CARDINAL INVARIANTS CONNECTED WITH QUOTIENTS OF REAL FUNCTIONS

BY

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Abstract. We study cardinal invariants related to quotients in the case of the complement in \(\mathbb{R}^\mathbb{R}\) of families of continuous, quasi-continuous, cliquish and Darboux functions.

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1. Introduction

The letters \(\mathbb{R}\), \(\mathbb{Q}\) and \(\mathbb{N}\) denote the real line, the set of rationals and the set of positive integers, respectively. The family of all functions from a set \(X\) into \(Y\) is denoted by \(Y^X\). The word function denotes a mapping from \(\mathbb{R}\) to \(\mathbb{R}\) unless otherwise explicitly stated. For each set \(A \subset \mathbb{R}\) the symbol \(\chi_A\) denotes the characteristic function of \(A\). We consider cardinals as ordinals not in one-to-one correspondence with the smaller ordinals. The symbol \(\text{card}\ X\) stands for the cardinality of a set \(X\). We write \(c = \text{card}\ \mathbb{R}\). For each set \(A \subset \mathbb{R}\) the symbols \(\text{int\ } A\) and \(\text{cl\ } A\) denote the interior and the closure of \(A\), respectively. Let \(\kappa\) be a cardinal number such that \(\omega \leq \kappa \leq c\). We say that a set \(A \subset \mathbb{R}\) is \(\text{bilaterally } \kappa\text{-dense in itself}\) if \(\text{card}(A \cap I) = \kappa\) for every nondegenerate interval \(I\) with \(A \cap I \neq \emptyset\).

Let \(f : \mathbb{R} \to \mathbb{R}\). The symbol \(C(f)\) denotes the set of points of continuity of \(f\). For each \(y \in \mathbb{R}\) let \([f = y] = \{x \in \mathbb{R} : f(x) = y\}\). Similarly we define the symbols \([f \neq y]\), \([f > y]\) and \([f < y]\).

The following symbols denote respective classes of functions:
\( \mathcal{D} \) – of all Darboux functions, i.e., \( f \in \mathcal{D} \) iff it has the intermediate value property;

\( \mathcal{C} \) – of all continuous functions;

\( \mathcal{Q} \) – of all quasi-continuous functions in the sense of Kempisty [6], i.e., \( f \in \mathcal{Q} \) iff for each \( x \in \mathbb{R} \) there is a sequence \( (x_n) \subset \mathcal{C}(f) \) such that \( x_n \to x \) and \( f(x_n) \to f(x) \) (see, e.g., [4] or [5, Lemma 2]);

\( \mathcal{C}_q \) – of all cliquish functions [16], i.e., \( f \in \mathcal{C}_q \) iff \( \text{cl} \mathcal{C}(f) = \mathbb{R} \) (see, e.g., [14]).

The following cardinal function has been defined for families \( \mathcal{A} \subset \mathbb{R}^\mathbb{R} \) (see [13]):

\[
a(\mathcal{A}) \overset{\text{df}}{=} \min \left\{ \{ \text{card} \mathcal{F} : \mathcal{F} \subset \mathbb{R}^\mathbb{R} \ \& \ -(\exists g \ \forall f \in \mathcal{F} \ (f + g \in \mathcal{A})) \} \cup \{ (2^\mathbb{C})^+ \} \right\}.
\]

The above value for different classes of real functions has been studied in several papers (see e.g. [2] and [3]).

The following cardinal function connected with quotients of functions has been defined for families \( \mathcal{A} \subset \mathbb{R}^\mathbb{R} \) in [8] (compare also [7] and [12]):

\[
q(\mathcal{A}) \overset{\text{df}}{=} \min \left\{ \{ \text{card} \mathcal{F} : \mathcal{F} \subset \mathcal{A}_A \ \& \ -(\exists g, \forall f \in \mathcal{F} \ (f/g \in \mathcal{A})) \} \cup \{ (\text{card} \mathcal{A}/\mathcal{A})^+ \} \right\},
\]

where \( \mathcal{A}_A \overset{\text{df}}{=} \{ f/g : f, g \in \mathcal{A}, g(x) \neq 0 \text{ for each } x \in \mathbb{R} \} \). In particular, were examined the values of \( q(\mathcal{D}) \) and \( q(\mathcal{Q}) \) (see [7, Theorem] and [8, Theorem 2.7]).

In the above definition, it is quite natural to restrict ourselves to sub-families of \( \mathcal{A}/\mathcal{A} \) only. Indeed, if there is a function \( g \) such that both \( f/g \) and \( 1/g \) are in \( \mathcal{A} \), then \( f \in \mathcal{A}/\mathcal{A} \).

We denote the complement of a family \( \mathcal{A} \subset \mathbb{R}^\mathbb{R} \) by \(-\mathcal{A}\).

In 1996, Jordan [9] examined the values of \( a(-\mathcal{A}) \), where classes \( \mathcal{A} \) are chosen from the classes of Darboux-like functions. Notice that \( a(-\mathcal{A}) \) has the following interpretation:

\( a(-\mathcal{A}) \) is the smallest cardinality of a family \( \mathcal{B} \subset \mathbb{R}^\mathbb{R} \) such that \( \mathcal{A} - \mathcal{B} = \mathbb{R}^\mathbb{R} \), where \( \mathcal{A} - \mathcal{B} = \{ f - g : f \in \mathcal{A} \ \& \ g \in \mathcal{B} \} \).

Similarly, the values of \( q(-\mathcal{A}) \), where \( \mathcal{A} \) is the family of peripherally continuous or closed graph functions, were studied in [10] and [11], respectively.

The purpose of this paper is to find the values of \( q \) for the families \(-\mathcal{C}, -\mathcal{Q}, -\mathcal{C}_q, -\mathcal{D}\). We obtained the following results:
• \( q(¬C) = q(¬Q) = 2^c \) (see Theorem 2.7);

• \( c < q(¬D) \leq 2^c \) (see Theorem 2.4);

• \( \omega < q(¬Cq) \leq \delta \leq c \) (see Corollary 3.4 and Theorem 3.6).

**Remark 1.1.** From [10, Theorem 3.1.] we obtain

**Corollary 1.2.** If \( A \in \{C, Q, Cq\} \), then \( a(¬A) = 2^c \).

Recall that the cardinal \( a(¬D) \) was examined by Jordan [9, Theorem 8].

2. Families \( ¬D, ¬C, ¬Q \)

First, we will find the values of \( q \) for the families \( ¬D, ¬C, ¬Q \). We will need the following lemma.

**Lemma 2.1.** Let \( \kappa \) be a cardinal number such that \( \omega \leq \kappa \leq c \) and assume that \( \{A_\alpha : \alpha < \kappa\} \) is a family of non-empty and \( \kappa \)-dense in itself subsets of \( \mathbb{R} \). There exists a family \( \{A_{\alpha\beta} : \alpha, \beta < \kappa\} \) of pairwise disjoint sets such that for each \( \alpha, \beta < \kappa \) we have \( A_{\alpha\beta} \subset A_\alpha \subset \operatorname{cl} A_{\alpha\beta} \).

**Proof.** Let \( \mathcal{A} \overset{\text{df}}{=} \{(p, q, \alpha, \beta) : p, q \in \mathbb{Q}, \alpha, \beta < \kappa, (p, q) \cap A_\alpha \neq \emptyset\} \). Arrange all elements of \( \mathcal{A} \) in a sequence \( \{\langle p_\gamma, q_\gamma, \alpha_\gamma, \beta_\gamma \rangle : \gamma < \kappa\} \). For each \( \gamma < \kappa \) we choose a point \( x_\gamma \in (p_\gamma, q_\gamma) \cap A_{\alpha_\gamma} \setminus \{x_\delta : \delta < \gamma\} \). For each \( \alpha, \beta < \kappa \) define \( A_{\alpha\beta} \overset{\text{df}}{=} \{x_\gamma : \gamma < \kappa, \alpha_\gamma = \alpha, \beta_\gamma = \beta\} \). We will show that the family \( \{A_{\alpha\beta} : \alpha, \beta < \kappa\} \) has all required properties. Fix \( \alpha, \beta < \kappa \). Clearly \( A_{\alpha\beta} \subset A_\alpha \).

Let \( p, q \in \mathbb{Q} \) and \( (p, q) \cap A_\alpha \neq \emptyset \). Then \( \langle p, q, \alpha, \beta \rangle = \langle p_\gamma, q_\gamma, \alpha_\gamma, \beta_\gamma \rangle \) for some \( \gamma < \kappa \). Since \( x_\gamma \in (p, q) \cap A_{\alpha\beta} \), we have \( (p, q) \cap A_{\alpha\beta} \neq \emptyset \). It follows that \( A_{\alpha\beta} \subset \operatorname{cl} A_{\alpha\beta} \).

Now assume that \( A_{\alpha_\gamma\beta_\gamma} \cap A_{\alpha_\gamma\beta_\gamma'} \neq \emptyset \) for some \( \alpha', \beta', \alpha'', \beta'' < \kappa \). There are \( \gamma', \gamma'' < \kappa \) such that \( x_{\gamma'} = x_{\gamma''}, \alpha_{\gamma'} = \alpha', \beta_{\gamma'} = \beta', \alpha_{\gamma''} = \alpha'' \) and \( \beta_{\gamma''} = \beta'' \). Since the sequence \( \{x_\gamma : \gamma < \kappa\} \) is injective, we have \( \gamma' = \gamma'' \). Thus, \( \alpha' = \alpha_{\gamma'} = \alpha_{\gamma''} = \alpha'' \) and \( \beta' = \beta_{\gamma'} = \beta_{\gamma''} = \beta'' \), and the proof is complete. \( \square \)

**Theorem 2.2.** Let \( \{f_\alpha : \alpha < c\} \subset \mathbb{R} \setminus \{\chi_0\} \). There exists a function \( g : \mathbb{R} \to \mathbb{R} \setminus \{0\} \) such that \( f_\alpha/g \in ¬D \cap ¬Q \) for each \( \alpha < c \).
Proof. We can assume that the set \([f_α \neq 0]\) is \(c\)-dense in itself for each \(α < c\). Indeed, otherwise for each function \(g: \mathbb{R} \to \mathbb{R} \smallsetminus \{0\}\) the set \([f_α/g \neq 0] = [f_α \neq 0]\) is not \(c\)-dense in itself and consequently \(f_α/g \in \neg D \cap \neg Q\).

By Lemma 2.1, there is a family \(\{A_{αi} : α < c, i \in \{0, 1\}\}\) of pairwise disjoint sets such that \(A_{αi} \subset [f_α \neq 0] \subset \text{cl} A_{αi}\) for each \(α < c\) and \(i \in \{0, 1\}\). Clearly we may assume that each set \(A_{αi}\) is countable. Arrange all elements of the set \(\mathbb{R} \smallsetminus \bigcup_{α < c} (A_{α0} \cup A_{α1})\) in a sequence \(\{x_β : β < κ\}\). Now, by transfinite induction we will define a transfinite sequence of real numbers \(\{y_α : α < c\}\) and the function \(g: \mathbb{R} \to \mathbb{R} \smallsetminus \{0\}\).

First, for each \(α < c\) define \(K_α \triangleq \bigcup_{β < α}(A_{β0} \cup A_{β1}) \cup \{x_β : β < \min\{α, κ\}\}\) One can easily see that card \(K_α < c\) for each \(α < c\).

Fix \(α < c\). Assume that for each \(β < α\) a point \(y_β\) is already chosen and the function \(g\) on the set \(K_α\) is already constructed such that:

i) \(y_β \in \mathbb{R} \setminus ((f_β/g)[K_α] \cup \{0\})\),

ii) \(A_{β0} \subset [f_β/g < y_β]\),

iii) \(A_{β1} \subset [f_β/g > y_β]\).

Choose any real number \(y_α\) such that

\[
(1) \quad y_α \in \mathbb{R} \setminus ((f_α/g)[K_α] \cup \{0\})
\]

and define \(g\) on \(K_{α+1} \setminus K_α\) with

\[
(2) \quad y_β \notin (f_β/g)[K_{α+1} \setminus K_α] \quad \text{for each } β ≤ α
\]

and \(A_{α0} \subset [f_α/g < y_α], A_{α1} \subset [f_α/g > y_α]\).

Fix \(β ≤ α\). Observe that, by conditions i) and (1), we obtain \(y_β \in \mathbb{R} \setminus ((f_β/g)[K_α] \cup \{0\})\). Furthermore, by condition (2), we have \(y_β \in \mathbb{R} \setminus ((f_β/g)[K_{α+1}] \cup \{0\})\). Clearly \(A_{β0} \subset [f_β/g < y_β]\) and \(A_{β1} \subset [f_β/g > y_β]\).

Then conditions i)–iii) hold for each \(β ≤ α\).

Fix \(α < c\). First we will show that \(f_α/g \in \neg D\).

Let \(x \in \mathbb{R}\). Since \(\bigcup_{β < c} K_β = \mathbb{R}\) and the sequence \(\{K_β : β < c\}\) is increasing, we have \(x \in K_β\) for some \(α < β < c\). By condition i), we have \(y_α \notin (f_α/g)[K_β]\). It follows that \((f_α/g)(x) \neq y_α\) and, consequently, \([f_α/g = y_α] = \emptyset\). Moreover \([f_α/g < y_α] \neq \emptyset \neq [f_α/g > y_α]\) (see conditions ii) and iii)). This implies that \(f_α/g \notin \neg D\).

Now, we will show that \(f_α/g \in \neg Q\). Let \(x_0 \in [f_α \neq 0] = [f_α/g \neq 0]\). Using the fact that \([f_α/g \neq 0] \subset \text{cl} A_{α0}\) and condition ii) we get that \(\liminf_{x \to x_0} (f_α/g)(x) ≤ y_α\). Similarly, using the fact that \([f_α/g \neq 0] \subset \text{cl} A_{α1}\) and condition iii)
cl $A_{\alpha 1}$ and condition iii) we get that \( \limsup_{x \to x_0} (f_{\alpha}/g)(x) \geq y_\alpha \). Furthermore, since \((f_{\alpha}/g)(x_0) \neq y_\alpha \) we have \( x_0 \notin C(f_{\alpha}/g) \). Hence \( C(f_{\alpha}/g) \subseteq [f_{\alpha}/g = 0] \). Thus, \( f_{\alpha}/g \notin \neg Q \), and the proof is complete. □

Corollary 2.3. If \( A \in \{\neg C, \neg D, \neg Q\} \), then \( A/\mathcal{A} = \mathbb{R}^\mathbb{R} \setminus \{\chi_0\} \).

Proof. Clearly \( \chi_0 \notin A/\mathcal{A} \). Let \( f \in \mathbb{R}^\mathbb{R} \setminus \{\chi_0\} \). Then, by Theorem 2.2, there is a function \( g \) such that \( f/g \in \mathcal{A} \) and \( 1/g \in \mathcal{A} \). Thus \( f = (f/g)/(1/g) \in A/\mathcal{A} \). □

Theorem 2.4. If \( A \in \{\neg C, \neg D, \neg Q\} \), then \( \mathfrak{c} < q(A) \leq 2^\mathfrak{c} \).

Proof. The inequality \( q(A) > \mathfrak{c} \) follows from Theorem 2.2 and Corollary 2.3. Let \( \mathcal{F} \overset{\text{df}}{=} (\mathbb{R} \setminus \{0\})^\mathbb{R} \). Clearly \( \mathcal{F} \subseteq A/\mathcal{A} \) (see Corollary 2.3). If \( g: \mathbb{R} \to \mathbb{R} \setminus \{0\} \), then evidently \( g \in \mathcal{F} \) and \( g/g = \chi_\mathbb{R} \notin A \). Consequently \( q(A) \leq \text{card} \mathcal{F} = 2^\mathfrak{c} \).

Using Theorem 2.4 we obtain

**Theorem 2.5.** If \( c^+ = 2^\mathfrak{c} \), then \( q(\neg D) = 2^\mathfrak{c} \).

The following problem is open

**Problem 1.** Can the equality \( q(\neg D) = 2^\mathfrak{c} \) be proved in ZFC?

Now, we will show that \( q(\neg C) = q(\neg Q) = 2^\mathfrak{c} \). We start with a useful lemma [10, Lemma 3.4.1]:

**Lemma 2.6.** Let \( A \subseteq \mathbb{R} \) and \( \text{card} \ A = \mathfrak{c} \). If \( \mathcal{F} \subseteq (\mathbb{R} \setminus \{0\})^A \) and \( \text{card} \mathcal{F} < 2^\mathfrak{c} \), there is a function \( g: A \to \mathbb{R} \setminus \{0\} \) such that the function \( f/g: A \to \mathbb{R} \setminus \{0\} \) is unbounded for each \( f \in \mathcal{F} \).

In the proof of the following theorem we shall use methods of the proof of [10, Theorem 3.5.1].

**Theorem 2.7.** If \( A \in \{\neg C, \neg Q\} \), then \( q(A) = 2^\mathfrak{c} \)

Proof. The inequality \( q(A) \leq 2^\mathfrak{c} \) follows from Theorem 2.4. Now, we will show that \( q(A) \geq 2^\mathfrak{c} \).

Let \( \mathcal{F} \subseteq A/\mathcal{A} = \mathbb{R}^\mathbb{R} \setminus \{\chi_0\} \) (c.f. Corollary 2.3) and assume that \( \text{card} \mathcal{F} < 2^\mathfrak{c} \). It is enough to show that there is a function \( g: \mathbb{R} \to \mathbb{R} \setminus \{0\} \) such that \( f/g \in A \) for each function \( f \in \mathcal{F} \). First recall that, if \( f \in \mathcal{F} \) and \( \text{cl}[f = 0] = \mathbb{R} \), then \( f/g \in A \) for each \( g: \mathbb{R} \to \mathbb{R} \setminus \{0\} \). So we may assume that \( \text{cl}[f = 0] \neq \mathbb{R} \) for each \( f \in \mathcal{F} \).
Choose a partition \( \{ S_\alpha : \alpha < c \} \) of \( \mathbb{R} \) into pairwise disjoint \( c \)-dense sets and let \( \{ I_\alpha : \alpha < c \} \) be an enumeration of the open intervals in \( \mathbb{R} \). Let \( A_\alpha \overset{\text{df}}{=} S_\alpha \cap I_\alpha \). Note that \( \text{card} \, A_\alpha = c \) and \( A_\alpha \cap A_\beta = \emptyset \) for \( \alpha < \beta < c \). Fix \( \alpha < c \). Let \( \mathcal{F}_\alpha \overset{\text{df}}{=} \{ f | A_\alpha : f \in \mathcal{F} \, \& \, A_\alpha \subset [f \neq 0] \} \). Evidently \( \text{card} \, \mathcal{F}_\alpha < 2^c \).

By Lemma 2.6 there is some \( g_\alpha : A_\alpha \rightarrow \mathbb{R} \setminus \{0\} \) such that \( f/g_\alpha \) is not bounded on \( A_\alpha \) for every \( f \in \mathcal{F} \) with \( A_\alpha \subset [f \neq 0] \). Let \( g : \mathbb{R} \rightarrow \mathbb{R} \setminus \{0\} \) extend \( \bigcup \{ g_\alpha : \alpha < c \} \). Observe that for each \( f \in \mathcal{F} \) there is a nondegenerate interval \( I_f \) such that \( C((f/g)|_{I_f}) = 0 \). Thus, \( f/g \in A \) for every \( f \in \mathcal{F} \) and the proof is complete. \( \square \)

### 3. The family \(-C_q\)

Now, we will examine the value of \( q(-C_q) \).

Denote by \( \mathcal{B} \) the family of all functions \( f \in \mathbb{R}^\mathbb{R} \) such that the set \( [f \neq 0] \) is not nowhere dense, i.e. \( \text{int} \, \text{cl}[f \neq 0] \neq \emptyset \). We start with a simple proposition.

**Proposition 3.1.** \(-C_q/-C_q \subset \mathcal{B}\).

**Proof.** Assume that there is a function \( f \in -C_q/-C_q \setminus \mathcal{B} \). Let \( f = g/h \), where \( g, h \in -C_q \). Since \( f \notin \mathcal{B} \) and \( [f \neq 0] = [g \neq 0] \), we have \( \text{cl} \text{int}[g = 0] = \mathbb{R} \). Consequently \( g \in C_q \), an impossibility. \( \square \)

**Theorem 3.2.** Let \( f_1, f_2, \ldots \in \mathcal{B} \). There exists a function \( g : \mathbb{R} \rightarrow \mathbb{R} \setminus \{0\} \) such that \( f_n/g \in -C_q \), for each \( n \in \mathbb{N} \).

**Proof.** For each \( n \in \mathbb{N} \) choose an open interval \( I_n \subset \text{cl}[f_n \neq 0] \). By Lemma 2.1, there is a family \( \{ A_{nk} : n \in \mathbb{N}, k \in \{0,1\} \} \) of pairwise disjoint sets such that \( A_{nk} \subset I_n \cap [f_n \neq 0] \subset \text{cl} A_{nk} \) for each \( n \in \mathbb{N} \) and \( k \in \{0,1\} \). Define a function \( g : \mathbb{R} \rightarrow \mathbb{R} \setminus \{0\} \) by formula

\[
g(x) = \begin{cases} (-1)^k f_n(x), & \text{if } x \in A_{nk}, \; n \in \mathbb{N}, \; k \in \{0,1\}, \\ 1, & \text{otherwise}. \end{cases}
\]

Fix \( n \in \mathbb{N} \). Notice that for each \( k \in \{0,1\} \), \( I_n \subset \text{cl} I_n = \text{cl}(I_n \cap \text{cl}[f_n \neq 0]) \subset \text{cl} \text{cl} A_{nk} = \text{cl} A_{nk} \). Moreover \( f_n/g = 1 \) on \( A_{n0} \) and \( f_n/g = -1 \) on \( A_{n1} \). Consequently \( C(f_n/g) \cap I_n = \emptyset \) and \( f_n/g \in -C_q \).

The next corollary follows from Proposition 3.1 and Theorem 3.2 (see also the proof of Corollary 2.3).
Corollary 3.3. \(-C_q/\neg C_q = B\).

From Corollary 3.3 and Theorem 3.2 we obtain the following

Corollary 3.4. \(q(\neg C_q) > \omega\).

Now, we need the following cardinal: \(d \overset{\text{df}}{=} \min\{\text{card} \ F : F \subset \omega^\omega, \forall g \in \omega^\omega, \exists f \in F \ (g \leq f)\}\). It is well-known that \(\omega < d \leq c\) and it is consistent with ZFC that \(d = \omega_1\) and \(c = \omega_2\) (see [1] or [15]).

Recall also the following Lemma [11, Lemma 2.6]

Lemma 3.5. There exists a family \(F \subset \mathbb{R}^\mathbb{R}\) of cardinality \(d\) such that:

a) \([f \neq 0] = \mathbb{Q}\) for each function \(f \in F\),

b) for each function \(g: \mathbb{R} \to \mathbb{R} \setminus \{0\}\) there exists a function \(f \in F\) such that \(\lim_{t \to x} (f/g)(t) = 0\) for each \(x \in \mathbb{R}\).

Theorem 3.6. \(q(\neg C_q) \leq d\).

Proof. Let \(F\) be a family of functions defined in Lemma 3.5. By Corollary 3.3 and condition a) of Lemma 3.5, we have \(F \subset \neg C_q/\neg C_q\).

Fix \(g: \mathbb{R} \to \mathbb{R} \setminus \{0\}\). By condition b) of Lemma 3.5, there exists a function \(f \in F\) such that \(\lim_{t \to x} (f/g)(t) = 0\) for each \(x \in \mathbb{R}\). Moreover by condition a) of Lemma 3.5 we have \([f/g = 0] = \mathbb{R} \setminus \mathbb{Q}\). Consequently \(C(f/g) = \mathbb{R} \setminus \mathbb{Q}\) and \(f/g \in C_q\). It follows that \(q(\neg C_q) \leq \text{card} \ F = d\). \(\square\)

Using Corollary 3.4, Theorems 3.6 and the inequality \(d \leq c\), we conclude that

Theorem 3.7. The Continuum Hypothesis implies \(q(\neg C_q) = c\).

The following problem is open:

Problem 2. Can the inequality \(q(\neg C_q) < d\) be a consequence of ZFC?

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