ON A THIRD ORDER RATIONAL SYSTEMS OF
DIFFERENCE EQUATIONS

BY

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Abstract. In this paper we investigate the form of the solutions of the following
systems of difference equations of order three
\[ x_{n+1} = \frac{x_{n-1}y_n}{y_n \pm y_{n-2}}, \quad y_{n+1} = \frac{x_n y_{n-1}}{x_n \pm x_{n-2}}, \]
with a nonzero real numbers initial conditions.

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1. Introduction

In this paper we deal with the solutions of the following systems of difference
equations
\[ x_{n+1} = \frac{x_{n-1}y_n}{y_n \pm y_{n-2}}, \quad y_{n+1} = \frac{x_n y_{n-1}}{x_n \pm x_{n-2}}, \]
with initial conditions \( x_{-2}, x_{-1}, x_0, y_{-2}, y_{-1} \) and \( y_0 \) are nonzero real numbers.

Recently, there has been great interest in studying difference equations systems. One of the reasons for this is the necessity for some techniques that can be used in investigating equations arising in mathematical models describing real life situations in population biology, economics, probability theory, genetics, psychology etc. There are many papers related to the difference equations systems for example, in [5] CINAR studied the solutions of
the system of difference equations \( x_{n+1} = \frac{1}{y_n}; \ y_{n+1} = \frac{y_n}{x_{n-1}y_{n-1}} \). Camouzis and Papaschinopoulos [4] studied the global asymptotic behavior of positive solutions of the system of rational difference equations \( x_{n+1} = 1 + \frac{x_n}{y_{n-m}}, \ y_{n+1} = 1 + \frac{y_n}{x_{n-m}} \). Clark et al. [8] investigated the global asymptotic stability of the system of difference equations \( x_{n+1} = \frac{x_n}{a + cy_n}, \ y_{n+1} = \frac{y_n}{b + dx_n} \). Elabbasy et al. [9] has obtained the solution of particular cases of the following general system of difference equations

\[
\begin{align*}
x_{n+1} &= \frac{a_1 + a_2 y_n}{a_3 z_n + a_4 x_{n-1} z_n}, \quad y_{n+1} = \frac{b_1 z_{n-1} + b_2 z_n}{b_3 x_n y_n + b_4 x_n y_{n-1}}, \\
z_{n+1} &= \frac{c_1 z_{n-1} + c_2 z_n}{c_3 x_{n-1} y_{n-1} + c_4 x_n y_n + c_5 x_n y_{n}},
\end{align*}
\]

Elsayed [13] has obtained the solutions of the following system of the difference equations \( x_{n+1} = \frac{1}{y_{n-k}}, \ y_{n+1} = \frac{y_{n-k}}{x_n y_n} \). Özban [26] has investigated the positive solutions of the system of rational difference equations \( x_{n+1} = \frac{1}{y_{n-k}}, \ y_{n+1} = \frac{y_n}{x_{n-m} y_{n-m-k}} \). Papaschinopoulos and Schinas [28] studied the oscillatory behavior, the boundedness of the solutions, and the global asymptotic stability of the positive equilibrium of the system of nonlinear difference equations \( x_{n+1} = A + \frac{y_n}{x_n y_n}; \ y_{n+1} = A + \frac{x_n}{y_n y_{n-q}} \). Simsek et al. [29] studied the behavior of the solutions of the following system of difference equations \( x_{n+1} = \text{max}\{\frac{A}{x_n}, \frac{y_n}{x_n}\}, \ y_{n+1} = \text{max}\{\frac{A}{y_n}, \frac{x_n}{y_n}\} \). Touafek et al. [30] investigated the periodic nature and gave the form of the solutions of the following systems of rational difference equations

\[
\begin{align*}
\pm 1 \pm x_{n-3} y_{n-1}^{-1}, \quad &y_{n+1} = \frac{y_{n-3}}{\pm 1 \pm x_{n-3} x_{n-1}}, \\
\pm 1 \pm x_{n-3} y_{n-1}^{-1}, \quad &y_{n+1} = \frac{y_{n-3}}{\pm 1 \pm x_{n-3} x_{n-1}}.
\end{align*}
\]

Yalçınkaya [32,34] investigated the sufficient condition for the global asymptotic stability of the following systems of difference equations

\[
\begin{align*}
z_{n+1} &= \frac{t_n z_{n-1} + a}{t_n + z_{n-1}}, \quad t_{n+1} = \frac{z_n t_{n-1} + a}{z_n + t_{n-1}}, \\
x_{n+1} &= \frac{x_n + y_{n-1}}{x_n y_{n-1} + 1}, \quad y_{n+1} = \frac{y_n + x_{n-1}}{y_n x_{n-1} - 1}.
\end{align*}
\]

Yang et al. [38] considered the behavior of the positive solutions of the system of difference equations \( x_n = \frac{a}{y_{n-p}}, \ y_n = \frac{b y_{n-p}}{x_{n-p} y_{n-p}} \). Similar to difference equations and nonlinear systems of rational difference equations were investigated (see [1]-[38]).
2. Main results

2.1. The system: \( x_{n+1} = \frac{x_{n-1}y_n}{y_n+y_{n-2}}, \ y_{n+1} = \frac{x_ny_{n-1}}{x_n+x_{n-2}} \).

We study in this section the form of the solutions of the system of the difference equations

\[
\begin{align*}
x_{n+1} &= \frac{x_{n-1}y_n}{y_n + y_{n-2}}, \quad y_{n+1} = \frac{x_ny_{n-1}}{x_n + x_{n-2}},
\end{align*}
\]

where \( n \in \mathbb{N}_0 \) and the initial conditions are arbitrary nonzero real numbers with \( x_{-2}, x_0, y_{-2} \) and \( y_0 \) are positive.

The following theorem is devoted to the form of the solutions of system (1).

**Theorem 2.1.** Suppose that \( \{x_n, y_n\} \) are solutions of system (1). Then for \( n = 0, 1, 2, \ldots \)

\[
\begin{align*}
x_{2n} &= \frac{x_{0}^{n+1}}{\prod_{i=0}^{n}((2i + 2)x_0 + x_{-2})}, \quad x_{2n-1} = \frac{x_{-1}y_0^n}{\prod_{i=0}^{n-1}((2i + 1)y_0 + y_{-2})},
\end{align*}
\]

\[
\begin{align*}
y_{2n} &= \frac{y_{0}^{n+1}}{\prod_{i=0}^{n}((2i + 2)y_0 + y_{-2})}, \quad y_{2n-1} = \frac{y_{-1}x_0^n}{\prod_{i=0}^{n-1}((2i + 1)x_0 + x_{-2})},
\end{align*}
\]

where \( \prod_{i=0}^{n-1}(a_i) = 1 \).

**Proof.** For \( n = 0, 1 \) the result holds. Now suppose that \( n > 1 \) and that our assumption holds for \( n - 2 \) and \( n - 1 \). That is,

\[
\begin{align*}
x_{2n-4} &= \frac{x_{0}^{n-1}}{\prod_{i=0}^{n-3}((2i + 2)x_0 + x_{-2})}, \quad x_{2n-5} = \frac{x_{-1}y_0^{n-2}}{\prod_{i=0}^{n-3}((2i + 1)y_0 + y_{-2})},
\end{align*}
\]

\[
\begin{align*}
y_{2n-4} &= \frac{y_{0}^{n-1}}{\prod_{i=0}^{n-3}((2i + 2)y_0 + y_{-2})}, \quad y_{2n-5} = \frac{y_{-1}x_0^{n-2}}{\prod_{i=0}^{n-3}((2i + 1)x_0 + x_{-2})},
\end{align*}
\]

\[
\begin{align*}
x_{2n-2} &= \frac{x_{0}^{n}}{\prod_{i=0}^{n-2}((2i + 2)x_0 + x_{-2})}, \quad x_{2n-3} = \frac{x_{-1}y_0^{n-1}}{\prod_{i=0}^{n-2}((2i + 1)y_0 + y_{-2})},
\end{align*}
\]

\[
\begin{align*}
y_{2n-2} &= \frac{y_{0}^{n}}{\prod_{i=0}^{n-2}((2i + 2)y_0 + y_{-2})}, \quad y_{2n-3} = \frac{y_{-1}x_0^{n-1}}{\prod_{i=0}^{n-2}((2i + 1)x_0 + x_{-2})}.
\end{align*}
\]

Let \( \alpha_i = (2i + 2)x_0 + x_{-2}, \ \beta_i = (2i + 2)y_0 + y_{-2}, \ \lambda_i = (2i + 1)x_0 + x_{-2}, \ \gamma_i = (2i + 1)y_0 + y_{-2} \).
It follows from Eq. (1) that

\[
x_{2n} = \frac{y_{2n-1} x_{2n-2}}{y_{2n-1} + y_{2n-3}} = \frac{x_{2n-2} y_{2n-3}}{x_{2n-2} + x_{2n-4}} x_{2n-2} = \frac{x_{2n-2}^2}{x_{2n-4} + 2 x_{2n-2}}
\]

\[
= \frac{x_{2n-1}^2}{\prod_{i=0}^{n-3} \alpha_i} + 2 \frac{x_{2n-3}}{\prod_{i=0}^{n-3} \alpha_i}
\]

\[
= \frac{x_{2n-1}^2}{\prod_{i=0}^{n-3} \alpha_i} + 2 \frac{x_{2n-3}}{\prod_{i=0}^{n-3} \alpha_i}
\]

\[
x_{2n} = \frac{y_{2n-1} y_{2n-2}}{y_{2n-1} + x_{2n-3}} = \frac{y_{2n-2} x_{2n-3}}{y_{2n-2} + y_{2n-4}} y_{2n-2} = \frac{y_{2n}^2}{y_{2n-4} + 2 y_{2n-2}}
\]

\[
= \frac{y_{2n-1} y_{2n-3}}{\prod_{i=0}^{n-3} \beta_i} + 2 \frac{y_{2n-3}}{\prod_{i=0}^{n-3} \beta_i}
\]

\[
y_{2n} = \frac{y_{2n-1} y_{2n-2}}{y_{2n-1} + x_{2n-3}} = \frac{y_{2n-2} x_{2n-3}}{y_{2n-2} + y_{2n-4}} y_{2n-2} = \frac{y_{2n}^2}{y_{2n-4} + 2 y_{2n-2}}
\]

\[
= \frac{y_{2n-1} y_{2n-3}}{\prod_{i=0}^{n-3} \beta_i} + 2 \frac{y_{2n-3}}{\prod_{i=0}^{n-3} \beta_i}
\]

\[
x_{2n-1} = \frac{y_{2n-2} y_{2n-3}}{y_{2n-2} + y_{2n-4}} = \frac{y_{2n-1} y_{2n-3}}{\prod_{i=0}^{n-3} \gamma_i} \cdot \frac{x_{2n-1}}{\prod_{i=0}^{n-3} \gamma_i}
\]

\[
= \frac{y_{2n-1} y_{2n-3}}{\prod_{i=0}^{n-3} \gamma_i} \cdot \frac{x_{2n-1}}{\prod_{i=0}^{n-3} \gamma_i}
\]
and

\[ y_{2n-1} = \frac{x_{2n-2}y_{2n-3}}{x_{2n-2} + x_{2n-4}} = \frac{x_n}{(\prod_{i=0}^{n-2} \lambda_i)} \times \frac{y_{1}^{n-1}}{(\prod_{i=1}^{n-2} \alpha_i)} = \frac{y_{1}^{n-1}}{(\prod_{i=0}^{n-2} \alpha_i)(x_0 + \alpha_{n-2})} \]

The proof is complete. \(\Box\)

**Lemma 1.** Every positive solution of system (1) is bounded and \(\lim_{n \to \infty} x_n = \lim_{n \to \infty} y_n = 0.\)

**Proof.** It follows from Eq. (1) that \(x_{n+1} = \frac{x_{n-1}y_n}{y_{n} + y_{n-2}} \leq \frac{x_{n-1}y_n}{y_n} = y_{n-1}, \) \(y_{n+1} = \frac{x_ny_{n-1}}{x_n + x_{n-2}} \leq \frac{x_ny_{n-1}}{x_n} = y_{n-1},\) or \(x_{n+1} \leq x_{n-1}, y_{n+1} \leq y_{n-1}.\) Then the subsequences \(\{x_{2n-1}\}_{n=0}^{\infty}, \{x_{2n}\}_{n=0}^{\infty}, \{y_{2n-1}\}_{n=0}^{\infty}, \{y_{2n}\}_{n=0}^{\infty}\) are decreasing and so are bounded from above by \(M, N\) respectively since \(M = \max\{x_{-1}, x_0\}, N = \max\{y_{-1}, y_0\}.\)

**Example 1.** We consider interesting numerical example for the difference system (1) with the initial conditions \(x_{-2} = 0.15, x_{-1} = 0.08, x_0 = 0.3, y_{-2} = 0.19, y_{-1} = 0.21\) and \(y_0 = 0.12\) (see Fig. 1).

![Figure 1](image-url)
2.2. The system: \( x_{n+1} = \frac{x_{n-1}y_n}{y_n + y_{n-2}}, \ y_{n+1} = \frac{x_ny_{n-1}}{x_n - x_{n-2}} \).

In this section, we investigate the solutions of the system of the difference equations

\[ x_{n+1} = \frac{x_{n-1}y_n}{y_n + y_{n-2}}, \ y_{n+1} = \frac{x_ny_{n-1}}{x_n - x_{n-2}}, \]

where \( n = 0, 1, 2, \ldots \) and the initial conditions are arbitrary nonzero real numbers with \( y_0 \neq \pm y_{-2}, \ x_0 \neq x_{-2} \) and \( 2x_0 \neq x_{-2} \).

**Theorem 2.2.** Suppose that \( \{x_n, y_n\} \) are solutions of system (2). Also, assume that \( x_{-2}, \ x_{-1}, \ x_0, \ y_{-2}, \ y_{-1} \) and \( y_0 \) are arbitrary nonzero real numbers with \( y_0 \neq \pm y_{-2}, \ x_0 \neq x_{-2} \) and \( 2x_0 \neq x_{-2} \). Then the solutions of Eq. (2) are given by the following formula for \( n = 0, 1, 2, \ldots \)

\[
\begin{align*}
    x_{4n-2} &= \frac{x_0^{2n}}{x_{-2}^{n-1}(2x_0 - x_{-2})^n}, \quad x_{4n-1} = \frac{x_{-1}y_0^{2n}}{(y_0 + y_{-2})^{n-1}(y_0 - y_{-2})^n}, \\
    x_{4n} &= \frac{x_0^{2n+1}}{x_{-2}^{n-1}(2x_0 - x_{-2})^n}, \quad x_{4n+1} = \frac{x_{-1}y_0^{2n+1}}{(y_0 + y_{-2})^{n+1}(y_0 - y_{-2})^n}, \\
    y_{4n-2} &= \frac{(-1)^n y_0^{2n}}{y_{-2}^{n-1}}, \quad y_{4n-1} = \frac{(-1)^n x_0^{2n-1}y_{-1}}{(x_0 - x_{-2})^{2n+1}}, \\
    y_{4n} &= \frac{(-1)^n y_0^{2n+1}}{y_{-2}^{n-1}}, \quad y_{4n+1} = \frac{(-1)^n x_0^{2n+1}y_{-1}}{(x_0 - x_{-2})^{2n+1}}.
\end{align*}
\]

**Proof.** For \( n = 0 \) the result holds. Now suppose that \( n > 0 \) and that our assumption holds for \( n - 1 \), that is,

\[
\begin{align*}
    x_{4n-6} &= \frac{x_0^{2n-2}}{x_{-2}^{n-2}(2x_0 - x_{-2})^{n-1}}, \quad x_{4n-5} = \frac{x_{-1}y_0^{2n-2}}{(y_0 + y_{-2})^{n-1}(y_0 - y_{-2})^{n-1}}, \\
    x_{4n-4} &= \frac{x_0^{2n-1}}{x_{-2}^{n-2}(2x_0 - x_{-2})^{n-1}}, \quad x_{4n-3} = \frac{x_{-1}y_0^{2n-1}}{(y_0 + y_{-2})^{n}(y_0 - y_{-2})^{n-1}}, \\
    y_{4n-6} &= \frac{(-1)^{n-1} y_0^{2n-2}}{y_{-2}^{n-3}}, \quad y_{4n-5} = \frac{(-1)^{n-1} x_0^{2n-2}y_{-1}}{(x_0 - x_{-2})^{2n-2}}, \\
    y_{4n-4} &= \frac{(-1)^{n-1} y_0^{2n-1}}{y_{-2}^{n-2}}, \quad y_{4n-3} = \frac{(-1)^{n-1} x_0^{2n-1}y_{-1}}{(x_0 - x_{-2})^{2n-1}}.
\end{align*}
\]
It follows from Eq.(2) that

\[ x_{n-2} = \frac{x_{4n-4}y_{4n-3}}{y_{4n-3} + y_{4n-5}} = \frac{x_{0}^{2n-1} \left(\frac{x_{n-1}y_{0}^{2n-1}}{(y_{0} - y_{-2})^{n-1}}\right)}{x_{n-2}^{2n-1}(2x_{0} - x_{-2})^{n-1}} \]

\[ = \frac{x_{0}^{2n-1}}{x_{n-2}^{2n-1}(2x_{0} - x_{-2})^{n-1}} + 1 \]

\[ = \frac{x_{n-1}(2x_{0} - x_{-2})^{n-1}(x_{0} + x_{0} - x_{-2})}{x_{0}^{2n}(x_{0} - x_{-2})^{n}} \]

\[ y_{n-2} = \frac{x_{4n-3}y_{4n-4}}{x_{4n-3} - x_{4n-5}} = \frac{x_{0}^{2n} \left(\frac{x_{n-1}y_{0}^{2n}}{(y_{0} + y_{-2})^{n-1}}\right)}{x_{n-2}^{2n-1}(y_{0} - y_{-2})^{n-1}} \]

\[ = \frac{x_{0}^{2n}}{(y_{0} + y_{-2})^{n-1}} - 1 \]

\[ = \frac{x_{n-1}y_{0}^{2n}}{y_{-2}^{2n}y_{0}^{2n}} \]

\[ x_{n-1} = \frac{x_{4n-3}y_{4n-2}}{y_{4n-2} + y_{4n-4}} = \frac{x_{0}^{2n} \left(\frac{x_{n-1}y_{0}^{2n}}{(y_{0} + y_{-2})^{n-1}}\right)}{y_{n-2}^{2n} \left(\frac{x_{0}^{2n}}{(y_{0} - y_{-2})^{n-1}}\right)} \]

\[ = \frac{x_{0}^{2n} \left(\frac{x_{n-1}y_{0}^{2n}}{(y_{0} + y_{-2})^{n-1}}\right)}{(y_{0} + y_{-2})^{n} \left(\frac{x_{0}^{2n}}{(y_{0} - y_{-2})^{n-1}}\right)} - 1 \]

\[ = \frac{x_{0}^{2n} \left(\frac{x_{n-1}y_{0}^{2n}}{(y_{0} + y_{-2})^{n-1}}\right)}{(y_{0} + y_{-2})^{n}} \]

\[ y_{n-1} = \frac{x_{4n-2}y_{4n-3}}{x_{4n-2} - x_{4n-4}} = \frac{x_{0}^{2n} \left(\frac{x_{n-1}y_{0}^{2n}}{(x_{0} - x_{-2})^{n-1}}\right)}{x_{n-2}^{2n-1} \left(\frac{x_{0}^{2n}}{(x_{0} - x_{-2})^{n}}\right)} \]

\[ = \frac{(x_{0} - x_{-2})^{n} \left(\frac{x_{n-1}y_{0}^{2n}}{(2x_{0} - x_{-2})^{n-1}}\right)}{(x_{0} - x_{-2})^{2n-1} \left(\frac{x_{0}^{2n}}{(2x_{0} - x_{-2})^{n}}\right)} - 1 \]

\[ = \frac{(x_{0} - x_{-2})^{2n-1}}{(x_{0} - x_{-2})^{2n}} \]
Similarly one can prove the other relations. The proof is complete.

Remark 1. There is no solution \( \{x_n, y_n\} \) of system (2) with \( \{y_n\} \) positive. In fact for example, from the formula of \( y_{4n} \) in Theorem 2.2, we get \( y_2 = -y_2 \left( \frac{y_0}{y_1} \right)^2, \quad y_6 = y_2 \left( \frac{y_0}{y_1} \right)^4 \) which are with opposite sign.

There are solutions \( \{x_n, y_n\} \) of system (2) with \( \{x_n\} \) positive and unbounded. Consider the initial conditions \( x_2 = 1, \quad x_0 = 3, \quad y_2 = 0, \quad y_0 = 2 \). According to Theorem 2.2, \( \{x_{4n+1}\} \) for \( i = \{-2, -1, 0, 1\} \) is positive and so \( \{x_n\} \) is positive. We then have that \( x_{4n-2} = x_{-2} \left( \frac{x_0^2}{2x_0x_{-2}-x_{-2}^2} \right)^2 = \left( \frac{9}{5} \right)^n \), and so \( \{x_n\} \) is unbounded.

Assume that \( x_0^2 < (x_0 - 2x_{-2})^2, \quad x_0^2 < |x_{-2} (2x_0 - x_{-2})|, \quad y_0^2 < y_{-2}^2, \quad y_0^2 < |y_0^2 - y_{-2}^2| \), then \( \lim_{n \to +\infty} x_n = \lim_{n \to +\infty} y_n = 0 \). The initial values \( x_{-2} = 3, \quad x_0 = 1, \quad y_{-2} = 3, \quad y_0 = 1 \) satisfied the above conditions.

Example 2. We consider interesting numerical example for the difference system (2) with the initial conditions \( x_2 = 5, \quad x_0 = 0.7, \quad x_0 = 2, \quad y_2 = 8, \quad y_1 = 4 \) and \( y_0 = 3 \) (see Fig. 2).

![Fig. 2: This figure shows the solution of the system (2) with the initial values as in example 2.](image)

2.3. The system: \( x_{n+1} = \frac{x_{n-1}y_n}{y_n - y_{n-2}}, \quad y_{n+1} = \frac{x_ny_{n-1}}{x_n + x_{n-2}} \).

In this section, we obtain the form of the solutions of the system of the
difference equations

\[ x_{n+1} = \frac{x_{n-1}y_n}{y_n - y_{n-2}}, \quad y_{n+1} = \frac{x_ny_{n-1}}{x_n + x_{n-2}}, \]

where \( n \in \mathbb{N}_0 \) and the initial conditions are arbitrary nonzero real numbers with \( y_0 \neq y_{-2}, \) \( 2y_0 \neq y_{-2} \) and \( x_{-2} \neq \pm x_0. \)

If we interchange between \( x_n \) and \( y_n \) in system (2) we get system (3), so the following result follows immediately from Theorem (2.2).

**Theorem 2.3.** Assume that \( \{x_n, y_n\} \) are solutions of system (3). Then the solutions of Eq. (3) are given by the following formula for \( n = 0, 1, 2, \ldots \)

\[
\begin{align*}
x_{4n-2} &= (-1)^n x_0^{2n} \frac{y_0^{2n}}{x_{-2}^{2n-1}}, \quad x_{4n-1} = (-1)^n x_{-1} y_0^{2n} \frac{y_0^{2n}}{(y_0 - y_{-2})^{2n}}, \\
x_{4n} &= (-1)^n x_0^{2n+1} y_0^{2n+1} \frac{y_0^{2n+1}}{x_{-2}^{2n+1}}, \quad x_{4n+1} = (-1)^n x_{-1} y_0^{2n+1} \frac{y_0^{2n+1}}{(y_0 - y_{-2})^{2n+1}}, \\
y_{4n-2} &= y_0^{2n-1} (2y_0 - y_{-2})^{2n}, \quad y_{4n-1} = x_0^{2n} y_0^{2n-1} \frac{y_0^{2n-1}}{(x_0 + x_{-2})^{2n}(x_{-2} - x_0)^n}, \\
y_{4n} &= y_0^{2n} (2y_0 - y_{-2})^{2n}, \quad y_{4n+1} = x_0^{2n+1} y_0^{2n+1} \frac{y_0^{2n+1}}{(x_0 + x_{-2})^{2n+1}(x_{-2} - x_0)^n},
\end{align*}
\]

where \( y_0 \neq y_{-2}, \) \( 2y_0 \neq y_{-2} \) and \( x_{-2} \neq \pm x_0. \)

**Remark 2.** There is no solution \( \{x_n, y_n\} \) of system (3) with \( \{x_n\} \) positive. In fact for example, from the formula of \( x_{4n-2} \) in Theorem 2.3, we get \( x_2 = -x_{-2}(x_{-2})^2, x_6 = x_{-2}(x_{-2})^4 \) which are with opposite sign.

There are solutions \( \{x_n, y_n\} \) of system (3) with \( \{y_n\} \) positive and unbounded. Consider the initial conditions \( y_{-2} = 1, y_{-1} = 1, y_0 = 3, x_{-2} = 1, x_0 = 2. \) According to Theorem 2.2, \( \{y_{4n+i}\} \) for \( i = \{-2,-1,0,1\} \) is positive and so \( \{y_n\} \) is positive. We then have that \( y_{4n-2} = y_2(\frac{y_0^2}{(y_0 - y_{-2})^2})^2 = (\frac{9}{5})^n, \) and so \( \{y_n\} \) is unbounded.

Assume that \( y_0^2 < (y_0 - y_{-2})^2, \) \( y_0^2 < |y_{-2}(2y_0 - y_{-2})|, \) \( x_0^2 < x_{-2}^2, \) \( x_0^2 < |x_0^2 - x_{-2}^2|, \) then \( \lim_{n \to +\infty} x_n = \lim_{n \to +\infty} y_n = 0. \) The initial values \( y_{-2} = 3, y_0 = 1, x_{-2} = 3, x_0 = 1 \) satisfied the above conditions.

**Example 3.** For the initial conditions \( x_{-2} = -3, x_{-1} = 0.7, x_0 = 2, \) \( y_{-2} = 8, y_{-1} = 4 \) and \( y_0 = 3 \) when we take the system (3) (see Fig. 3).
2.4. The system: $x_{n+1} = \frac{x_{n-1}y_n}{y_n - y_{n-2}}, y_{n+1} = \frac{x_ny_{n-1}}{x_n - x_{n-2}}$.

In this section, we deal with the expression of the solutions of the following system of the difference equations

$$(4)\quad x_{n+1} = \frac{x_{n-1}y_n}{y_n - y_{n-2}}, \quad y_{n+1} = \frac{x_ny_{n-1}}{x_n - x_{n-2}},$$

where $n \in \mathbb{N}_0$ and the initial conditions are arbitrary nonzero real numbers with $x_0 \neq x_2$ and $y_2 \neq y_0$.

**Theorem 2.4.** Suppose that $\{x_n, y_n\}$ are solutions of system (4). Then for $n = 0, 1, 2, \ldots$, $x_{2n-2} = \frac{x_0^n}{x_2^{-2}}, x_{2n-1} = x_1 \left(\frac{y_0}{y_0 - y_2}\right)^n, y_{2n-2} = \frac{y_0^n}{y_2^{-2}}, y_{2n-1} = y_1 \left(\frac{x_0}{x_0 - x_2}\right)^n$.

**Proof.** For $n = 0, 1$ the result holds. Now suppose that $n > 1$ and that our assumption holds for $n - 2$ and $n - 1$ that is,

\begin{align*}
x_{2n-6} &= \frac{x_0^{n-2}}{x_2^{-2}}, \quad x_{2n-5} = x_1 \left(\frac{y_0}{y_0 - y_2}\right)^{n-2}, \quad y_{2n-6} = \frac{y_0^{n-2}}{y_2^{-2}}, \\
y_{2n-5} &= y_1 \left(\frac{x_0}{x_0 - x_2}\right)^{n-2}, \quad x_{2n-4} = \frac{x_0^{n-1}}{x_2^{-2}}, \quad x_{2n-3} = x_1 \left(\frac{y_0}{y_0 - y_2}\right)^{n-1}, \\
y_{2n-4} &= \frac{y_0^{n-1}}{y_2^{-2}}, \quad y_{2n-3} = y_1 \left(\frac{x_0}{x_0 - x_2}\right)^{n-1}.
\end{align*}
It follows from Eq.(4) that

\[
x_{2n-2} = \frac{x_{2n-4} y_{2n-3}}{y_{2n-3} - y_{2n-5}} = \left(\frac{x_0}{x_{-2}}\right)^{n-1} \left(\frac{x_0}{x_{-2}}\right)^{n-1} - \left(\frac{x_0}{x_{-2}}\right)^{n-2} \left(\frac{x_0}{x_{-2}}\right)^{n-1}
\]

\[
y_{2n-2} = \frac{y_{2n-4} x_{2n-3}}{x_{2n-3} - x_{2n-5}} = \left(\frac{y_0}{y_{-2}}\right)^{n-1} \left(\frac{y_0}{y_{-2}}\right)^{n-1} - \left(\frac{y_0}{y_{-2}}\right)^{n-2} \left(\frac{y_0}{y_{-2}}\right)^{n-1}
\]

Similarly one can prove the other relations. The proof is complete.

**Example 4.** Consider the difference equations system (4) with the initial conditions \(x_{-2} = 0.5, x_{-1} = 0.9, x_0 = 0.2, y_{-2} = 0.8, y_{-1} = -0.7,\) and \(y_0 = 0.3\) (see Fig. 4).

![Figure 4: This figure shows the solution of the system (4) with the initial values as in example 4.](image)

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