ON SOME NONLINEAR INTEGRAL INEQUALITIES FOR VOLterra INTEGRAL EQUATIONS

BY

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Abstract. In this paper, we have stated and proved some results on nonlinear integral inequalities and its applications which provide an explicit bound on unknown function and can be used as a tool in the study of certain nonlinear retarded Volterra integral equations.

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1. Introduction

Integral inequalities involving functions of one independent variable, which provide explicit bounds on unknown functions play an important role in the development of the theory of linear and nonlinear differential and integral equations. In recent years nonlinear integral inequalities have received considerable attention because of the important applications to a variety of problems in diverse fields of nonlinear differential and integral equations. Basic inequalities are established by Gronwall [5], Bellman [3] and Pachpatte [8] provide explicit bounds on solutions of a class of differential and integral equations which are further studied by many mathematicians see [1, 2, 4, 7, 9, 10].

In this paper, we extend and improve some of the results reported in [6] to obtain a new generalization for some inequalities, which can be used as handy tools to study the qualitative as well as the quantitative properties of
solutions of some nonlinear integral equations. Some applications are also
given to convey the importance of our results.

Before proceeding to the statement of our main result, we state some
important basic integral inequalities.

**Theorem 1.1 ([5]).** If \( u(t) \) is a continuous function defined on the
interval \( J = [\alpha, \alpha + h] \) and

\[
0 \leq u(t) \leq \int_{\alpha}^{t} [b u(s) + a] ds, \quad t \in J
\]

(1.1) where \( a \) and \( b \) are nonnegative constants, then

\[
0 \leq u(t) \leq ah^b, \quad t \in J.
\]

**Theorem 1.2 ([3]).** Let \( u(t) \) and \( f(t) \) be a nonnegative continuous func-
tions defined on \( J = [\alpha, \alpha + h] \) and \( c \) be nonegative constant. If

\[
u(t) \leq c + \int_{\alpha}^{t} f(s) u(s) ds, \quad t \in J
\]

(1.3) then

\[
u(t) \leq c \exp \left( \int_{\alpha}^{t} f(s) ds \right), \quad t \in J.
\]

**Theorem 1.3 ([1]).** We assume that \( x(t), f(t) \) and \( h(t) \) be nonnegative
real valued continuous functions defined on \( R_+ = [0, \infty) \), and satisfy the
inequality

\[
x^p(t) \leq x_0 + \int_{0}^{t} f(s) x^p(s) ds + \int_{0}^{t} h(s) x^q(s) ds,
\]

(1.5) for all \( t \in R_+ \), where \( p > q \geq 0 \), are constants. Then

\[
x(t) \leq \exp \left( \frac{1}{p} \int_{0}^{t} f(s) ds \right) \cdot
\]

(1.6) \[
x_0^p + p_1 \int_{0}^{t} h(s) \exp \left( -p_1 \int_{0}^{s} f(\lambda) d\lambda \right) ds \left[ \frac{1}{p-1} \right]
\]

for all \( t \in R_+ \), where \( p_1 = \frac{p-q}{p} \).
2. Main results

In this section, we state and prove some new nonlinear integral inequalities of more general type and we obtain different and simple bound on unknown function, which can be used in the analysis of various problems in the theory of nonlinear ordinary differential and integral equations.

**Theorem 2.1.** Let $u$ be a nonnegative real valued continuous function on $R_+ = [0, \infty)$, $\alpha \in C^1(R_+, R_+)$ is nondecreasing and $\alpha(t) \leq t$ for $t \geq 0$, $a \in C(R_+ \times R_+, R_+)$ with $(t, s) \rightarrow \partial_t a(t, s) \in C(R_+ \times R_+, R_+)$ and $k \geq 1$. If

\[ u^p(t) \leq k + \int_0^{\alpha(t)} a(t, s) u(s) ds, \quad \text{for} \quad t \geq 0, \quad (2.1) \]

then

\[ u(t) \leq \left[ k^{\frac{p-1}{p}} + \frac{p-1}{p} \left( \int_0^{\alpha(t)} a(t, s) ds \right)^{\frac{1}{p-1}} \right]^{\frac{1}{p}}, \quad \text{for} \quad t \geq 0, \quad (2.2) \]

where $p > 1$.

**Proof.** Define a function $z(t)$ by

\[ z^p(t) = k + \int_0^{\alpha(t)} a(t, s) u(s) ds, \quad (2.3) \]

then $u(t) \leq z(t)$ and $z^p(0) = k + \int_0^{\alpha(0)} a(t, s) u(s) ds = k$. Differentiating (2.3) and using the fact that $z(t)$ is monotone nondecreasing for $t \in R_+$, we obtain

\[ pz^{p-1}(t) z'(t) \leq \int_0^{\alpha(t)} a(t, s) z(\alpha(t)) ds + \alpha'(t) a(t, \alpha(t)) z(\alpha(t)) \]

\[ \leq \left( \int_0^{\alpha(t)} a_t(t, s) ds + \alpha'(t) a(t, \alpha(t)) \right) z(t). \]

Now, we have

\[ z^{p-2}(t) z'(t) \leq \frac{1}{p} \left( \int_0^{\alpha(t)} a_t(t, s) ds + \alpha'(t) a(t, \alpha(t)) \right) \]

\[ \frac{1}{p - 1} \frac{\partial}{\partial t} \left( z^{p-1}(t) \right) \leq \frac{1}{p - 1} \frac{\partial}{\partial t} \left( \int_0^{\alpha(t)} a(t, s) ds \right). \]
Integrating (2.4) from 0 to $t$, we get

$$z^{p-1}(t) \leq z^{p-1}(0) + \frac{p-1}{p} \left( \int_0^t a(t,s)ds - \int_0^0 a(t,s)ds \right)$$

$$w^{p-1}(t) \leq k^{p-1} + \frac{p-1}{p} \left( \int_0^t a(t,s)ds \right)$$

$$u(t) \leq \left[ k^{p-1} + \frac{p-1}{p} \left( \int_0^t a(t,s)ds \right) \right]^{\frac{1}{p-1}}.$$

Which is a desired bound for $u(t)$ given by (2.2). The proof is complete. □

**Theorem 2.2.** Let $u$ be a nonnegative real valued continuous function on $R_+ = [0, \infty)$, $k \in C^1(R_+, R_+)$ is nondecreasing and $k(t) \geq 1$ for $t \geq 0$, $\alpha \in C^1(R_+, R_+)$ is nondecreasing and $\alpha(t) \leq t$ for $t \geq 0$ and $a \in C(R_+ \times R_+, R_+)$ with $(t,s) \mapsto \partial_t a(t,s) \in C(R_+ \times R_+, R_+)$. If

$$u^p(t) \leq k(t) + \int_0^t a(t,s)u(s)ds, \quad \text{for } t \geq 0,$$

then

$$u(t) \leq \left[ k^{\frac{p-1}{p}} + \frac{p-1}{p} \left( k(t) - k(0) + \int_0^t a(t,s)ds \right) \right]^{\frac{1}{p-1}},$$

for $t \geq 0$, where $p > 1$.

**Proof.** Using the same method, as in the proof of Theorem 2.1 with suitable modifications, we can obtain the explicit bound on $u(t)$ given by (2.6).

**Remark 2.1.** Theorem 2.2 reduces to one of the most useful inequality in the development of theory of differential and integral equations obtained in (see [8], p. 233), if $p = 2, k(t) = c^2$ and $\alpha(t) = t$.

**Theorem 2.3.** Let $u$ be a nonnegative real valued continuous function on $R_+ = [0, \infty)$, $b \in C^1(R_+, R_+), \quad k \in C^1(R_+, R_+)$ is nondecreasing and $k(t) \geq 1$ for $t \geq 0$, $\alpha \in C^1(R_+, R_+)$ is nondecreasing and $\alpha(t) \leq t$ for $t \geq 0$ and $a \in C(R_+ \times R_+, R_+)$ with $(t,s) \mapsto \partial_t a(t,s) \in C(R_+ \times R_+, R_+)$. If

$$w^p(t) \leq k(t) + b(t) \int_0^t a(t,s)u(s)ds, \quad \text{for } t \geq 0,$$
then
\begin{equation}
(2.8) \quad u(t) \leq \left( k \frac{1}{p} (0) + \frac{p-1}{p} \left( k(t) - k(0) + b(t) \int_0^\alpha a(t,s)ds \right) \right)^{\frac{1}{p-1}},
\end{equation}
for \( t \geq 0 \), where \( p > 1 \).

**Proof.** Define a function \( z(t) \) by
\begin{equation}
(2.9) \quad z^p(t) = k(t) + b(t) \int_0^\alpha a(t,s)u(s)ds.
\end{equation}
Differentiating (2.9) and using the properties of \( z(t), k(t), \alpha(t) \) and \( b(t) \) for \( t \in R_+ \), we obtain
\begin{equation}
(2.10) \quad \frac{1}{p-1} \frac{\partial}{\partial t} \left( z^{p-1}(t) \right) \leq \frac{1}{p} \left( k'(t) + b(t) \int_0^\alpha a(t,s)ds \right) + \alpha'(t)b(t)u(t) + b'(t) \int_0^\alpha a(t,s)ds.
\end{equation}
Integrating (2.10) from 0 to \( t \), we get a desired bound for \( u(t) \) given by (2.8). The proof is complete. \( \square \)

**Theorem 2.4.** Let \( u \) be a nonnegative real valued continuous function on \( R_+ = [0, \infty) \), \( k \in C^1(R_+, R_+) \) is nondecreasing and \( k(t) \geq 1 \) for \( t \geq 0 \), \( \alpha \in C^1(R_+, R_+) \) is nondecreasing and \( \alpha(t) \leq t \) for \( t \geq 0 \) and \( \alpha \in C(R_+ \times R_+, R_+) \) with \( (t,s) \mapsto \partial_2 a(t,s) \in C(R_+ \times R_+, R_+) \). If
\begin{equation}
(2.11) \quad u^p(t) \leq k(t) + \int_0^\alpha a(t,s)u^q(s)ds, \quad \text{for } t \geq 0,
\end{equation}
then
\begin{equation}
(2.12) \quad u(t) \leq \left[ k^{\frac{p-q}{p}} (0) + \frac{p-q}{p} \left( k(t) - k(0) + \int_0^\alpha a(t,s)ds \right) \right]^{\frac{1}{p-q}},
\end{equation}
for \( t \geq 0 \), where \( p > q > 0 \).
Proof. Define a function \( z(t) \) by

\[
z^p(t) = k(t) + \int_0^\alpha(t) a(t, s)u^q(s)ds.
\]

Differentiating (2.13) and using the properties of \( z(t), k(t) \) and \( \alpha(t) \) for \( t \in R_+ \), we obtain

\[
pz^{p-1}(t)z'(t) \leq \left( k'(t) + \int_0^\alpha(t) a(t, s)ds + \alpha'(t)\alpha(t) \right) z^q(t)
\]

(2.14)

\[
z^{p-q-1}(t)z'(t) \leq \frac{1}{p} \left( k'(t) + \int_0^\alpha(t) a(t, s)ds + \alpha'(t)\alpha(t) \right)
\]

\[
\frac{1}{p-q} \frac{\partial}{\partial t}(z^{p-q}(t)) \leq \frac{1}{p} \frac{\partial}{\partial t} \left( k(t) + \int_0^\alpha(t) a(t, s)ds \right).
\]

Integrating equation (2.14) from 0 to \( t \), we get

\[
z^{p-q}(t) \leq k \frac{p-q}{p} (0) + \frac{p-q}{p} \left( k(t) - k(0) + \int_0^\alpha(t) a(t, s)ds \right)
\]

\[
u^{p-q}(t) \leq k \frac{p-q}{p} (0) + \frac{p-q}{p} \left( k(t) - k(0) + \int_0^\alpha(t) a(t, s)ds \right)
\]

\[
u(t) \leq \left[ k \frac{p-q}{p} (0) + \frac{p-q}{p} \left( k(t) - k(0) + \int_0^\alpha(t) a(t, s)ds \right) \right]^{\frac{1}{p-q}}.
\]

Thus we have obtained a desired bound for \( u(t) \) given by (2.12). This completes the proof.

Remark 2.2. Note that for \( p \geq 2, k(t) = x_0 \geq 1, q = 1 \) and \( \alpha(t) = t \) in Theorem 2.4. We get an inequality obtained in [4].

Corollary 2.1. Let \( u \) be a nonnegative real valued continuous function on \( R_+ = [0, \infty) \), \( \alpha \in C^1(R_+, R_+) \) is nondecreasing and \( \alpha(t) \leq t \) for \( t \geq 0 \), \( a \in C(R_+ \times R_+, R_+) \) with \( (t, s) \mapsto \partial_\alpha a(t, s) \in C(R_+ \times R_+, R_+) \) and \( k \geq 1 \). If

\[
u^p(t) \leq k + \int_0^\alpha(t) a(t, s)u^q(s)ds, \quad \text{for } t \geq 0,
\]

(2.15)
then
\[(2.16)\quad u(t) \leq \left[ k^{\frac{p-2q}{p}} + \frac{p-q}{p} \left( \int_0^a \alpha(t) ds \right) \right]^{\frac{1}{p-q}},\]
for \( t \geq 0, \) where \( p > q > 0. \)

**Proof.** The proof follows immediately from Theorem 2.4 by substituting \( k(t) = k. \)

**Theorem 2.5.** Let \( u \) be a nonnegative real valued continuous function on \( R_+ = [0, \infty), k \in C^1(R_+, R_+) \) is nondecreasing and \( k(t) \geq 1 \) for \( t \geq 0, \)
\( \alpha \in C^1(R_+, R_+) \) is nondecreasing and \( \alpha(t) \leq t \) for \( t \geq 0 \) and \( a \in C(R_+ \times R_+ \times R_+). \) If
\[(2.17)\quad u^p(t) \leq \left( k(t) + \int_0^a a(t, s) u(s) ds \right)^q, \quad \text{for } t \geq 0,\]
then
\[(2.18)\quad u(t) \leq \left( k^{\frac{q(p-2q+1)}{p}}(0) + \frac{q(p-2q+1)}{p} \right) \left[ \left( k(t) + \int_0^a a(t, s) ds \right)^q - k^q(0) \right]^{\frac{1}{p-2q+1}},\]
for \( t \geq 0, \) where, \( p + 1 > 2q > 0. \)

**Proof.** Define a function \( z(t) \) by
\[(2.19)\quad z^p(t) = \left( k(t) + \int_0^a a(t, s) u^q(s) ds \right)^q,\]
then \( u(t) \leq z(t) \) and \( z(0) = k^q(0). \) Differentiating (2.19) and using the fact that \( z(t) \) is monotone nondecreasing for \( t \in R_+, \) we obtain
\[pz^{p-1}(t)z'(t) = q \left( k(t) + \int_0^a a(t, s) u^q(s) ds \right)^{q-1}\]
\[\times \left( k'(t) + \int_0^a a'(t, s) u^q(s) ds + \alpha'(t) a(t, \alpha(t)) u^q(\alpha(t)) \right),\]
Hence, we have

\[
\frac{1}{p - 2q + 1} \frac{\partial}{\partial t} \left( z^{p-2q+1}(t) \right) \leq \frac{q}{p} \frac{\partial}{\partial t} \left( k(t) + \int_0^t a(t,s) ds \right)^q.
\]

Integration (2.20) from 0 to \( t \), we get

\[
z^{p-2q+1}(t) \leq k \frac{q(p-2q+1)}{p} (0) + \frac{q(p-2q+1)}{p}.
\]

Consequently, we have obtained a desired bound for \( u(t) \) given by (2.18). This completes the proof.

**Corollary 2.2.** Let \( u \) be a nonnegative real valued continuous function on \( R_+ = [0, \infty) \), \( \alpha \in C^1(R_+,R_+) \) is nondecreasing and \( \alpha(t) \leq t \) for \( t \geq 0 \), \( a \in C(R_+ \times R_+,R_+) \) with \( (t,s) \mapsto \partial a(t,s) \in C(R_+ \times R_+,R_+) \) and \( k \geq 1 \). If

\[
u^p(t) \leq \left( k + \int_0^t a(t,s) u(s) ds \right)^q, \quad \text{for } t \geq 0,
\]
\[
\begin{equation}
\tag{2.22}
\begin{aligned}
&u(t) \leq \left( k^{\frac{q(p-2q+1)}{p}} + \frac{q(p-2q+1)}{p} \right) \\
&\cdot \left[ \left( k + \int_0^t a(t, s) ds \right)^q - k^q \right]^{-\frac{1}{p-2q+1}}
\end{aligned}
\end{equation}
\]

for \( t \geq 0 \), where \( p + 1 > 2q > 0 \).

**Proof.** The proof follows immediately from Theorem 2.5 by substituting \( k(t) = k \).

\[\square\]

### 3. Some applications

In this section, we give the applications of our main results to study the boundedness and asymptotic behaviour of the solutions of nonlinear retarded Volterra integral equations.

**Example 3.1.** We calculate the explicit bound on the solution of the nonlinear retarded Volterra integral equation

\[
(3.1) \quad u^5(t) = 5 + \int_0^t \sqrt{s} u(s) ds,
\]

where \( u \) be defined as in Theorem (2.1) and we assume that the solution \( u(t) \) of (3.1) exists on the maximal interval \([0, T]\) with \( T \in (0, \infty) \). Applying Theorem 2.1, we have

\[
u(t) \leq (5^{\frac{9}{20}} + \frac{9t}{2} \int_0^t s^3 ds)^{\frac{2}{3}} = (5^{\frac{9}{20}} + \frac{9t}{2}[\frac{t^2}{2}])^{\frac{2}{3}},\]

on \([0, T]\). Since blow-up does not occur in finite time, the solution is globally defined, that is, \( T = \infty \).

**Example 3.2.** Consider the nonlinear retarded Volterra integral equation

\[
(3.2) \quad u^4(t) = \left( e^t + \int_0^{t^{1/9}} tsu(s) ds \right)^2,
\]

where \( u \) be defined as in Theorem 2.5 and we assume that solution \( u(t) \) of (3.2) exists on the maximal interval \([0, T]\) with \( T \in (0, \infty) \). Applying Theorem 2.5, we have

\[
u(t) \leq (1 + \frac{1}{2}[(e^t + \int_0^{t^{1/9}} tsds)^2 - 1])^{\frac{1}{2}} = \frac{1}{2} \left[ \frac{1}{2} + \frac{1}{2}(e^t + \int_0^{t^{1/9}} tsds)^2 \right],\]

on \([0, T]\). Since blow-up does not occur in finite time, the solution is globally defined, that is, \( T = \infty \). Which is the desired bound for the solution of (3.2).
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