \leq \max_{m+1 \leq j \leq [\lambda m]} \left( \sum_{r=m+1}^{j} \sum_{s=n+1}^{k} \Delta_{r,s} V_{rs}^{(11)}(\Delta u) \right) + \max_{m+1 \leq j \leq [\lambda m]} \left| \sum_{r=m+1}^{j} \Delta_{r} V_{rn}^{(11)}(\Delta u) \right| \]
\[ + \max_{n+1 \leq s \leq [\lambda n]} \left| \sum_{r=m+1}^{k} \Delta_{s} V_{ms}^{(11)}(\Delta u) \right|. \]

Taking lim sup of both sides of the inequality (10) as \( m, n \to \infty \), the first term on the right-hand side of the inequality (10) vanishes by Lemma 4. We then have
\[ \limsup_{m,n \to \infty} |V_{mn}^{(11)}(\Delta u) - \sigma_{mn}^{(11)}(V^{(11)}(\Delta u))| \]
\[ \leq \limsup_{m,n \to \infty} \max_{m+1 \leq j \leq [\lambda m]} \left( \sum_{r=m+1}^{j} \sum_{s=n+1}^{k} \Delta_{r,s} V_{rs}^{(11)}(\Delta u) \right) \]
\[
+ \limsup_{m,n \to \infty} \max_{m+1 \leq j \leq \lfloor km \rfloor} \left| \sum_{r=m+1}^{j} \Delta_r V^{(11)}_{rn}(\Delta u) \right|
\]
\[
+ \limsup_{m,n \to \infty} \max_{n+1 \leq j \leq \lfloor kn \rfloor} \left| \sum_{s=n+1}^{k} \Delta_s V^{(11)}_{ms}(\Delta u) \right|.
\]
Since \((u_{mn})\) is slowly oscillating in sense \((1, 1), (1, 0), \) and \((0, 1)\),
\[
\limsup_{m,n \to \infty} |V^{(11)}_{mn}(\Delta u) - \sigma^{(11)}_{mn}(V^{(11)}(\Delta u))| \leq 0.
\]
The Kronecker identity completes the proof. \(\square\)

The following assertion is an extended version of the generalized Littlewood theorem for the \((C, 1)\) summability method given by Hardy [7, Theorem 68]. It deserves to be mentioned that in Theorem 13 the sequence \((u_{mn})\) is not necessarily \((C, 1, 0)\) nor \((C, 0, 1)\) summable.

**Theorem 13.** If \((u_{mn})\) is \((C, 1, 1)\) summable to \(s\), and \((u_{mn})\) is slowly oscillating in sense \((1, 1), (1, 0), \) and \((0, 1)\), then \((u_{mn})\) is \(P\)-convergent to \(s\).

**Proof.** It is clear that \((\sigma^{(11)}_{mn}(u))\) is \(P\)-convergent to \(s\) since \((u_{mn})\) is \((C, 1, 1)\) summable to \(s\). If we apply similar calculation for \((V^{(11)}_{mn}(\Delta u))\) in the proof of Theorem 12 to the sequence \((u_{mn})\), one can obtain that \((u_{mn})\) is \(P\)-convergent to \(s\). \(\square\)

Analogously to the case of Theorem 10, Theorem 12 can be reformulated as follows with the aid of Theorem 13.

**Theorem 14.** Let the double sequence \((u_{mn})\) be bounded. If \((u_{mn})\) is summable to \(s\), and

\((V^{(11)}_{mn}(\Delta u))\) is slowly oscillating in sense \((1, 1), (1, 0), \) and \((0, 1)\),
\((V^{(10)}_{mn}(\Delta u))\) is slowly oscillating in sense \((1, 1), (1, 0), \) and \((0, 1)\),
\((V^{(01)}_{mn}(\Delta u))\) is slowly oscillating in sense \((1, 1), (1, 0), \) and \((0, 1)\),

then \((u_{mn})\) is \(P\)-convergent to \(s\).

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