PERMANENT WEAK MODULE AMENABILITY OF SEMIGROUP ALGEBRAS

BY

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Abstract. We employ the fact that $L^1(G)$ is $n$-weakly amenable for each $n \geq 1$ to show that for an inverse semigroup $S$ with the set of idempotents $E$, $\ell^1(S)$ is $n$-weakly module amenable as an $\ell^1(E)$-module with trivial left action. We study module amenability and weak module amenability of the module projective tensor products of Banach algebras.

Mathematics Subject Classification 2010: 43A20, 46H25.

Key words: Banach modules, module derivation, $n$-weak module amenability, inverse semigroup, module projective tensor product.

1. Introduction

A Banach algebra $A$ is amenable if $H^1(A, X^*) = \{0\}$ for every Banach $A$-module $X$, where $H^1(A, X^*)$ is the first Hochschild cohomology group of $A$ with coefficients in $X^*$. The notion is introduced by Johnson in [17]. Dales et al introduced the notion of $n$-weak amenability of Banach algebras in [12]. A Banach algebra $A$ is $n$-weakly amenable if $H^1(A, A^{(n)}) = \{0\}$, where $A^{(n)}$ is $n$th dual space of $A$ (1-weak amenability is called weak amenability). A Banach algebra is called permanently weakly amenable if it is $n$-weakly amenable for each positive integer $n$. It is well known that for any locally compact group $G$, $L^1(G)$ is $n$-weakly amenable whenever $n \in \mathbb{N}$ (see [10], [12] and [18]). As for $A \hat{\otimes} B$, for amenable Banach algebras $A$ and $B$, that is always amenable (see [17]). However, $A \hat{\otimes} B$ is weakly amenable when $A$ and $B$ are commutative and weakly amenable (see [15]).

In [3], the second author and Bagha extended the notion of weak amenability for a Banach algebra $A$ to the case that there is an extra $A$-module
structure on \(\mathcal{A}\) and showed that \(\ell^1(S)\) is weakly module amenable, as an \(\ell^1(E)\)-module, when \(S\) is a commutative inverse semigroup with the set of idempotents \(E\). The same is true for an arbitrary inverse semigroup with trivial left action (see [5]). Also Bodaghi et al in [8] showed that \(\ell^1(S)\) is \(n\)-weakly module amenable as an \(\ell^1(E)\)-module (with trivial left action) when \(n\) is odd. It is proved in [4] that the module projective tensor product \(\ell^1(S)\hat{\otimes}_{\ell^1(E)}\ell^1(S)\) is module amenable when \(S\) is amenable (the module contractibility case is shown in [6]).

In this paper, we investigate \(n\)-weak module amenability of semigroup algebras and using the fact that for a locally compact group \(G\), \(L^1(G)\) is \(n\)-weakly amenable for all \(n \in \mathbb{N}\), we show that the inverse semigroup algebra \(\ell^1(S)\) is \(n\)-weakly module amenable as an \(\ell^1(E)\)-module for all \(n \in \mathbb{N}\). We also investigate module amenability and weak module amenability of the module projective tensor product \(\mathcal{A}\hat{\otimes}\mathcal{B}\) (as \(\mathfrak{A}\)-module) under certain conditions. In particular, we show that \(\ell^1(S)\hat{\otimes}_{\ell^1(E)}\ell^1(S)\) is weakly module amenable.

2. \(n\)-weak module amenability of semigroup algebras

Let \(\mathfrak{A}\) and \(\mathcal{A}\) be Banach algebras such that \(\mathcal{A}\) is a Banach \(\mathfrak{A}\)-bimodules with following compatible actions \(\alpha \cdot (ab) = (\alpha \cdot a)b, (ab) \cdot \alpha = a(b \cdot \alpha)\), for every \(a, b \in \mathcal{A}, \alpha \in \mathfrak{A}\). If \(\mathfrak{A}\) is a unital Banach algebra, the Banach algebra \(\mathcal{A}\) is said to be a unital Banach \(\mathfrak{A}\)-module if \(e_{\mathfrak{A}} \cdot a = a \cdot e_{\mathfrak{A}} = a\) for every \(a \in \mathcal{A}\). Let \(\mathcal{A}\) and \(\mathcal{B}\) be Banach \(\mathfrak{A}\)-bimodules with compatible actions. An \(\mathfrak{A}\)-module map is a mapping \(\varphi : \mathcal{A} \rightarrow \mathcal{B}\) with following properties

1. \(\varphi(a \pm b) = \varphi(a) \pm \varphi(b)\) (\(a, b \in \mathcal{A}\));

2. \(\varphi(\alpha \cdot a) = \alpha \cdot \varphi(a), \varphi(a \cdot \alpha) = \varphi(a) \cdot \alpha\) (\(a \in \mathcal{A}, \alpha \in \mathfrak{A}\)).

One should note that \(\varphi\) is not linear, in general. Let \(X\) be a Banach \(\mathcal{A}\)-bimodule and a Banach \(\mathfrak{A}\)-bimodule with following compatible actions: \(\alpha \cdot (a \cdot x) = (\alpha \cdot a) \cdot x, a(\alpha \cdot x) = (a \cdot \alpha) \cdot x, (\alpha \cdot x) \cdot a = \alpha \cdot (x \cdot a), (a \cdot x) \cdot \alpha = a \cdot (x \cdot \alpha), (a \cdot \alpha) \cdot x = a \cdot (\alpha \cdot x), (x \cdot a) \cdot \alpha = x \cdot (a \cdot \alpha)\), for every \(x \in X, a \in \mathcal{A}, \alpha \in \mathfrak{A}\). In this case, we say \(X\) is a Banach \(\mathcal{A}\)-\(\mathfrak{A}\)-module. If \(\alpha \cdot x = x \cdot \alpha\), for every \(x \in X\) and \(\alpha \in \mathfrak{A}\), then \(X\) is called a commutative Banach \(\mathcal{A}\)-\(\mathfrak{A}\)-module. Moreover, if \(a \cdot x = x \cdot a\), for every \(x \in X\) and \(a \in \mathfrak{A}\), then \(X\) is called a bi-commutative Banach \(\mathcal{A}\)-\(\mathfrak{A}\)-module. It is clear that \(\mathcal{A}\) is a Banach \(\mathcal{A}\)-\(\mathfrak{A}\)-module. Also, if \(\mathcal{A}\) is a commutative \(\mathfrak{A}\)-bimodule, then \(\mathcal{A}\)
is a commutative $\mathcal{A}$-$\mathfrak{A}$-module, and so is the $n$-th dual of $\mathcal{A}$. If moreover $\mathcal{A}$ is a commutative Banach algebra, then it is a bi-commutative Banach $\mathcal{A}$-$\mathfrak{A}$-module, and the same holds for its $n$-th dual.

An $\mathfrak{A}$-module map $D : \mathcal{A} \rightarrow X$ is called a module derivation if $D(ab) = a \cdot D(b) + D(a) \cdot b$, ($a, b \in \mathcal{A}$). A module derivation $D$ is called bounded if there exists $M > 0$ such that $\|D(a)\| \leq M\|a\|$, for every $a \in \mathcal{A}$. Note that boundedness of $D$ implies its norm continuity while $D$ can be non-linear. We use the notations $Z_{\mathfrak{A}}(\mathcal{A}, X)$ and $N_{\mathfrak{A}}(\mathcal{A}, X)$ for the set of all continuous module derivations and continuous inner module derivations from $\mathcal{A}$ to $X$, respectively. Also the quotient space $Z_{\mathfrak{A}}(\mathcal{A}, X)/N_{\mathfrak{A}}(\mathcal{A}, X)$ (which we call the first $\mathfrak{A}$-module cohomology group of $\mathcal{A}$ with coefficients in $X$) is denoted by $H^1_{\mathfrak{A}}(\mathcal{A}, X)$. From now on, by a module derivation we mean a continuous module derivation.

The Banach algebra $\mathcal{A}$ is called module amenable (as an $\mathfrak{A}$-module) if for any commutative Banach $\mathcal{A}$-$\mathfrak{A}$-module $X$, each module derivation $D : \mathcal{A} \rightarrow X^*$ is inner [1], in other word, $\mathcal{A}$ is module amenable if $H^1_{\mathfrak{A}}(\mathcal{A}, X^*) = \{0\}$, for each commutative Banach $\mathcal{A}$-$\mathfrak{A}$-module $X$ (see [1]) and it is super module amenable (module contractible) if $H^1_{\mathfrak{A}}(\mathcal{A}, X) = \{0\}$, for each commutative Banach $\mathcal{A}$-$\mathfrak{A}$-module $X$ (see [6, 21]). Also $\mathcal{A}$ is weakly module amenable (as an $\mathfrak{A}$-module) if for any subset $Y$ of $\mathcal{A}^*$ which is $\mathcal{A}$-submodule and commutative Banach $\mathfrak{A}$-submodule, each module derivation from $\mathcal{A}$ to $Y$ is inner [5]. It is proved in [5] that $\mathcal{A}$ is weakly module amenable if every module derivation from $\mathcal{A}$ to $J^\perp$ is inner when $\mathcal{A}/J$ is a commutative $\mathcal{A}$-$\mathfrak{A}$-module in which $J$ is the closed ideal of $\mathcal{A}$ generated by elements of the form $(a \cdot \alpha)b - a(\alpha \cdot b)$ for $\alpha \in \mathfrak{A}, a, b \in \mathcal{A}$. Note that if $\mathcal{A}$ is a commutative Banach algebra and bi-commutative $\mathcal{A}$-$\mathfrak{A}$-module then the above definition coincides with definition 2.1 of [3].

Let $G$ be a locally compact and $X$ be a Banach space. Suppose that $G$ acts on $X$ by homomorphisms, i.e., we have the continuous mappings $(s, x) \mapsto s \cdot x$ from $G \times X$ into $X$ and $(x, s) \mapsto x \cdot s$ from $X \times G$ into $X$ such that $(s \cdot t) \cdot x = s \cdot (t \cdot x)$, $s \cdot (x \cdot t) = (s \cdot x) \cdot t$, $x \cdot (s \cdot t) = x \cdot (s \cdot t)$, $(s, t \in G, x \in X)$. A map $\partial : G \rightarrow X$ is called a $G$-derivation if $\partial(st) = s \cdot \partial(t) + \partial(s) \cdot t$, for every $s, t \in G$. The $G$-derivation $\partial$ is called inner if there exists $x \in X$ such that $\partial(s) = s \cdot x - x \cdot s$, for every $s \in S$. In this case we write $\partial = ad_x$. A map $\psi : G \rightarrow X$ is called a crossed homomorphism if $\psi(st) = s \cdot \psi(t) \cdot s^{-1} + \psi(s)$, for every $s, t \in G$, and $\psi$ is called principal if there exists $x \in X$ such that $\psi(s) = s \cdot x \cdot s^{-1} - x$, for every $s \in G$. Let $\partial : G \rightarrow X$ be a $G$-derivation, and set $\psi(s) = \partial(s) \cdot s^{-1}$, for $s \in G$. Then $\psi$ is a crossed homomorphism,
Let \( \psi : G \to X \) be a crossed homomorphism. Set \( \delta(s) = \psi(s) \cdot s \) for \( s \in G \). Then \( \delta \) is a \( G \)-derivation, and \( \delta \) is inner if \( \psi \) is principal. Let \( D : \ell^1(G) \to X^* \) be continuous derivation. Set \( \delta(s) = D(\delta_s) \) for every \( s \in G \). Then \( \delta \) is a \( G \)-derivation, and it is clear that if \( D \) is an inner derivation then so is \( \delta \).

Choose \( \delta \)-continuous \( \alpha \)-homomorphism. Set \( \psi = \alpha(\delta) \) for some \( \alpha \). Similarly \( \ell^1 = \alpha(\delta) = \alpha(\delta_1) = \alpha(\delta_2) \).

\[ \ell^1(G) \ni \ell \to (\alpha(\delta) \cdot \ell)_G \]

Let \( \ell^1 = \alpha(\delta)_G \).

Theorem 2.1. Let \( G \) be a discrete group. Then \( \ell^1(G) \) is \( n \)-weakly amenable for all \( n \in \mathbb{N} \).

Proof. The odd case is proved in [12, Theorem 4.1] and [18]. Therefore it is sufficient to prove the result in the even case (compare with [10]). Let \( D : \ell^1(G) \to \ell^1(G)(2n) \) be a bounded derivation. Since \( G \) is discrete, the group algebra \( \ell^1(G) \) has an identity. Hence \( \ell^1(G) = M(G) \). Put \( \Omega_0 = G \). On the other hand, \( \ell^\infty(G) \) is a commutative unital \( C^* \)-algebra so by Gelfand-Naimark Theorem we have \( \ell^\infty(G) \cong C(\Omega_1) \), where \( \Omega_1 \) is \( w^* \)-compact. Similarly \( \ell^\infty(G)^{**} \cong C(\Omega_2) \), for a \( w^* \)-compact space \( \Omega_2 \). Now if \( n \geq 2 \) and \( \ell^1(G) \cong M(\Omega_{n-1}) \), for some \( w^* \)-compact space \( \Omega_{n-1} \), then \( \ell^1(G)(2n) \cong \ell^1(G)(2n-2) \cong C(\Omega_{n-2}) \), for some \( w^* \)-compact space \( \Omega_{n-2} \). Let us show that \( G \) acts on \( \Omega_1 \) by homeomorphisms. Each \( x \in G \) induces a map \( \alpha_x : \Omega_1 \to \Omega_1 \) defined by \( \alpha_x(\omega)(f) = \omega(\ell_x f) \) for \( f \in \ell^\infty(G) \), where \( \ell_x \) is the left translation by \( x \). It is clear that if \( \omega \) is a continuous character on \( \ell^\infty(G) \), so is \( \omega_x \) and \( \alpha_x \) is \( w^* \)-continuous. Also \( \alpha_x^{-1} = \alpha_{x^{-1}} \) and \( \alpha_x \) is an \( \alpha_x \)-homomorphism. Similarly \( G \) acts on \( \Omega_n \) by homeomorphisms and \( M(\Omega_n) \) is a Banach \( \ell^1(G) \)-module and the above isomorphism of Banach spaces is indeed an isomorphism of Banach \( \ell^1(G) \)-modules. By Johnson’s Theorem ([11, Theorem 5.6.39]), \( D' = D(\delta) \) defines a bounded crossed homomorphism \( \psi \) from \( G \) into \( M(\Omega_n) \), and by Theorem 1.1 of [20], \( \psi \) is principal, and therefore \( D \) is inner.

Johnson and Ringrose in [19] showed that for a discrete group \( G \), \( H^1(\ell^1(G), \ell^1(G)) = \{0\} \). This is not true for arbitrary locally compact groups. Let \( G \) be an infinite compact (which is unimodular), noncommutative group. Choose \( x \in G \) which is not in the center of \( G \), then the continuous derivation \( D : L^1(G) \to L^1(G) : f \mapsto \delta_x \ast f - f \ast \delta_x \) is not inner. Indeed, if \( D = ad_g \) for some \( g \in L^1(G) \), then take a local basis \( U_a \) of identity.
element $e$ such that $xU_a$ is not equal to $(U_a)x$ ($x$ is not in the center), and let $f_a$ be the characteristic function of $U_a$ divided by Haar measure of $U_a$, then $f_a$ is a bounded approximate identity, and if we consider $D(f_a) = ad_g(f_a)$ and let $a \rightarrow \infty$ then the right hand side goes to zero, but the left hand side does not. However, if $D : L^1(G) \rightarrow L^1(G)$ is a bounded derivation then a similar argument as in the proof of the above theorem shows that $D = ad_\mu$ for some $\mu \in M(G)$ and $ad_\mu - ad_\nu$ is an inner derivation on $L^1(G)$ if and only if $\mu - \nu \in L^1(G)$, where $L^1(G)$ is considered as a closed ideal of $M(G)$.

When $n \geq 1$, as in the above argument we get $L^1(G)^{(2n)} = M(\Omega_n)$ for some $G$-space $\Omega_n$ and Johnson’s Theorem ([11, Theorem 5.6.39]) implies that $D$ is inner. Summing up:

**Theorem 2.2.** Let $G$ be a locally compact group. Then $H^1(L^1(G), L^1(G)) = M(G)/L^1(G)$ and $H^1(L^1(G), L^1(G)^{(2n)}) = \{0\}$, for $n \geq 1$.

Recall that a left Banach $A$-module $X$ is called a right essential $A$-module if the linear span of $X \cdot A = \{x \cdot a : a \in A, x \in X\}$ is dense in $X$. Left essential $A$-modules and (two-sided) essential $A$-bimodules are defined similarly. The following result is proved in [8, Theorem 3.14] when $n$ is an odd number but essentially the same proof works for an even natural number. So we have:

**Theorem 2.3.** Let $n \in \mathbb{N}$. Let $A/J$ has a left or right identity and $\mathfrak{A}$ acts trivially on $A$ from left. If $A$ is $n$-weakly module amenable, then $A/J$ is $n$-weakly amenable. The converse is true if $A$ is a right essential $\mathfrak{A}$-module.

A semigroup $S$ is called an inverse semigroup if for each $s \in S$ there exists unique $s^* \in S$ with $ss^*s = s$, $ss^* = s^*$. More details on inverse semigroups may be found in [16]. The mapping $s \mapsto s^*$ is an involution on $S$, i.e. $s^{**} = s$ and $(st)^* = t^*s^*$ for all $s, t \in S$. We denote the set of idempotents in $S$ by $E$. Each idempotent of $S$ is self-adjoint, and $E$ is a commutative idempotent subsemigroup of $S$ and a semilattice. In particular, $\ell^1(E)$ is a subalgebra of $\ell^1(S)$. We can consider $\ell^1(S)$ as a Banach module over $\ell^1(E)$ with trivial left action and multiplication as the right action (see [1]), that is $\delta_e \cdot \delta_s = \delta_s$, $\delta_s \cdot \delta_e = \delta_s \ast \delta_e = \delta_{se}$, $(s \in S, e \in E)$.

Considering $\mathcal{A} = \ell^1(S)$ and $\mathfrak{A} = \ell^1(E)$ in the above Theorem, $J$ is the closed ideal of $\ell^1(S)$ generated by $\{\delta_{se} - \delta_{st} : s, t \in S, e \in E_S\}$. We consider an equivalence relation on $S$ as follows: $s \approx t \iff \delta_s - \delta_t \in J$, for every
For an inverse semigroup $S$, the quotient $S/ \approx$ is a discrete group (see [2] and [21]), and by Theorem 3.3 of [22], we have $\ell^1(S)/J \cong \ell^1(S/ \approx)$.

**Corollary 2.4.** Let $n \in \mathbb{N}$ and let $S$ be an inverse semigroup with the set of idempotents $E$. Then $\ell^1(S)$ is $n$-weakly module amenable as an $\ell^1(E)$-module with trivial left action.

**Proof.** The result is proved in Theorem 3.15 of [8] when $n$ is odd. In the above action of $\ell^1(E)$ on $\ell^1(S)$, $\ell^1(S)$ is a right essential $\ell^1(E)$-module (see the proof of [8, Theorem 3.15]). According to above statement, $S/ \approx$ is a discrete group, so Theorem 2.1 implies that $\ell^1(S/ \approx)$ is $2n$-weakly module amenable. Now by applying Theorem 2.3, $\ell^1(S)$ is $2n$-weakly module amenable. □

It is well known that amenability of $\ell^1(S)$ implies amenability of $S$ (see [11, Theorem 5.6.1]). In general, $\ell^1(S)$ is not even weakly amenable if $S$ is amenable (a concrete example is bicyclic inverse semigroup). However, if $S$ is inverse semigroup, then $\ell^1(S)$ is amenable if and only if $S$ has only finitely many idempotents and every subgroup of $S$ is amenable (see [13]). In the case that $S$ is commutative:

(i) $\ell^1(S)$ is amenable if and only if $S$ is a finite semilattice of amenable groups (see [14]);

(ii) If every element of $S$ is idempotent, then $\ell^1(S)$ is spanned by its idempotents and so it is weakly amenable by [11, Proposition 2.8.72].

Let $S$ be a Clifford semigroup (an inverse semigroup whose idempotents are central). Then $H^1(\ell^1(S), \ell^\infty(S)) = \{0\}$ [9, Theorem 2.1], but $H^1_{\ell^1(E)}(\ell^1(S), \ell^1(S))$ is not zero in general. We also have $H^1(\ell^1(S), \ell^\infty(S)) = \{0\}$ when $S$ has only finitely many idempotents [9, Theorem 3.2]. Consider $\ell^1(S)$ as a $\ell^1(E)$-module with the following action:

\[ \delta_e \cdot \delta_s = \delta_s \cdot \delta_e = \delta_s \ast \delta_e = \delta_{se}. \]

Since every idempotent commutes with the elements of $S$, the proof of [8, Theorem 3.15] shows that $\ell^1(S)$ an essential $\ell^1(E)$-module. Now, it follows from the proof of [5, Theorem 3.14] that every module derivation $D$ from $\ell^1(S)$ into $\ell^\infty(S)$ is inner and thus we have the following result:

**Theorem 2.5.** If $S$ is a Clifford semigroup, then $\ell^1(S)$ is weakly module amenable.
It is proved in [3, Theorem 3.1] that \( \ell^1(S) \) is \( \ell^1(E) \)-weakly module amenable with the actions (2.1) when \( S \) is commutative. Note that in the proof of this result the commutativity of \( S \) is not needed and the same proof works if each idempotent is central.

Let \( C \) be the bicyclic inverse semigroup generated by \( a \) and \( b \), that is \( C = \{a^m b^n : m, n \geq 0\} \); \((a^m b^n)^* = a^n b^m\), \( E_C = \{a^n b^n : n = 0, 1, \ldots\}\). It is shown in [2] that \( \ell^1(C) \) is \( \ell^1(E_C) \)-module amenable, and so it is weakly module amenable. Now, it follows from [9, Theorem 3.6] and the above discussions that \( H_{\ell^1(E_C)}(\ell^1(C), \ell^1(C)) = \{0\} \).

The Brandt inverse semigroup corresponding to group \( G \) and non-empty set \( I \) is denoted by \( S = M(G, I) \) which is the collection of all \( I \times I \) matrices \((g)_{jk} \) with \( g \in G \) in the \((j, k)^{th}\) place and zero elsewhere and the \( I \times I \) zero matrix 0. It is shown in [21, Example 3.2] that \( \ell^1(S) \) is super module amenable (as \( \ell^1(E) \)-module) and so \( H_{\ell^1(E)}(\ell^1(S), \ell^1(S)) = \{0\} \).

**Lemma 2.6.** Let \( A \) be a essential bi-commutative \( A \)-\( \mathfrak{A} \)-module. Then \( A \) is weakly module amenable (as an \( \mathfrak{A} \)-module) if and only if for each bi-commutative Banach \( A \)-\( \mathfrak{A} \)-module \( X \), all bounded module derivations from \( A \) into \( X \) are zero.

**Proof.** We follow the standard argument in [11, Theorem 2.8.63]. Assume that there exists \( D \in \mathfrak{Z}_\mathfrak{A}(A, X) \) with \( D \neq 0 \). Since \( \mathfrak{A}^2 = \mathfrak{A} \), there exists \( a_0 \in A \) such that \( D(a_0^2) \neq 0 \). We have \( a_0 \cdot D(a_0) \neq 0 \) and thus \( f \in X^* \) with \( f(a_0 \cdot D(a_0)) = 1 \). Set \( R : X \rightarrow A^* \) defined by \( R(x)(a) = f(a \cdot x) \) where \( a \in A \), \( x \in X \). It is easy to check that \( R \circ D \in \mathfrak{Z}_\mathfrak{A}(A, A^*) \). We get \( (R \circ D(a_0), a_0) = (f, a_0 \cdot D(a_0)) = 1 \), and so \( R \circ D \neq 0 \). This shows that \( A \) is not weakly module amenable. The converse is clear. \( \square \)

**Theorem 2.7.** Let \( n \in \mathbb{N} \) and let \( S \) be a commutative inverse semigroup with the set of idempotents \( E \). Then \( \ell^1(S) \) is \( n \)-weakly module amenable as an \( \ell^1(E) \)-module with the actions (2.1).

**Proof.** For any semigroup \( S \), the semigroup algebra \( \ell^1(S) \) is commutative if and only if \( S \) is commutative. Since \( \ell^1(S) \) is a bi-commutative Banach \( \ell^1(S)-\ell^1(E) \)-module, so is \( \ell^1(S)^{(n)} \). By [3, Theorem 3.1], \( \ell^1(S) \) is weakly module amenable as an \( \ell^1(E) \)-module. The semigroup algebra \( \ell^1(S) \) is essential, in fact \( \ell^1(S) = \ell^1(S) \star \ell^1(E) \subseteq \ell^1(S) \star \ell^1(S) \subseteq \ell^1(S) \) (see the proof of [8, Theorem 3.15]). Now, it follows from Lemma 2.6 that every module derivation from \( \ell^1(S) \) into \( \ell^1(S)^{(n)} \) is zero. This shows that \( \ell^1(S) \) is \( n \)-weakly module amenable. \( \square \)
Corollary 2.8. Let $S$ be a commutative inverse semigroup with the set of idempotents $E$. Then $H^1_{\ell^1(E)}(\ell^1(S),\ell^1(S)) = \{0\}$.

3. Module amenability and weak module amenability

Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras and Banach $\mathfrak{A}$-bimodules. Consider the Banach space $\mathcal{A} \hat{\otimes} \mathcal{B} = \mathcal{A} \hat{\otimes} \mathcal{B}/N$, where $N$ is the closed linear span of $\{a \cdot \alpha \otimes b - a \otimes \alpha \cdot b \mid \alpha \in \mathfrak{A}, a \in \mathcal{A}, b \in \mathcal{B}\}$. By the following actions $\mathcal{A} \hat{\otimes} \mathcal{B}$ becomes a Banach $\mathfrak{A}$-bimodule

$$\alpha \cdot (a \otimes b) = \alpha \cdot a \otimes \mathfrak{A} b$$

and

$$(a \otimes \mathfrak{A} b) \cdot \alpha = a \otimes \mathfrak{A} b \cdot \alpha,$$

for every $\alpha \in \mathfrak{A}, a \in \mathcal{A}$ and $b \in \mathcal{B}$. By the above actions $N$ becomes $\mathfrak{A}$-submodule of $\mathcal{A} \hat{\otimes} \mathcal{B}$. We define multiplication on $\mathcal{A} \hat{\otimes} \mathcal{B}$ by usual algebraic tensor product on rings and modules as follows:

$$(a \otimes \mathfrak{A} b)(c \otimes \mathfrak{A} d) = (a \otimes b + N)(c \otimes d + N) = ac \otimes bd + N = ac \otimes \mathfrak{A} bd,$$

for every $a,c \in \mathcal{A}$ and $b,d \in \mathcal{B}$. Also, $\mathcal{A} \hat{\otimes} \mathcal{B}$ is a Banach $\mathcal{A}$-bimodule by following actions

$$(3.1) \quad a \cdot (a' \otimes b') = aa' \otimes b \quad \text{and} \quad (a' \otimes b') \cdot a = a'a \otimes b',$$

for every $a,a' \in \mathcal{A}$ and $b,b' \in \mathcal{B}$. Similarly we can see $\mathcal{A} \hat{\otimes} \mathcal{B}$ as a Banach $\mathcal{B}$-bimodule. We call $\mathcal{A} \hat{\otimes} \mathcal{B}$ the module projective tensor product of $\mathcal{A}$ and $\mathcal{B}$.

Module amenability of module projective tensor product of Banach algebras and semigroup algebras are studied in [4] and [7], respectively. We say that $X$ is a Banach $\mathcal{A}$-$\mathcal{B}$-$\mathfrak{A}$-module if $X$ is a Banach $\mathcal{A}$-$\mathfrak{A}$-module and a Banach $\mathcal{B}$-$\mathfrak{A}$-module.

**Theorem 3.1.** Let $\mathcal{A}$ and $\mathcal{B}$ be unital Banach $\mathfrak{A}$-modules. If $\mathcal{A}$ and $\mathcal{B}$ are module amenable then every continuous module derivation from $\mathcal{A} \hat{\otimes} \mathcal{B}$ into $X^*$ is inner, where $X$ is a Banach $\mathcal{A}$-$\mathcal{B}$-$\mathfrak{A}$-module.

**Proof.** Let $D : \mathcal{A} \hat{\otimes} \mathcal{B} \rightarrow X^*$ be a continuous module derivation. Set $\mathcal{A}_1 = \mathcal{A} \hat{\otimes} e_\mathcal{B}$ and $\mathcal{B}_1 = e_\mathcal{A} \hat{\otimes} \mathcal{B}$. Then the map $D_1 : \mathcal{A} \rightarrow X^*$ defined by $D_1(a) = D(a \otimes e_\mathcal{B} + N)$ is a continuous module derivation, hence there
exists \( f \in X^* \) such that \( D(a \otimes e_B + N) = (a \otimes e_B + N) \cdot f - f \cdot (a \otimes e_B + N) = D_f(a \otimes e_B + N), (a \in A) \).

It is clear that \( D - D_f|_{A_1} = 0 \). Therefore for every \( a \in A \) and \( b \in B \) we have

\[
(D - D_f)(a \otimes b + N) = (D - D_f)((a \otimes e_B + N)(e_A \otimes b + N))
\]

\[
= (a \otimes e_B + N) \cdot (D - D_f)(e_A \otimes b + N)
\]

\[
+ (D - D_f)(a \otimes e_B + N) \cdot (e_A \otimes b + N)
\]

(3.2)

\[
= (a \otimes e_B + N) \cdot (D - D_f)(e_A \otimes b + N)
\]

\[
= (D - D_f)((e_A \otimes b + N)(a \otimes e_B + N))
\]

\[
= (e_A \otimes b + N) \cdot (D - D_f)(a \otimes e_B + N)
\]

\[
+ (D - D_f)(e_A \otimes b + N) \cdot (a \otimes e_B + N)
\]

(3.3)

Thus, (3.2) and (3.3) imply that \( (a \otimes e_B + N) \cdot (D - D_f)(e_A \otimes b + N) = (D - D_f)(e_A \otimes b + N) \cdot (a \otimes e_B + N) \) for every \( a \in A \) and \( b \in B \). Let \( Y \) be the annihilator of \((D - D_f)(e_A \otimes b + N)\) in \( X \) then \( Y \) is a \( B \)-submodule. By relations (3.2) and (3.3) we have \((a \otimes e_B + N) \cdot y - y \cdot (a \otimes e_B + N), (D - D_f)(e_A \otimes b + N) = 0, \) for every \( y \in Y, a \in A \) and \( b \in B \). By a similar argument \( X/Y \) is a Banach \( \mathfrak{A} \)-\( B \)-bimodule. Since the restriction of \( D - D_f \) to \( B_1 \) defines a continuous module derivation from \( B \) into \( Y^* \subseteq X^* \) and since \( B \) is module amenable, there is \( g \in Y^* \) such that \( D - D_f = D_g \) and \( D - D_f - D_g|_{B_1} = 0 \). Since \((A \hat{\otimes}_\mathfrak{A} B) \cup (e_A \hat{\otimes}_\mathfrak{A} B)\) generates \( A \hat{\otimes}_\mathfrak{A} B \), \( D - D_f - D_g|_{A \hat{\otimes}_\mathfrak{A} B} = 0 \). This shows that \( D = D_f + D_g \), and so \( D \) is inner. \( \square \)

**Corollary 3.2.** Let \( A \) and \( B \) be unital commutative Banach algebras and unital Banach \( \mathfrak{A} \)-modules. If \( A \) and \( B \) are module amenable then \( A \hat{\otimes}_\mathfrak{A} B \) is module amenable.

**Proof.** Let \( D : A \hat{\otimes}_\mathfrak{A} B \rightarrow X^* \) be a continuous module derivation, where \( X \) is a Banach \( A \hat{\otimes}_\mathfrak{A} B \)-\( 
\mathfrak{A} \)-module. Since \( A \) and \( B \) are commutative, \( X \) is a Banach \( A \)-\( B \)-\( \mathfrak{A} \)-module. Similar to the proof of Theorem 3.1, set \( A_1 = A \hat{\otimes}_\mathfrak{A} e_B \) and \( B_1 = e_A \hat{\otimes}_\mathfrak{A} B \). Then the mapping \( D_1 : A \rightarrow X^* \) defined by \( D_1(a) = D(a \otimes e_B + N) \) is a continuous module derivation. Then \( D_1|_{A_1} = 0 \). Therefore for every \( a \in A \) and \( b \in B \) we have

\[
D(a \otimes b + N) = D((a \otimes e_B + N)(e_A \otimes b + N))
\]

\[
= (a \otimes e_B + N) \cdot D(e_A \otimes b + N)
\]

\[
+ D(a \otimes e_B + N) \cdot (e_A \otimes b + N)
\]
(3.4) 
\[ (a \otimes e_B + N) \cdot D(e_A \otimes b + N) = D((e_A \otimes b + N)(a \otimes e_B + N)) \]
\[ = (e_A \otimes b + N) \cdot D(a \otimes e_B + N) + D(e_A \otimes b + N) \cdot (a \otimes e_B + N) \]
\[ = D(e_A \otimes b + N) \cdot (a \otimes e_B + N). \]

Then (3.4) and (3.5) imply that 
\[ (a \otimes e_B + N) \cdot D(e_A \otimes b + N) = D(e_A \otimes b + N) \cdot (a \otimes e_B + N) \]
for every \(a \in A\) and \(b \in B\). Let \(Y\) be the annihilator of \(D(e_A \otimes b + N)\) in \(X\) then \(Y\) is a \(B\)-submodule. By relations (3.4) and (3.5) we have 
\[ \langle (a \otimes e_B + N) \cdot y - y \cdot (a \otimes e_B + N), D(e_A \otimes b + N) \rangle = 0, \]
for every \(y \in Y\), \(a \in A\) and \(b \in B\). Since \(X/Y\) is a Banach \(B\)-\(A\)-module, and the restriction of \(D\) to \(B_1\) defines a continuous module derivation from \(B\) into \(Y^*\), we have \(D|_{B_1} = 0\). Therefore \(D|_{A \hat{\otimes}_A B} = 0\).

Replacing \(A \hat{\otimes}_A B\) by \(X\) in the Corollary 3.2, and viewing \(A \hat{\otimes}_A B\) as a Banach \(A\)-\(B\)-\(A\)-module (relations (3.1)), we have the following result:

**Corollary 3.3.** Let \(A\) and \(B\) be unital commutative Banach algebras and unital Banach \(A\)-modules. If \(A\) and \(B\) are weakly module amenable then \(A \hat{\otimes}_A B\) is weakly module amenable.

Next we study weak module amenability of module projective tensor products of Banach algebras.

**Theorem 3.4.** Let \(A\) be a unital Banach \(A\)-bimodule and \(\varphi : A \to A\) be a continuous surjective map. Suppose that \(ab = \varphi(a) \cdot b\), for each \(a, b \in A\).
Then \(A\) is weakly module amenable.

**Proof.** Let \(D : A \to A^*\) be a continuous module derivation. Then
\[ \varphi(a) \langle c, Db \rangle = \langle c, D(ab) \rangle = \langle c, a \cdot D(b) + D(a) \cdot b \rangle = \langle ca, Db \rangle + \langle bc, Da \rangle \]
(3.6) 
\[ = \varphi(c) \langle a, Db \rangle + \varphi(b) \langle c, Da \rangle, \]
for each \(a, b, c \in A\). Let \(\lambda \in A^*\), and let \(\delta_\lambda : A \to A^*\) be the inner derivation specified by \(\lambda\). Hence
\[ \langle b, \delta_\lambda(a) \rangle = \langle b, a \cdot \lambda - \lambda \cdot a \rangle = \langle ba, \lambda \rangle - \langle ab, \lambda \rangle = \varphi(b) \langle a, \lambda \rangle - \varphi(a) \langle b, \lambda \rangle, \]
(3.7) 
for each \(a, b \in A\). Choose \(a_0 \in A\) with \(\varphi(a_0) = e_A\), and set \(\lambda(a) = \langle a_0, Da \rangle\) for each \(a \in A\). Clearly \(\lambda\) is linear. Using (3.6) and (3.7) we have 
\[ \langle b, \delta_\lambda(a) \rangle = \varphi(b) \langle a, \lambda \rangle - \varphi(a) \langle b, \lambda \rangle = \varphi(b) \langle a_0, Da \rangle - \varphi(a) \langle a_0, Db \rangle = \varphi(a_0) \langle b, Da \rangle - \varphi(a_0) \langle b, Da \rangle = \langle b, Da \rangle, \]
therefore \(D = \delta_\lambda\), and so \(A\) is weakly module amenable. \(\square\)
Example 3.5. Let \( \mathbb{N} \) be set of the positive integers. We can see it as a semigroup by denoting the product of two elements to be their maximum. The resulting semigroup, which we denote by \( \mathbb{N}_\vee \), is a semilattice. The semilattice \( \mathbb{N}_\vee \), is a commutative semigroup in which every element is idempotent. If we denote the set of idempotent elements of \( \mathbb{N}_\vee \) by \( E(\mathbb{N}_\vee) \), then \( E(\mathbb{N}_\vee) = \mathbb{N}_\vee \). We may then form the \( \ell^1 \)-convolution algebra \( \ell^1(\mathbb{N}_\vee) \). For every \( t \in \mathbb{N}_\vee \) we denote the point mass concentrated at \( t \) by \( \delta_t \).

The definition of multiplication in \( (3.9) \) ensures that \( \delta_n \ast \delta_m = \delta_{\max(n,m)} \) for all \( m, n \in \mathbb{N} \). Consider \( \ell^1(\mathbb{N}_\vee) \) as a Banach \( \ell^1(\mathbb{N}_\vee) \)-bimodule. Define \( \varphi: \ell^1(\mathbb{N}_\vee) \to \ell^1(\mathbb{N}_\vee) \) by \( \varphi(\delta_n) = \delta_1 \). Then \( \delta_n \ast \delta_m = \delta_m \ast \delta_n = \varphi(\delta_\alpha) \ast \delta_\beta \), where \( \alpha = \min\{m,n\} \) and \( \beta = \max\{m,n\} \). Then by Theorem 3.4, \( \ell^1(\mathbb{N}_\vee) \) is weakly module amenable.

Theorem 3.6. Let \( A \) and \( B \) be unital commutative Banach algebras and unital Banach \( \mathfrak{A} \)-modules, \( \varphi: A \to \mathfrak{A} \) and \( \psi: B \to \mathfrak{A} \) be continuous surjective maps. Suppose that \( ab = \varphi(a) \cdot b \), \( cd = \psi(c) \cdot d \), for each \( a, b \in A \) and \( c, d \in B \). Then \( A \otimes \mathfrak{A}B \) is weakly module amenable.

Proof. Let \( e_A \) and \( e_B \) be the unit elements of \( A \) and \( B \), respectively. Let \( D: A \otimes \mathfrak{A}B \to (A \otimes \mathfrak{A}B)^* \) be a continuous module derivation. For each \( a, c, e \in A \) and \( b, d, f \in B \) we have

\[
\langle c \otimes d + N, D(\{ e \otimes bf + N \}) \rangle \\
= \langle c \otimes d + N, D((a \otimes b + N)(e \otimes f + N)) \rangle \\
= \langle c \otimes d + N, (a \otimes b + N) \cdot D(e \otimes f + N) \rangle \\
+ \langle c \otimes d + N, D(a \otimes b + N) \cdot (e \otimes f + N) \rangle \\
= \langle ca \otimes db + N, D(e \otimes f + N) \rangle + \langle ec \otimes fd + N, D(a \otimes b + N) \rangle \\
= \varphi(c)\psi(d)\langle a \otimes b + N, D(e \otimes f + N) \rangle \\
+ \varphi(e)\psi(f)\langle c \otimes d + N, D(a \otimes b + N) \rangle \\
= \varphi(a)\psi(b)\langle c \otimes d + N, D(e \otimes f + N) \rangle.
\]

Fix \( b_0 \in B \) with \( \psi(b_0) = e_\mathfrak{A} \). Then by (3.8), we may write

\[
\langle e_A \otimes e_B + N, D(a \otimes b + N) \rangle \\
= \langle e_A \otimes e_B + N, D((a \otimes b_0 + N)(e_A \otimes b + N)) \rangle \\
= \varphi(a)\psi(b_0)\langle e_A \otimes e_B + N, D(e_A \otimes b + N) \rangle \\
= \langle a \otimes b_0 + N, D(e_A \otimes b + N) \rangle,
\]

\[
\text{(3.9)}
\]
for each \( a \in A \) and \( b \in B \). Hence there exists \( \lambda \in (A \hat{\otimes}_\mathbb{R} B)^* \) such that

\[
\langle a \otimes b + N, \lambda \rangle = \langle a \otimes b_0 + N, D(e_A \otimes b + N) \rangle,
\]

for each \( a \in A \) and \( b \in B \). Let \( \delta_\lambda : A \hat{\otimes}_\mathbb{R} B \rightarrow (A \hat{\otimes}_\mathbb{R} B)^* \) be the inner module derivation induced by \( \lambda \). Take \( a \in A \) and \( b, c \in B \). Then

\[
\langle a \otimes c + N, \delta_\lambda(e_A \otimes b + N) \rangle = \langle a \otimes c + N, (e_A \otimes b + N) \cdot \lambda - \lambda \cdot (e_A \otimes b + N) \rangle = \langle a \otimes cb + N, \lambda \rangle - \langle a \otimes bc + N, \lambda \rangle = \langle a \otimes b_0 + N, \psi(c)D(e_A \otimes b + N) \rangle - \langle a \otimes b_0 + N, \psi(b)D(e_A \otimes c + N) \rangle = \langle a \otimes (b_0 + N)(e_A \otimes c + N) \rangle = \langle a \otimes c + N, D(e_A \otimes b + N) \rangle.
\]

Therefore, \( D(e_A \otimes b + N) = \delta_\lambda(e_A \otimes b + N) \) for each \( b \in B \). We claim that

\[
D(a \otimes e_B + N) = \delta_\lambda(a \otimes e_B + N)
\]

for each \( a \in A \). Choose \( a_0 \in A \) with \( \varphi(a_0) = e_\mathbb{R} \). Then by (3.10) we have

\[
\langle a \otimes b + N, \lambda \rangle = \langle a \otimes b_0 + N, D(e_A \otimes b + N) \rangle = \langle a_0 \otimes b + N, D(e_A \otimes b + N) \rangle,
\]

for each \( a \in A \) and \( b \in B \). Now by (3.11) we get

\[
\langle c \otimes b + N, \delta_\lambda(a \otimes e_B + N) \rangle = \langle c \otimes b + N, (a \otimes e_B + N) \cdot \lambda - \lambda \cdot (a \otimes e_B + N) \rangle = \langle c \otimes b + N, (a \otimes e_B + N) \cdot \lambda \rangle - \langle c \otimes b + N, \lambda \rangle = \langle c \otimes b + N, \varphi(c)D(a \otimes e_B + N) \rangle - \langle c \otimes b + N, \varphi(a)D(c \otimes e_B + N) \rangle = \langle c \otimes b + N, D(a \otimes e_B + N) \rangle.
\]

Hence \( D(a \otimes e_B + N) = \delta_\lambda(a \otimes e_B + N) \) for each \( a \in A \). Thus for each \( a \in A \) and \( b \in B \) we have

\[
D(a \otimes b + N) = D((a \otimes e_B + N)(e_A \otimes b + N)) = (a \otimes e_B + N) \cdot D(e_A \otimes b + N) + D(a \otimes e_B + N) \cdot (e_A \otimes b + N) = (a \otimes b + N) \cdot \lambda - \lambda \cdot (a \otimes b + N) = \delta_\lambda(a \otimes b + N).
\]

Therefore, \( D = \delta_\lambda \) on \( A \hat{\otimes}_\mathbb{R} B \). Hence \( A \hat{\otimes}_\mathbb{R} B \) is weakly module amenable. \( \square \)

It is shown in [1, Lemma 3.1] that if \( \ell^1(E) \) acts on \( \ell^1(S) \) by multiplication from right and trivially from left, then \( \ell^1(S) \hat{\otimes}_{\ell^1(E)} \ell^1(S) \cong \ell^1(S \times S)/I \), where \( I \) is the closed ideal of \( \ell^1(S \times S) \) generated by the set of elements of the form \( \delta_{(s,t,u)} - \delta_{(s,t,u)} \), where \( s, t, u \in S \) and \( e \in E \).

**Corollary 3.7.** Let \( S \) be commutative and unital inverse semigroup with the set of idempotents \( E \). Then \( \ell^1(S) \hat{\otimes}_{\ell^1(E)} \ell^1(S) \) is weakly module amenable.

**Proof.** The result follows from [3, Theorem 3.1] and Corollary 3.3. \( \square \)
Acknowledgement. The authors sincerely thank the anonymous reviewer for his/her careful reading, constructive comments and fruitful suggestions to improve the quality of the first draft.

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