ON A CLASS OF VERTEX COVER IDEALS

BY

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Abstract. We investigate vertex cover ideals associated to a significative class of connected graphs. These ideals are stated to be Cohen-Macaulay and, using the notion of linear quotients, standard algebraic invariants for them are computed.

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Introduction

In this paper we are interested in studying standard properties of monomial ideals arising from a class of graphs in order to explain how algebraic models, built through graph theory, are useful for setting suitable solutions of several problems in network areas. In the last years, commutative and combinatorial algebra have paid particular attention to some monomial ideals deriving from graphs. Such ideals turn out to be interesting from a combinatorial point of view. A classical topic is the algebraic graph theory, in particular the study of edge ideals. Moreover, using the notion of minimal covering for the nodes of a graph, recent results show that it is possible to attach to graphs the so-called vertex cover ideals.

Let $G$ be a graph having $n$ vertices $v_1, \ldots, v_n$. If $G$ is simple, its edge ideal $I(G)$ is a monomial ideal of the polynomial ring $R=K[x_1, \ldots, x_n]$, $K$ a field, generated by squarefree monomials of degree two: $I(G)=\langle \{x_ix_j\mid \{v_i, v_j\} \text{ is an edge of } G \rangle \rangle$. The vertex cover ideal associated to $G$, denoted by $I_c(G)$, is the ideal of $R$ generated by all the monomials $x_{i_1} \ldots x_{i_k}$ such that $(x_{i_1}, \ldots, x_{i_k})$ is an associated prime ideal of $I(G)$ (see [10]). In some cases
$I(\mathcal{G})$ reflects properties of $I_c(\mathcal{G})$ and the study of $I(\mathcal{G})$ states some properties of $I_c(\mathcal{G})$. In particular, the Cohen-Macaulay property of the ideals $I_c(\mathcal{G})$ was studied in [8] using graph theoretical methods; the central notion of chordal graph is the right one to express homological properties of edge ideals and vertex cover ideals.

We consider a class of squarefree edge ideals associated to the connected graphs $\mathcal{H}$ which consist of the union of a complete graph $K_m$ with star graphs whose centers are the vertices of $K_m$ (see [6]). Such a class of graphs is an example of a suitable geometric model for wireless sensor networks, an important technology for large-scale monitoring, providing sensor measurements at high temporal and spatial resolution.

In the following we examine some algebraic properties of the ideals $I_c(\mathcal{H})$. We prove that such ideals have linear quotients (Proposition 2.1) and investigate conditions under which they have a linear resolution. Moreover, by studying the linear quotients of these ideals, as previously employed in [7], standard algebraic invariants such as projective dimension, depth, Krull dimension, Castelnuovo-Mumford regularity, are calculated (Corollary 2.2). Starting from theoretic properties of the edge ideal of $\mathcal{H}$ we also prove that for $I_c(\mathcal{H})$ the Cohen-Macaulay property holds.

1. Graphs and sensor networks

The aim of this section is to explain how algebraic and geometric models, built through graph theory, are useful for setting suitable solutions of several problems concerning the wireless sensor networks.

A wireless sensor network consists of spatially distributed autonomous sensors (called sensor nodes) to monitor physical or environmental conditions and to pass their data through the network to a main location. The sensors collect data of interest and send them to special gateway nodes, which send them to a base station. Current research (see [1]) indicates high developments for this type of wireless networks which are already in use in many fields. Useful applications of them are environmental, military and commercial.

Consider a sensor network in which each sensor must send information it has collected to the nearest gateway node. The problem is the placement of as few as possible gateway nodes so that each sensor is near to at least one of them. This problem is related to the energy saving of the sensors. The formalization of the model to represent the sensor network is given
by a graph $G = (V(G), E(G))$, $V(G) = \{v_1, \ldots, v_n\}$ is the set of nodes and $E(G) = \{\{v_i, v_j\} \mid v_i \neq v_j, v_i, v_j \in V(G)\}$ is the set of edges (if two nodes $v_1$ and $v_2$ are adjacent then $\{v_1, v_2\} \in E(G)$). Our problem consists of finding a set of nodes leaders (gateway nodes) to be responsible to send messages to nodes belonging to the higher level. Such leaders must have the feature of being near to all nodes. This means that we want to find a subset of nodes $C \subseteq V(G)$ such that each node in $V(G) \setminus C$ is joined to a node in $C$. This is a problem of minimum vertex cover. The optimization problem of minimum vertex cover is looking for a covering of vertices of a graph that has minimum cardinality. The corresponding decision problem of vertex cover is, given a graph $G$ and an integer $k$, if there is a vertex covering of $G$ of minimal cardinality $k$ or of cardinality at most $k$.

We investigate this problem considering as a network topology a class of simple graphs: the union of a complete graph and star graphs.

Let’s introduce some preliminary notions.

A cycle of length $q$ of a graph $G$ is a subgraph $C_q$ such that $E(C_q) = \{\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{q-1}, v_q\}, \{v_q, v_1\}\}$, where $\{v_1, \ldots, v_q\} \in V(G)$ and $v_{i+1} \neq v_i$ if $i \neq q$.

A chord of a cycle $C_q \subseteq G$ is an edge $\{v_{i,j}, v_{i,k}\}$ of $G$ such that $v_{i,j}$ and $v_{i,k}$ are vertices of $C_q$ with $\{v_{i,j}, v_{i,k}\} \notin E(C_q)$.

A graph $G$ is said chordal or triangulated if every cycle $C_q$ in $G$, $q > 3$, has a chord.

A graph $G$ with $n$ vertices $v_1, \ldots, v_n$ is said complete if there exists an edge for every pair $\{v_i, v_j\}$ of vertices of $G$. It is denoted by $K_n$.

A graph $G$ is said bipartite if its vertex set $V = [n]$ can be partitioned into disjoint subsets, $V_1 = [n_1]$ and $V_2 = [n_2]$, $n_1 + n_2 = n$, such that every edge of $G$ joins $V_1$ with $V_2$. It is denoted by $K_{n_1, n_2}$.

A star graph on vertex set $\{\{v_i\}, \{v_1, \ldots, v_{i-1}, v_{i+1}, v_n\}\}$ with center $v_i$, denoted by star$_i(n)$, $i = 1, \ldots, n$, is a complete bipartite graph of the form $K_{1, n-1}$.

The complementary graph of a graph $G$ on $\{v_1, \ldots, v_n\}$ is the graph $\overline{G}$ on $\{v_1, \ldots, v_n\}$ whose edge set $E(\overline{G})$ consists of those 2-element subset $\{v_i, v_j\}$ for which $\{v_i, v_j\} \notin E(G)$.

If $V(G) = \{v_1, \ldots, v_n\}$ and $R = K[x_1, \ldots, x_n]$ is the polynomial ring over a field $K$ such that each variable $x_i$ corresponds to the vertex $v_i$, the edge ideal $I(G)$ associated to $G$ is the ideal $\langle \{x_ix_j \mid \{v_i, v_j\} \in E(G)\} \rangle \subset R$.

**Definition 1.1.** Let $G$ be a graph with vertex set $V(G) = \{v_1, \ldots, v_n\}$. 

A subset $C$ of $V(G)$ is said a minimal vertex cover for $G$ if:

1. every edge of $G$ is incident with one vertex in $C$;
2. there is no proper subset of $C$ with this property.

Starting from the notion of minimal vertex cover, it is possible to attach to graphs the so-called vertex cover ideals (see [10]). There exists a one to one correspondence between the minimal vertex covers of $G$ and the minimal prime ideals of $I(G)$. In fact, $\varphi$ is a minimal prime ideal of $I(G)$ if and only if $\varphi = (C)$, for some minimal vertex cover $C$ of $G$ (see [10], Prop. 6.1.16).

**Definition 1.2.** A vertex cover ideal associated to $G$, denoted by $I_c(G)$, is the ideal of $R$ generated by all the monomials $x_{i_1}\ldots x_{i_k}$ such that \{\$v_{i_1}, \ldots, v_{i_k}\$\} is a minimal vertex cover of $G$, that is $\{x_{i_1}, \ldots, x_{i_k}\}$ is a minimal prime ideal of $I(G)$.

Now we study this ideal for a class of connected graphs and we show how it is possible to apply the results to the problem related to wireless sensor networks.

More precisely, $H$ is the connected graph on $n$ vertices $v_1,\ldots, v_n$ consisting of the union of a complete graph $K_m$ and star graphs in the vertices of $K_m$: $H = K_m \cup \text{star}_j(k)$, where $K_m$ is the complete graph on $m$ vertices $v_1,\ldots, v_m$, $m < n$, and $\text{star}_j(k)$ is the star graph on $k$ vertices with center $v_j$, for some $j = 1,\ldots, m$, $k \leq n - m$. We observe that $V(\text{star}_j(k)) \cap V(K_m) = \{v_j\}$ for $j = 1,\ldots, m$. We denote $|E(\text{star}_j(k))| = i_j$, for $j = 1,\ldots, m$.

The edge ideal $I(H) \subset R$ has generators, as shown in [6]:

$I(H) = (x_1x_{m+1}, x_1x_{m+2}, \ldots, x_1x_{m+i_1}, x_1x_2, x_2x_{m+i_1+1}, x_2x_{m+i_1+2}, \ldots, x_2x_{m+i_1+i_2}, x_2x_3, x_1x_3, x_3x_{m+i_1+i_2+1}, x_3x_{m+i_1+i_2+2}, \ldots, x_3x_{m+i_1+i_2+i_3}, x_3x_{m+i_1+i_2+i_3}, x_3x_{m+i_1+i_2+i_3+1}, \ldots, x_{m}x_{m+i_1+i_2+i_3+i_{m-1}+1}, x_{m}x_{m+i_1+i_2+i_3+i_{m-1}+2}, \ldots, x_{m}x_{m+i_1+i_2+i_3+i_{m-1}+i_m}).$

**Remark 1.1.** $I(H)$ has a linear resolution (see [6], Proposition 1.1). R. Fröberg proved that the edge ideal of a simple graph has a linear resolution if and only if its complementary graph is chordal (see [3], [4], Theorem 9.2.3). Hence $\overline{H}$ is chordal.

The structure of $I_c(H)$ is given in [6]. In order to describe $I_c(H)$ we rename the vertices of the complete graph $K_m$ with $v_{\alpha_1},\ldots, v_{\alpha_m}$, $1 \leq \alpha_1 < \alpha_2 < \ldots < \alpha_m = n$, and $\alpha_1 = 1$ if there is no star graph with center $v_{\alpha_1}$. 
The monomial ideal $I_c(\mathcal{H})$ has $m + 1$ monomial squarefree generators of degree $\geq m$, \(\{x_{a_1}x_{a_2}\ldots x_{a_m}, x_1\ldots x_{a_1-1}x_{a_2}\ldots x_{a_m}, x_{a_1}x_{a_1+1}\ldots x_{a_2-1}x_{a_3}\ldots x_{a_m}, \ldots, x_1\ldots x_{a_m-2}x_{a_m-2+1}\ldots x_{a_m-1}x_{a_m}, x_{a_1}\ldots x_{a_m-1}x_{a_m-1+1}\ldots x_{a_m-1}\}\).

Hence we give the answer to the decision problem of vertex cover for the graph $\mathcal{H}$: there is a vertex cover of $\mathcal{H}$ of minimum cardinality $m$, where $m$ is the number of vertices of the complete graph $K_m$. Moreover the generators of $I_c(\mathcal{H})$ give the structure of each vertex covering of $\mathcal{H}$.

**Example 1.1.** Let $\mathcal{H}$ be the simple graph with $V(\mathcal{H}) = \{v_1, \ldots, v_{11}\}$ \(\mathcal{H} = K_4 \cup \text{star}_2(2) \cup \text{star}_6(4) \cup \text{star}_8(2) \cup \text{star}_{11}(3)\).

The displayed graph is the model of a wireless sensor network:

$I_c(\mathcal{H})$ is generated by \(\{x_1x_6x_8x_{11}, x_2x_5x_8x_{11}, x_2x_6x_7x_{11}, x_2x_6x_8x_9x_{10}, x_2x_3x_4x_5x_8x_{11}\}\).

The minimal vertex covers of $\mathcal{H}$ are:
- $C_1 = \{v_1, v_6, v_8, v_{11}\}$,
- $C_2 = \{v_2, v_6, v_8, v_{11}\}$,
- $C_3 = \{v_2, v_7, v_{11}\}$,
- $C_4 = \{v_2, v_6, v_8, v_9, v_{10}\}$,
- $C_5 = \{v_2, v_3, v_4, v_5, v_8, v_{11}\}$.

Choose $C_1$ among the three vertex covers with minimum cardinality. $C_1$ gives the position of the leaders in the sensor network that $\mathcal{H}$ represents.

### 2. Algebraic properties of $I_c(\mathcal{H})$

$I_c(\mathcal{H})$ is an interesting class of monomial ideals for its good algebraic properties. In this section we study some of them.

**Definition 2.1.** A monomial ideal $I \subset R$ has linear quotients if there is an ordering $u_1, \ldots, u_t$ of monomials belonging to its unique minimal set
of monomial generators \( G(I) \) such that the colon ideal \((u_1, \ldots, u_{j-1}) : (u_j)\) is generated by a subset of \( \{x_1, \ldots, x_n\} \), for \( 2 \leq j \leq t \).

For a monomial ideal \( I \) of \( R \) having linear quotients with respect to the ordering \( u_1, \ldots, u_t \) of the monomials of \( G(I) \), let \( q_j(I) \) denote the number of the variables which is required to generate the ideal \((u_1, \ldots, u_{j-1}) : (u_j)\).

Set \( q(I) = \max_{2 \leq j \leq t} q_j(I) \). The integer \( q(I) \) is independent on the choice of the ordering of the generators that gives linear quotients (see [5]).

**Proposition 2.1.** \( I_c(\mathcal{H}) \) has linear quotients.

**Proof.** 1) If at least a vertex of \( K_m \) has degree \( m - 1 \) (see [6], Proposition 2.1), then \( I_c(\mathcal{H}) \) has \( m \) monomial squarefree generators. We have the following cases:

a) \( \mathcal{H} \) is formed by the union of a complete graph \( K_m, m < n \), with vertices \( v_{\alpha_1}, \ldots, v_{\alpha_m}, 1 \leq \alpha_1 < \alpha_2 < \ldots < \alpha_m = n \), and of a star graph \( \text{star}_{\alpha_1}(\alpha_1) \) on vertices \( v_1, \ldots, v_{\alpha_1} \). Then we set the following order for the monomial generators of \( I_c(\mathcal{H}) \): \( f_1 = x_{\alpha_1} \cdots x_{\alpha_{m-1}}, f_2 = x_{\alpha_1} \cdots x_{\alpha_{m-2}} x_{\alpha_m}, \ldots, f_{m-1} = x_{\alpha_1} x_{\alpha_3} \cdots x_{\alpha_m}, f_m = x_1 \cdots x_{\alpha_{m-1}} x_{\alpha_m} \). It follows: \((f_1) : (f_2) = (x_{\alpha_{m-1}}), (f_1, f_2) : (f_3) = (x_{\alpha_{m-2}}), \) and so on up to \((f_1, \ldots, f_{m-1}) : (f_m) = (x_{\alpha_1})\).

b) \( \mathcal{H} \) is formed by the union of a complete graph \( K_m, m < n \), with vertices \( v_{\alpha_1}, \ldots, v_{\alpha_m}, 1 \leq \alpha_1 < \alpha_2 < \ldots < \alpha_m = n \), and of a star graph \( \text{star}_{\alpha_1}(\alpha_i - \alpha_{i-1}) \), for some \( i = 2, \ldots, m \). Then we set the following order for the monomial generators of \( I_c(\mathcal{H}) \): \( f_1 = x_{\alpha_1} \cdots x_{\alpha_{m-1}}, f_2 = x_{\alpha_1} \cdots x_{\alpha_{m-2}} x_{\alpha_m}, \ldots, f_{m-1} = x_{\alpha_1} x_{\alpha_2} \cdots x_{\alpha_m}, f_m = x_{\alpha_2} \cdots x_{\alpha_m} \). Hence: \((f_1) : (f_2) = (x_{\alpha_{m-1}}), (f_1, f_2) : (f_3) = (x_{\alpha_{m-2}}), \) and so on up to \((f_1, \ldots, f_{m-1}) : (f_m) = (x_{\alpha_1})\).

2) If all the vertices of \( K_m \) have degree at least \( m \) (see [6], Theorem 2.1), then we set the following order for the monomial generators of \( I_c(\mathcal{H}) \): \( f_1 = x_{\alpha_1} x_{\alpha_2} \cdots x_{\alpha_m}, f_2 = x_{\alpha_1} \cdots x_{\alpha_{m-1}} x_{\alpha_{m-1}+1} \cdots x_{\alpha_m-1}, f_3 = x_{\alpha_1} \cdots x_{\alpha_{m-2}} x_{\alpha_{m-2}+1} \cdots x_{\alpha_{m-1}} x_{\alpha_m}, \ldots, f_m = x_{\alpha_1} x_{\alpha_2} \cdots x_{\alpha_{m-1}} x_{\alpha_m}, f_{m+1} = x_1 \cdots x_{\alpha_{m-1}} x_{\alpha_m} \). It follows: \((f_1) : (f_2) = (x_{\alpha_m}), (f_1, f_2) : (f_3) = (x_{\alpha_{m-1}}), (f_1, f_2, f_3) : (f_4) = (x_{\alpha_{m-2}}), \) and so on up to \((f_1, \ldots, f_m) : (f_{m+1}) = (x_{\alpha_1})\).

Hence \( I_c(\mathcal{H}) \) has linear quotients.

**Corollary 2.1.** Let \( I_c(\mathcal{H}) \subset R \). Then \( q(I_c(\mathcal{H})) = 1 \).

**Proof.** By Proposition 2.1, the number of the variables which is required to generate the ideal \((f_1, \ldots, f_{h-1}) : (f_h)\) is 1, for all \( h = 1, \ldots, t \), \( t \leq m + 1 \).
**Remark 2.1.** If a monomial ideal $I$ of $R$ generated in one degree has linear quotients, then $I$ has a linear resolution (see [2], Lemma 4.1).

By a result in [9], $I_c(H)$ has linear quotients, but in general it has not a linear resolution because it is not generated in one degree.

Using some properties of the graph $H$ and of its edge ideal $I(H)$ we can obtain the conditions for which $I_c(H)$ has a linear resolution.

**Proposition 2.2.** Let $H = K_m \cup star_j(k)$. Then:
1) $I(H)$ is Cohen Macaulay if and only if $k = 2$.
2) $I_c(H)$ has linear resolution if and only if $k = 2$.

**Proof.** 1) It is an immediate consequence of [6], Theorem 1.1.
2) In order to have a linear resolution, generators must be of the same degree, thus $k$ needs to be 2. The converse holds by Proposition 2.1. □

Following [10], the vertex cover ideal $I_c(G)$ is unmixed of height 2. But generally it is not Cohen-Macaulay. The Cohen-Macaulay property of the ideal $I_c(G)$ was studied in [8]. The notion of chordal graph is used to express this algebraic property and to study an interesting duality between $I(G)$ and $I_c(G)$ (see [3]).

**Proposition 2.3.** $I_c(H)$ is Cohen-Macaulay.

**Proof.** $I(H)$ has linear resolution (see [6], Proposition 1.1). Hence the complementary graph $\overline{H}$ of $H$ is chordal (see [4], Theorem 9.2.3). By [8], this is equivalent to say that $I_c(H)$ is Cohen-Macaulay. □

The study of the linear quotients is useful in order to investigate some algebraic invariants of $R/I_c(H)$.

**Corollary 2.2.** Let $R = K[x_1, \ldots, x_n]$ and $I_c(H) \subset R$. Then:
1) $pd_R(R/I_c(H)) = 2$
2) $\text{depth}_R(R/I_c(H)) = n - 2$
3) $\text{dim}_R(R/I_c(H)) = n - 2$
4) $\text{reg}_R(R/I_c(H)) = \max\{\deg(v_{\alpha_i})\} - 1, i = 1, \ldots, m$.

**Proof.** 1), 2) and 3) follow by the result $pd_R(R/I_c(H)) = q(I_c(H)) + 1$ (see [5], Corollary 1.6) and by Auslander-Buchsbaum formula in the Cohen-Macaulay case.
4) By [2], Lemma 4.1, the Castelnuovo-Mumford regularity is 
\[ \text{reg}_R(R/I_c(\mathcal{H})) = \max\{\deg(f)| f \text{ minimal generator of } I_c(\mathcal{H})\} - 1. \]
By the structure of generators of \( I_c(\mathcal{H}) \) it follows that 
\[ \max\{\deg(f)| f \text{ minimal generator of } I_c(\mathcal{H})\} = m - 1 - \max\{\deg(v_{\alpha_i})\} - (m - 1) = \max\{\deg(v_{\alpha_i})\}, \]
i = 1, \ldots, m. Hence 
\[ \text{reg}_R(R/I_c(\mathcal{H})) = \max\{\deg(v_{\alpha_i})\} - 1, \ i = 1, \ldots, m. \]
\[ \square \]

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