RINGS WHOSE MODULES ARE $\oplus$-COFINITELY SUPPLEMENTED

BY

ERGÜL TÜRKMEN

Abstract. It is known that a commutative ring $R$ is an artinian principal ideal ring if and only if every left $R$-module is $\oplus$-supplemented. In this paper, we show that a commutative ring $R$ is a serial ring if every left $R$-module is $\oplus$-cofinitely supplemented. The converse holds if $R$ is a max ring. Moreover, we study maximally $\oplus$-supplemented modules as a proper generalization of $\oplus$-cofinitely supplemented modules. Using these modules, we also prove that a ring $R$ is semiperfect if and only if every projective left $R$-module with small radical is supplemented.

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1. Introduction

Throughout this study by a ring $R$ we mean an associative ring with identity and by an $R$-module we mean an arbitrary unital left $R$-module. A submodule $N$ of an $R$-module $M$ will be denoted by $N \leq M$. A submodule $N \leq M$ is called small (in $M$) and written by $N << M$, if $M \neq N + L$ for every proper submodule $L$ of $M$. Let $M$ be a module. $M$ is called supplemented if every submodule $N$ of $M$ has a supplement, that is a submodule $K$ minimal with respect to $N + K = M$. $K$ is a supplement of $N$ in $M$ if and only if $N + K = M$ and $N \cap K << K$ ([17]). Clearly, supplements are a generalization of direct summands. In [12], a module $M$ is called $\oplus$-supplemented if every submodule $N$ of $M$ has a supplement that is a direct summand of $M$. 
Let $M$ be a module and $N \leq M$. $N$ is said to be cofinite if $\frac{M}{N}$ is finitely generated. Maximal submodules are cofinite. $M$ is said to be cofinitely supplemented if every cofinite submodule of $M$ has a supplement in $M$ ([1]). It is clear that every supplemented module is cofinitely supplemented. It is shown in [1, Theorem 2.8] that a module $M$ is cofinitely supplemented if and only if every maximal submodule has a supplement in $M$. In light of the characterization, we call a module $M$ maximally $\oplus$-supplemented if every maximal submodule of $M$ has a supplement that is a direct summand of $M$.

ÇALIŞICI and PANCAR [5] call a module $M \oplus$-cofinitely supplemented if every cofinite submodule $N$ of $M$ has a supplement that is a direct summand of $M$. From these definitions, every $\oplus$-cofinitely supplemented module is maximally $\oplus$-supplemented. Also, a maximally $\oplus$-supplemented module with the property (SSP) is $\oplus$-cofinitely supplemented by [5, Theorem 2.3], but it is not generally that every maximally $\oplus$-supplemented module is $\oplus$-cofinitely supplemented (see Example 2.2).

Let $f : P \rightarrow M$ be an epimorphism. XUE [18] calls $f$ a (generalized) cover if $(\text{Ker}(f) \leq \text{Rad}(P)) \text{Ker}(f) \ll P$, and calls a (generalized) cover $f$ a (generalized) projective cover if $P$ is a projective module. In the spirit of [18], a module $M$ is said to be (generalized) semiperfect if every factor module of $M$ has a (generalized) projective cover.

A ring $R$ is left (semi) perfect if every (finitely generated) left $R$-module has a projective cover ([10]).

The aim of this paper is to characterize the rings whose modules are $\oplus$-cofinitely supplemented. In particular, we prove that, for a commutative ring $R$, if every left $R$-module is $\oplus$-cofinitely supplemented, then $R$ is a serial ring. If $R$ is a max ring, the converse holds. It follows that a commutative max ring $R$ is an artinian principal ideal ring if and only if every left $R$-module is $\oplus$-cofinitely supplemented if and only if every left $R$-module is maximally $\oplus$-supplemented. Moreover, we obtain various properties of maximally $\oplus$-supplemented modules. Every projective maximally $\oplus$-supplemented, w-local module is local. We also prove that a ring $R$ is semiperfect if and only if every projective left $R$-module with small radical is supplemented.

2. Maximally $\oplus$-supplemented modules

In this section, we develop some properties of maximally $\oplus$-supplemented
modules. Using these modules, we give a new characterization of semiperfect rings.

Let $R$ be any ring and let $M$ be an $R$-module. By $\text{Rad}(M)$, we denote the radical of $M$. $M$ is called local if $\text{Rad}(M)$ is a maximal submodule and $M$ is finitely generated. Local modules are $\oplus$-supplemented. Note that, from [17, 41.1(3)], supplements of a maximal submodule are local. A ring $R$ is local if $R_R$ (or $R_R$) is local.

To show that maximally $\oplus$-supplemented modules are a proper generalization of $\oplus$-cofinitely supplemented modules, we need the following simple Lemma.

**Lemma 2.1.** Let $M$ be a module and $N$ be a proper submodule of $M$. If $K$ is a local submodule of $M$ such that $M = N + K$, then $K$ is a supplement of $N$ in $M$.

**Proof.** By our assumption, it suffices to prove that $N \cap K$ is small in $K$. To see this, let $N \cap K + L = K$ for some submodule $L$ of $K$. Since $K$ is finitely generated, the submodule $N \cap K$ is contained in $\text{Rad}(K)$ and so we have $K = N \cap K + L = \text{Rad}(K) + L$. This implies that $L = K$. Hence, $K$ is a supplement of $N$ in $M$. \[\square\]

**Example 2.2.** Let $R$ be a commutative local ring which is not valuation. Let $M$ be the $R$-module in [9, Example 2.3]. $M$ is not $\oplus$-supplemented. Since finitely generated $\oplus$-cofinitely supplemented modules are $\oplus$-supplemented, $M$ is not a $\oplus$-cofinitely supplemented module. Let $N$ be any maximal submodule of $M$. By [9, Remark 3.4], there exists a local direct summand $K$ of $M$ such that $M = N + K$. It follows from Lemma 2.1 that $N$ has a supplement that is a direct summand of $M$. Thus $M$ is maximally $\oplus$-supplemented.

A module $M$ is said to have the Summand Sum Property (SSP) if the sum of two direct summands of $M$ is again a direct summand of $M$.

Let $M$ be a non-zero module. $M$ is called indecomposable if the only direct summands of $M$ are 0 and $M$. Note that an indecomposable module has the property (SSP).

**Corollary 2.3.** Let $M$ be an indecomposable module. $M$ is $\oplus$-cofinitely supplemented if and only if it is maximally $\oplus$-supplemented.

For the next result see [5, Lemma 2.2].
Corollary 2.4. A module $M$ is maximally $\oplus$-supplemented if and only if $\frac{M}{\text{Loc}^\oplus(M)}$ has no a maximal submodule, where $\text{Loc}^\oplus(M)$ is the direct sum of local direct summands of $M$.

A module $M$ is called coatomic if every proper submodule of $M$ is contained in a maximal submodule of $M$. For Theorem 2.6, we need the following key Lemma.

Lemma 2.5. Let $M$ be a projective module. $M$ is maximally $\oplus$-supplemented if and only if every simple factor module of $M$ has a projective cover.

Proof. Let $N$ be any maximal submodule of $M$. By [17, 42.1], $\frac{M}{N}$ has a projective cover if and only if there exists a direct summand $K$ of $M$ such that $M = N + K$ and $N \cap K << K$. Then the proof follows. □

Theorem 2.6. For a projective coatomic module $M$, the following are equivalent.

1. $M$ is maximally $\oplus$-supplemented.
2. $M$ is $\oplus$-cofinitely supplemented.
3. $M$ is $\oplus$-supplemented.
4. $M$ is semiperfect.

Proof. (4) $\implies$ (3) $\implies$ (2) $\implies$ (1) are obvious.

(1) $\implies$ (4) Let $M$ be a maximally $\oplus$-supplemented module. By Lemma 2.5, every simple factor module of $M$ has a (generalized) projective cover. Since $M$ is coatomic, $M$ is generalized semiperfect according to [2, Theorem 2.4]. It follows from [14, Theorem 1.3] that $M$ is semiperfect. □

Let $R$ be a ring. $R$ is called semilocal if $\frac{R}{\text{rad}(R)}$ is semisimple. Note that semilocal rings properly contains semiperfect rings.

Corollary 2.7. Let $R$ be a semilocal ring and let $M$ be a projective $R$-module with small radical. $M$ is semiperfect if and only if it is maximally $\oplus$-supplemented.

Proof. Necessity is clear. Conversely, suppose that $M$ is maximally $\oplus$-supplemented. Let $N$ be any proper submodule of $M$. Note that $N + \text{Rad}(M) \neq M$. Since $R$ is semilocal, by [11, Theorem 3.5], $\frac{M}{\text{Rad}(M)}$ is
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(⇐) By Theorem 2.8. □

The following example shows that, in general, a factor module of a maximally ⊕-supplemented module need not be maximally ⊕-supplemented.

Example 2.10. Let $R$ be a commutative local ring which is not valuation ring. Then we have a finitely presented factor module $M$ of $R^n$ ($n \geq 2$) such that $M$ can not be generated by fewer then $n$ elements. According to Theorem 2.8, $R^n$ is maximally ⊕-supplemented. Suppose that $M$ is maximally ⊕-supplemented. Since $M$ is indecomposable and finitely generated, the notions of ⊕-supplemented and being maximally ⊕-supplemented coincide by Corollary 2.3. Therefore $M$ is ⊕-supplemented. This is a contradiction by [9, Example 2.2]. Hence $M$ is not maximally ⊕-supplemented.

In [4], a module $M$ is said to be weakly distributive if every submodule $N$ of $M$ is weak distributive, i.e. $N = U \cap N + V \cap N$ whenever $M = U + V$. Now we prove that any factor module of a weakly distributive maximally ⊕-supplemented module is maximally ⊕-supplemented.

Proposition 2.11. Let $M$ be a module. Suppose that $M$ is weakly distributive. If $M$ is maximally ⊕-supplemented, then $\frac{M}{U}$ is maximally ⊕-supplemented for every submodule $U \leq M$. In particular, every direct summand of $M$ is maximally ⊕-supplemented.

Proof. Let $\frac{N}{U}$ be a maximal submodule of $\frac{M}{U}$. So $N$ is a maximal submodule of $M$. Then there exist submodules $K$ and $L$ such that $M = N + K$, $N \cap K < U$ and $K \cap L = M$. By [17, 41.1(7)], $\frac{K + U}{U}$ is a supplement of $\frac{N}{U}$ of $\frac{M}{U}$, i.e. $\frac{N}{U} = \frac{N}{U} + \frac{(K + U)}{U}$ and $\frac{N}{U} \cap \frac{(K + U)}{U} \ll \frac{(K + U)}{U}$. It remains to show that $\frac{K + U}{U}$ is a direct summand of $\frac{M}{U}$. Since $M$ is a weakly distributive module, we have that $U = K \cap U + L \cap U$. Then,

$$\left( \frac{K + U}{U} \right) \cap \left( \frac{L + U}{U} \right) = \frac{(K + U) \cap L + U}{U} = \frac{(K + L \cap U) \cap L + U}{U} = \frac{K \cap L + L \cap U + U}{U} = 0.$$

Hence $\frac{K + U}{U}$ is a direct summand of $\frac{M}{U}$. □

It is well known that a ring $R$ is left perfect if and only if every projective left $R$-module is ⊕-supplemented. Now we prove an analogue for semiperfect rings.
Theorem 2.12. A ring $R$ is semiperfect if and only if every projective left $R$-module is maximally $\oplus$-supplemented.

Proof. Let $M$ be any projective module. Since $R$ is semiperfect, each simple factor module of $M$ has a projective cover by [17, 42.6]. From Lemma 2.5, we obtain that $M$ is maximally $\oplus$-supplemented. Conversely, suppose that $RR$ is maximally $\oplus$-supplemented. By Theorem 2.6, $R$ is semiperfect.

Recall that every projective supplemented module is $\oplus$-supplemented. The following Corollary is an immediate consequence of Corollary 2.7 and Theorem 2.12.

Corollary 2.13. Let $R$ be a ring. $R$ is semiperfect if and only if every projective left $R$-module with small radical is supplemented.

Proof. Suppose that $R$ is a semiperfect ring. Let $M$ be a projective $R$-module with small radical. By Theorem 2.12, $M$ is maximally $\oplus$-supplemented. Note that $R$ is semilocal. It follows from Corollary 2.7 that $M$ is semiperfect. Hence, $M$ is a supplemented module. The converse is clear.

In [3], a module $M$ is called $w$-local if $\text{Rad}(M)$ is maximal in $M$. Clearly, every local module is $w$-local but a $w$-local module need not be local. Gerasimov and Sakhaev gave in [7] an example of a projective $w$-local module, which is not local. R. Ware proved that if $R$ is commutative or Noetherian, then every projective $w$-local module is local ([16]). Using maximally $\oplus$-supplemented modules, we obtain the following fact.

Proposition 2.14. A projective maximally $\oplus$-supplemented, $w$-local module is local.

Proof. By the hypothesis, there exists a local direct summand $N$ of $M$ such that $M = \text{Rad}(M) + N$ and $N \cap \text{Rad}(M) \ll N$. Let $M = N \oplus K$. It follows from [17, 21.6.(5)] that $\text{Rad}(M) = \text{Rad}(N) \oplus \text{Rad}(K)$. Then, $M = \text{Rad}(M) + N = \text{Rad}(N) \oplus \text{Rad}(K) + N = \text{Rad}(K) \oplus N$. So $K = \text{Rad}(K)$. Therefore, $K = 0$ because $K$ is projective according to [10, 5.3.4 (b)]. Hence $N = M$. That is, $M$ is local.

3. Rings whose modules are $\oplus$-cofinitely supplemented

It is shown in [8, Theorem 1] that a commutative ring $R$ is an artinian
principal ideal ring if and only if every left $R$-module is $\oplus$-supplemented. Our aim is to prove that if every left $R$-module is $\oplus$-cofinitely supplemented, then $R$ is a serial ring. It follows that a commutative max ring $R$ is an artinian principal ideal ring if and only if every left $R$-module is $\oplus$-cofinitely supplemented if and only if every left $R$-module is maximally $\oplus$-supplemented.

Throughout this section, unless otherwise stated, it is assumed that $R$ is a commutative ring.

Recall that a submodule $N$ of $M$ is said to be essential (or large) in $M$, denoted by $N \trianglelefteq M$, if $N \cap K \neq 0$ for every non-zero submodule $K$ of $M$. A module $M$ is said to be uniform if every non-zero submodule of $M$ is essential in $M$. $M$ is called uniserial if its submodules are linearly ordered by inclusion. We call a commutative ring $R$ uniserial if $R$ is uniserial.

**Proposition 3.1.** Let $R$ be a local ring and let $M$ be a uniform $R$-module. Suppose that every submodule of $M$ is maximally $\oplus$-supplemented. Then $M$ is uniserial.

**Proof.** By [15, Lemma 6.2], it suffices to show that every finitely generated submodule of $M$ is local. Let $K$ be any finitely generated submodule of $M$. Then, $K$ contains a maximal submodule $N$. By the assumption, $N$ has a supplement $V$ in $K$ such that $V \oplus V' = K$ for some submodule $V'$ of $K$. Note that $V' \subseteq M$. Therefore $V$ is local according to [17, 41.1(3)]. Since $M$ is uniform and $N$ is maximal, we have $V' = 0$. In conclusion $V = K$. □

We denote by $E(M)$ the injective hull of $M$.

**Corollary 3.2.** Let $R$ be local ring. If every submodule of $E(\frac{R}{\text{Rad}(R)})$ is maximally $\oplus$-supplemented, $R$ is a uniserial ring.

**Proof.** Since $E(\frac{R}{\text{Rad}(R)})$ is uniform, the proof follows from Proposition 3.1 and [15, Corollary of Lemma 6.2]. □

**Theorem 3.3.** If every left $R$-module is maximally $\oplus$-supplemented, then $R$ is a serial ring. In particular, if every left $R$-module is $\oplus$-cofinitely supplemented, then $R$ is a serial ring.

**Proof.** Assume that every left $R$-module is maximally $\oplus$-supplemented. Applying Theorem 2.12, we conclude that $R$ is semiperfect. Then, there exist orthogonal idempotents $\{e_1, e_2, \ldots, e_n\}$ in $R$ such that $R = Re_1 \oplus$
$Re_2 \oplus ... \oplus Re_n$ and each $Re_i$ is local. It is easy to see that every left $Re_i$-module is maximally $\oplus$-supplemented. It follows from Corollary 3.2 that each $Re_i$ is uniserial. Therefore $R$ is a serial ring. The rest of the proof is clear by definitions.

A ring $R$ is said to be a max ring if every non-zero left $R$-module has a maximal submodule. Note that a semilocal ring $R$ is left perfect if and only if it is a max ring. Let $M$ be a module.

**Corollary 3.4.** For a commutative max ring $R$ the following statements are equivalent:

1. $R$ is an artinian principal ideal ring.
2. Every left $R$-module is maximally $\oplus$-supplemented.
3. Every left $R$-module is $\oplus$-cofinitely supplemented.

**Proof.** (1) $\implies$ (3) and (3) $\implies$ (2) are clear.

(2) $\implies$ (1) By Theorem 3.3, we have $R = Re_1 \oplus Re_2 \oplus ... \oplus Re_n$, where $Re_i$ is uniserial. Since $R$ is a max ring, it is perfect. Therefore, for all $i \in \{1,2,...,n\}$, $Re_i$ is perfect. It follows that every non-zero ideal $I$ of $Re_i$ has a maximal submodule, say $K$. By the hypothesis and the proof of Theorem 3.3, every $Re_i$-module is maximally $\oplus$-supplemented, and so there exist ideals $L$ and $N$ of $Re_i$ such that $L$ is a supplement of $K$ and $Re_i = L \oplus N$. Then $L$ is local. Since $Re_i$ is uniserial, we obtain that $N = 0$. Therefore $K$ is local. So $Re_i$ is a principal ideal ring. Hence $R$ is an artinian principal ideal ring by [10, 11.6.4.(c)]. This completes the proof.

A module $M$ is called $\oplus$-cofinitely Rad-supplemented if every cofinite submodule $N$ of $M$ has a Rad-supplement that is a direct summand of $M$. Here a submodule $K$ of $M$ is called Rad-supplement of $N$ in $M$ if $M = N + K$ and $N \cap K \leq Rad(K)$ ([13]). Every $\oplus$-cofinitely supplemented module is $\oplus$-cofinitely Rad-supplemented; however the converse is not always true (see [13, Example 2.2]). The following Lemma can be easily proven.

**Lemma 3.5.** Let $N, K \leq M$ be modules and let $K$ be a Rad-supplement of $N$ in $M$. If $Rad(K) \ll K$, then $K$ is a supplement of $N$ in $M$.

**Proposition 3.6.** Every $\oplus$-cofinitely Rad-supplemented module over a max ring is $\oplus$-cofinitely supplemented.
Proof. Let $R$ be a max ring and let $M$ be any $\oplus$-cofinitely Rad-supplemented $R$-module. Suppose that $N$ is a cofinite submodule of $M$. By the hypothesis, there exists a direct summand $K$ of $M$ such that $M = N + K$ and $N \cap K \leq \text{Rad}(K)$. Since $R$ is a max ring, $\text{Rad}(K)$ is small in $K$. Then, by Lemma 3.5, $K$ is a supplement of $N$ in $M$. Hence $M$ is $\oplus$-cofinitely supplemented. □

Using this fact and Corollary 3.4, we obtain the next result.

Corollary 3.7. For a commutative max ring $R$ the following statements are equivalent:

1. $R$ is an artinian principal ideal ring.
2. Every left $R$-module is maximally $\oplus$-supplemented.
3. Every left $R$-module is $\oplus$-cofinitely supplemented.
4. Every left $R$-module is $\oplus$-cofinitely Rad-supplemented.

Proposition 3.8. Let $M$ be a weakly distributive module and $\Phi : M \to N$ be an epimorphism of modules. If $M$ is $\oplus$-cofinitely (Rad-) supplemented module, then $N$ is $\oplus$-cofinitely (Rad-) supplemented.

Proof. The proof is as in the case of maximally $\oplus$-supplemented modules. □

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