WEAKLY \((\tau_q, m)\)-CONTINUOUS FUNCTIONS

BY

UĞUR ŞENGÜL

Abstract. In this paper we introduce a new class of functions called weakly \((\tau_q, m)\)-continuous functions. Some characterizations and several properties concerning weak \((\tau_q, m)\)-continuity are obtained.

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1. Introduction

Semi-open sets, preopen sets, \(\alpha\)-sets, \(b\)-open sets, \(\beta\)-open sets play an important role for generalization of continuity in topological spaces. By using these sets several authors introduced and studied various modifications of continuity such as weak continuity, almost \(s\)-continuity ([22]), \(p(\theta)\)-continuity ([7]). Popa and Noiri [30] introduced the notions of minimal structures. After this work, various mathematicians turned their attention in introducing and studying diverse classes of sets and functions defined on an structure, because this notions are a natural generalization of many well known results related with generalized sets and several weaker forms of continuity such as ([20], [21], [32], [33], [40]). The notion of weakly \(M\)-continuous and weakly \((\tau, m)\)-continuous functions are introduced and studied by Popa and Noiri ([28], [29]) for unifying weak continuity types using minimal conditions. They also defined weakly \((\tau, \beta)\)-continuous functions as a special case of weak \((\tau, m)\)-continuity. Weak \((\tau, \beta)\)-continuity is also studied by present author [39] and by Basu and Ghosh [4] (under
the name of $(\theta, \beta)$-continuous functions). Recently SON, PARK and LIM introduced and studied weakly clopen functions ([36]). In fact this type of functions can be unified as weakly $M$-continuous function from a space with quasi-topology $\tau_q$, to a space with an $m$-structure, that is a function $f : (X, \tau_q) \rightarrow (Y, m_Y)$ which can be named as weakly $(\tau_q, m)$-continuous functions. The purpose of this paper is to introduce and investigate the notion of weakly $(\tau_q, m)$-continuous functions. In addition we also discuss possible generalizations of the concept of almost clopen functions due to EKICI [11], which is recently studied in detail by various authors ([15],[16]).

2. Preliminaries

Throughout this paper $(X, \tau)$ and $(Y, \sigma)$ (or simply $X$ and $Y$) represent nonempty topological spaces on which no separation axioms are assumed, unless otherwise mentioned. For a subset $S$ of $(X, \tau)$, $cl(S)$ and $int(S)$ represent, the closure of $S$ and the interior of $S$, respectively. A subset $S$ of a space $(X, \tau)$ is said to be regular open ([38]) (resp. regular closed ([38])) if $S = int(cl(S))$ (resp. $S = cl(int(S))$). A point $x$ of $X$ is called a $\theta$-cluster ([41]) point of $A$ if $cl(U) \cap A \neq \emptyset$ for every open set $U$ of $X$ containing $x$. The set of all $\theta$-cluster points of $A$ is called the $\theta$-closure ([41]) of $A$ and is denoted by $cl_\theta(A)$. A set $A$ is said to be $\theta$-closed if $A = cl_\theta(A)$. The complement of a $\theta$-closed set is said to be $\theta$-open. A subset $S$ of a space $(X, \tau)$ is said to be semi-open ([17]) (resp. preopen ([19]), $\alpha$-open ([27]), semi-preopen ([2]) or $\beta$-open ([1]), $b$-open ([3]) if $S \subset cl(int(S))$ (resp. $S \subset int(cl(S))$, $S \subset int(cl(int(S)))$, $S \subset cl(int(cl(S)))$). The family of all semi-open (resp.preopen, $\alpha$-open, $\beta$-open, $b$-open) sets of $X$ is denoted by $SO(X)$ (resp. $PO(X)$, $\alpha O(X)$, $\beta O(X)$, $BO(X)$). The complement of a semi-open (resp.preopen, $\alpha$-open, $\beta$-open, $b$-open) set is said to be semi-closed (resp. preclosed, $\alpha$-closed, $\beta$-closed, $b$-closed). If $S$ is a subset of a space $X$, then the $b$-closure of $S$, denoted by $bcl(S)$, is the smallest $b$-closed set containing $S$. The semiclosure (resp. preclosure, $\alpha$-closure, $\beta$-closure) of $S$ is similarly defined and is denoted by $scl(S)$ (resp. $pcl(S)$, $\alpha Cl(S)$, $bCl(S)$). A point $x \in X$ is said to be in the semi-$\theta$-closure ([9]) (resp. $\beta$-$\theta$-closure or sp-$\theta$-closure ([24])) of $A$, denoted by $scl_\theta(A)$ (resp. by $\beta cl_\theta(A)$), if $A \cap scl(V) \neq \emptyset$ (resp. $A \cap \beta cl(V) \neq \emptyset$) for every $V \in SO(X, x)$ (resp. $V \in BO(X, x)$). If $scl_\theta(A) = A$ (resp. $\beta cl_\theta(A) = A$), then $A$ is said to be semi-$\theta$-closed (resp. $\beta$-$\theta$-closed or sp-$\theta$-closed ([24])). The complement of a semi-$\theta$-closed (resp. $\beta$-$\theta$-closed) set is said to be semi-$\theta$-open (resp.
The quasi-component ([10]) of a point \( x \in X \) is the intersection of all clopen subsets of \( X \) which contain the point \( x \). The quasi-topology \( \tau_q \) on \( X \) is the topology having as base clopen subsets of \( (X, \tau) \). The closure of each point in quasi-topology is precisely the quasi-component of that point.

The open (resp. closed) subsets of the quasi-topology is called quasi-open ([10]) (resp. quasi-closed ([10])). For a space \( (X, \tau) \) the space \( (X, \tau_q) \) is called by Staum [37] the ultraregular kernel of \( X \) and denoted by \( X_q \) for simplicity.

A space \( (X, \tau) \) is called ultraregular ([37]) if \( \tau = \tau_q \).

Definition 1. A subfamily \( m_X \) of the power set \( \wp(X) \) of a nonempty set \( X \) is called a minimal structure (briefly \( m \)-structure) ([30]) on \( X \). By \( (X, m_X) \), we denote a nonempty subset \( X \) with a minimal structure \( m_X \) on \( X \). Each member of \( m_X \) is said to be \( m_X \)-open and the complement of \( m_X \)-open set is said to be \( m_X \)-closed.

Definition 2. A subset \( S \) is said to be \( m_X \)-regular if it is \( m_X \)-open and \( m_X \)-closed. The family of all \( m_X \)-regular sets of \( X \) is denoted by \( mR(X) \) and the family of all \( m_X \)-open (resp. \( m_X \)-regular or \( m_X \)-clopen) sets of \( X \) containing a point \( x \in X \) is denoted by \( mO(X, x) \) (resp. \( mR(X, x) \)).

Remark 1. Let \( (X, \tau) \) be a topological space. Then the families \( \tau, \tau_q, SO(X), PO(X), \alpha O(X), \beta O(X), SR(X), \beta R(X) \) are all \( m \)-structures on \( X \).

Definition 3. Let \( X \) be a nonempty set and \( m_X \) an \( m \)-structure on \( X \). For a subset \( A \) of \( X \), the \( m_X \)-closure of \( A \) and the \( m_X \)-interior of \( A \) are defined in [18] as follows:

\[
\begin{align*}
(\text{a})\ m_X-\text{Cl}(V) &= \cap\{F : A \subset F, X - F \in m_X\} \\
(\text{b})\ m_X-\text{Int}(V) &= \cup\{U : U \subset A, U \in m_X\}.
\end{align*}
\]

Remark 2. Let \( (X, \tau) \) be a topological space and \( A \) a subset of \( X \). If \( m_X = \tau, (\text{resp. } \tau_q, SO(X), PO(X), \alpha O(X), \beta O(X), SR(X), \beta R(X)) \) then we have:

\[
\begin{align*}
(\text{a})\ m_X-\text{Cl}(V) &= cl(V) \ (\text{resp. } cl_q(V), scl(A), pcl(A), \alpha Cl(A), \beta cl(A), scl_\theta(X), \beta cl_\theta(X)).
\end{align*}
\]
Lemma 1 ([18]). Let $X$ be a nonempty set and $m_X$ a minimal structure on $X$. For subsets $A$ and $B$ of $X$, the following hold:

(a) $m_X$-$\text{Cl}(X - A) = X - (m_X$-$\text{Int}(A))$ and $m_X$-$\text{Int}(X - A) = X - (m_X$-$\text{Cl}(A))$.

(b) If $X - A \in m_X$, then $m_X$-$\text{Cl}(A) = A$ and if $A \in m_X$, then $m_X$-$\text{Int}(A) = A$.

(c) $m_X$-$\text{Cl}(\emptyset) = \emptyset$, $m_X$-$\text{Cl}(X) = X$, $m_X$-$\text{Int}(\emptyset) = \emptyset$ and $m_X$-$\text{Int}(X) = X$.

(d) If $A \subset B$, then $m_X$-$\text{Cl}(A) \subset m_X$-$\text{Cl}(B)$ and $m_X$-$\text{Int}(A) \subset m_X$-$\text{Int}(B)$.

(e) If $A \subset m_X$-$\text{Cl}(A)$ and $m_X$-$\text{Int}(A) \subset A$.

(f) $m_X$-$\text{Cl}(m_X$-$\text{Cl}(A)) = m_X$-$\text{Cl}(A)$ and $m_X$-$\text{Int}(m_X$-$\text{Int}(A)) = m_X$-$\text{Int}(A)$.

Lemma 2 ([30]). Let $X$ be a nonempty set with a minimal structure $m_X$ and $A$ a subset of $X$. Then $x \in m_X$-$\text{Cl}(A)$ if and only if $U \cap A \neq \emptyset$, for every $U \in m_X$ containing $x$.

A point $x \in X$ is called a $m_\theta$-adherent point ([29]) of $S$ if $m_X$-$\text{Cl}(U) \cap S \neq \emptyset$ for every $m_X$-open set $U$ containing $x$. The set of all $m_\theta$-adherent points of $S$ is denoted by $m\text{Cl}_\theta(S)$. A subset $S$ is said to be $m_\theta$-closed if $S = m\text{Cl}_\theta(S)$. The complement of a $m_\theta$-closed set is said to be $m_\theta$-open.

Definition 4 ([18]). A minimal structure $m_X$ on a nonempty set $X$ is said to have property (B) if the union of any family of subsets belonging to $m_X$ belongs to $m_X$.

Lemma 3 ([30]). Let $X$ be a nonempty set and $m_X$ a minimal structure on $X$ satisfying the property (B). For a subset $A$ of $X$, the following properties hold:

(a) $A \in m_X$ if and only if $m_X$-$\text{Int}(A) = A$. 

(b) A is $m_X$-closed if and only if $m_X\text{-Cl}(A) = A$.

(c) $m_X\text{-Int}(A) \in m_X$ and $m_X\text{-Cl}(A)$ is $m_X$-closed.

**Definition 5.** A minimal structure $m_X$ on a nonempty set $X$ satisfying the property $(B)$ is said to have property $(mR)$ if for any subset $A$ of $X$ the following two conditions are true:

(a) $m_X\text{-Cl}(m_X\text{-Int}(m_X\text{-Cl}(A))) = m_X\text{-Int}(m_X\text{-Cl}(A))$.

(b) $m_X\text{-Int}(m_X\text{-Cl}(m_X\text{-Int}(A))) = m_X\text{-Cl}(m_X\text{-Int}(A))$.

**Remark 3.** Let $X$ be a nonempty set and $m_X$ a minimal structure on $X$. For the case $m_X = SO(X)$, $m_X$ satisfies equalities in Definition 5 by [1], for $m_X \in \{BO(X), \beta O(X)\}$ $m_X\text{-Cl}(m_X\text{-Int}(A)) = m_X\text{-Int}(m_X\text{-Cl}(A))$ is true. This statement implies conditions of Definition 5.

**Lemma 4.** Let $X$ be a nonempty set and $m_X$ a minimal structure on $X$ satisfying the property $(B)$ and have property $(mR)$, then the following properties hold:

(a) If $V \in m_X$ then $m_X\text{-Cl}(V)$ is $m_X$-regular.

(b) If $F$ is $m_X$-closed then $m_X\text{-Int}(A)$ is $m_X$-regular.

**Proof.** (a) If $V \in m_X$ then by property $(mR)$ (b), $m_X\text{-Int}(m_X\text{-Cl}(V)) = m_X\text{-Cl}(V)$, that is $m_X\text{-Cl}(V)$ is both $m_X$-open and $m_X$-closed.

That is the $m_X$-closure of every $m_X$-open set is $m_X$-open, then $m_X$ is $m$-extremely disconnected (see [40] Definition 3.14).

(b) If $F$ is $m_X$-closed then by property $(mR)$ (a), $m_X\text{-Cl}(m_X\text{-Int}(F)) = m_X\text{-Int}(F)$, that is $m_X\text{-Int}(F)$ is both $m_X$-open and $m_X$-closed. □

**Definition 6.** A function $f : (X, m_X) \rightarrow (Y, m_Y)$, where $X$ and $Y$ are nonempty sets with minimal structures $m_X$ and $m_Y$, respectively, is said to be weakly $M$-continuous ([28]) ($M$-continuous, ([30]), almost $M$-continuous ([5])) at $x \in X$ if for each $V \in m_Y$ containing $f(x)$ there exist $U \in m_X$ containing $x$ such that $f(U) \subset m_Y\text{-Cl}(V)$ (resp. $f(U) \subset V, f(U) \subset m_Y\text{-Int}(m_Y\text{-Cl}(V))$). A function $f : (X, m_X) \rightarrow (Y, m_Y)$ is said to be weakly $M$-continuous (resp. $M$-continuous, almost $M$-continuous) if it has the property at each point $x \in X$.

**Definition 7.** A function $f : (X, m_X) \rightarrow (Y, m_Y)$, is said to be $M^*$-continuous ([20]) if for every $V \in m_Y, f^{-1}(V) \in m_X$. 
Remark 4. Let $Y$ be a nonempty set and $m_Y$ a minimal structure on $Y$ for which satisfying the property $(mR)$. For a function $f : (X, m_X) \to (Y, m_Y)$ the following properties are equivalent:

(a) $f$ is weakly $M$-continuous.

(b) $f$ is almost $M$-continuous.

(c) For $m_Y^* = m_Y(Y)$, $f : (X, m_X) \to (Y, m_Y^*)$ is $M$-continuous.

Proof. Let $V \in m_Y$, then by property $(mR)(b)$, $m_X-\text{Int}(m_X-\text{Cl}(V)) = m_X-\text{Cl}(V)$, that is $m_X-\text{Int}(m_X-\text{Cl}(V))$ and $m_X-\text{Cl}(V)$ are both $m_X$-regular and equal.

Lemma 5 ([30]). For a function $f : (X, m_X) \to (Y, m_Y)$ the following properties are equivalent:

(a) $f$ is $M$-continuous.

(b) $f^{-1}(V) = m_X-\text{Int}(f^{-1}(V))$ for every $V \in m_Y$.

(c) $f(m_X-\text{Cl}(A)) \subset m_Y-\text{Cl}(f(A))$ for every subset $A$ of $X$.

(d) $m_X-\text{Cl}(f^{-1}(B)) \subset f^{-1}(m_Y-\text{Cl}(B))$ for every subset $B$ of $Y$.

(e) $f^{-1}(m_X-\text{Int}(B)) \subset m_X-\text{Int}(f^{-1}(B))$ for every subset $B$ of $Y$.

(f) $m_X-\text{Cl}(f^{-1}(K)) = f^{-1}(K)$ for every $m_Y$-closed set $K$ of $Y$.

Definition 8. A function $f : X \to Y$ is $(\tau, m)$-continuous ([29]), (resp. weakly $(\tau, m)$-continuous ([29]), almost $(\tau, m)$-continuous) for each $x \in X$ and each $m_Y$-open set $V$ containing $f(x)$, there exists an open set $U$ containing $x$, such that $f(U) \subset V$ (resp. $f(U) \subset m_Y-\text{cl}(V)$, $f(U) \subset m_Y-\text{Int}(m_Y-\text{cl}(V))$).

Definition 9. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be;

(a) almost continuous ([35]) (resp. $(\theta, b)$-continuous, almost $s$-continuous ([22]), weakly $(\tau, \beta)$-continuous ([29]), $p(\theta)$-continuous ([7])), if for each $x \in X$ and each open (resp. $b$-open, semiopen, $\beta$-open, preopen) set $V$ containing $f(x)$, there exists an open set $U$ containing $x$ such that $f(U) \subset \text{int}(\text{cl}(V))$ (resp. $f(U) \subset b\text{cl}(V)$, $f(U) \subset s\text{cl}(V)$, $f(U) \subset \beta\text{cl}(V)$, $f(U) \subset p\text{cl}(V)$).
(b) almost clopen ([11]) if for each \( x \in X \) and each open set \( V \) in \( Y \) containing \( f(x) \), there exists a clopen set \( U \) containing \( x \) such that \( f(U) \subset \text{int}(\text{cl}(V)) \).

**Remark 5.** Let \((X, \tau)\) and \((Y, \sigma)\) be topological spaces.

(a) We put \( m_X = \tau \) and \( m_Y = \tau \) (resp. \( SO(Y), PO(Y), \beta O(Y) \)). Then, a weakly \( M \)-continuous function \( f : (X, \tau) \to (Y, m_Y) \) is weakly continuous (resp. almost \( s \)-continuous, \( p(\theta) \)-continuous, weakly \((\tau, \beta)\)-continuous).

(b) We put \( m_X = \tau_q \) and \( m_Y = \tau \). Then, an almost \( M \)-continuous \( f : (X, \tau_q) \to (Y, m_Y) \) is almost clopen.

**Definition 10.** Let \( Y \) be a nonempty set and \( m_Y \) a minimal structure on \( Y \). A function \( f : X \to Y \), is said to be \((\tau_q, m)\)-continuous (resp. weakly \((\tau_q, m)\)-continuous, almost \((\tau_q, m)\)-continuous) at \( x \in X \), if for each \( V \in m_Y \) containing \( f(x) \) there exists a clopen set \( U \) containing \( x \) such that \( f(U) \subset V \) (resp. \( f(U) \subset m_Y \cdot \text{Cl}(V) \), \( f(U) \subset m_Y \cdot \text{Int}(m_Y \cdot \text{Cl}(V)) \)).

A function \( f : (X, \tau_q) \to (Y, m_Y) \) is said to be \((\tau_q, m)\)-continuous (resp. weakly \((\tau_q, m)\)-continuous, almost \((\tau_q, m)\)-continuous) if it has the property at each point \( x \in X \).

We will write for \((\tau_q, m)\)-continuous (resp. weakly \((\tau_q, m)\)-continuous, almost \((\tau_q, m)\)-continuous) briefly \((\tau_q, m)\).c (resp. \( w.(\tau_q, m).c \), \( a.(\tau_q, m).c \)).

**Proposition 1.** A function \( f : (X, \tau_q) \to (Y, m_Y) \) is \((\tau_q, m)\)-continuous (resp. weakly \((\tau_q, m)\)-continuous, almost \((\tau_q, m)\)-continuous) if and only if \( f : (X, \tau_q) \to (Y, m_Y) \) is \( M \)-continuous (resp. weakly \( M \)-continuous, almost \( M \)-continuous).

**Proof.** \((\Rightarrow)\) Let \( x \in X \) and \( V \) be a \( m_Y \)-open set in \( Y \) containing \( f(x) \). Then by definition there exists a clopen set \( U \) containing \( x \) such that \( f(U) \subset V \) (resp. \( f(U) \subset m_Y \cdot \text{Cl}(V) \), \( f(U) \subset m_Y \cdot \text{Int}(m_Y \cdot \text{Cl}(V)) \)). Since every clopen set is quasi-open we have \( f : (X, \tau_q) \to (Y, m_Y) \) is \( M \)-continuous (resp. weakly \( M \)-continuous, almost \( M \)-continuous).

\((\Leftarrow)\) Let \( x \in X \) and \( V \) be a \( m_Y \)-open set containing \( f(x) \) then there exists a quasi-open set \( U \) containing \( x \), such that \( f(U) \subset V \) (resp. \( f(U) \subset m_Y \cdot \text{Cl}(V) \), \( f(U) \subset m_Y \cdot \text{Int}(m_Y \cdot \text{Cl}(V)) \)). Since \( U \) is quasi open there exists a clopen set \( W \) in \( U \) containing \( x \) such that \( f(W) \subset V \) (resp. \( f(W) \subset m_Y \cdot \text{Cl}(V) \), \( f(W) \subset m_Y \cdot \text{Int}(m_Y \cdot \text{Cl}(V)) \)) and by Definition 10, \( f \) is \((\tau_q, m)\).c. (resp. \( w.(\tau_q, m)\).c., \( a.(\tau_q, m)\).c.). □
Theorem 1. For a function \( f : (X, \tau_q) \to (Y, m_Y) \) the following properties are equivalent:

(a) \( f \) is \((\tau_q, m)\)-continuous.
(b) \( f^{-1}(V) = \text{int}_q(f^{-1}(V)) \) for every \( V \in m_Y \).
(c) \( f(\text{cl}_q(A)) \subset m_Y-\text{Cl}(f(A)) \) for every subset \( A \) of \( X \).
(d) \( \text{cl}_q(f^{-1}(B)) \subset f^{-1}(m_Y-\text{Cl}(B)) \) for every subset \( B \) of \( Y \).
(e) \( f^{-1}(m_Y-\text{Int}(B)) \subset \text{int}_q(f^{-1}(B)) \) for every subset \( B \) of \( Y \).
(f) \( \text{cl}_q(f^{-1}(K)) = f^{-1}(K) \) for every \( m_Y \)-closed set \( K \) of \( Y \).

Proof. Here we will use same techniques with the proof of Lemma 5 ([30]).

(a)\(\Rightarrow\)(b) Let \( V \in m_Y \) and \( x \in f^{-1}(V) \). Then \( f(x) \in V \). There exists \( U \in \tau_q \) containing \( x \) such that \( f(U) \subset V \). Thus \( x \in U \subset f^{-1}(V) \). This implies that \( x \in \text{int}_q(f^{-1}(V)) \). This shows that \( f^{-1}(V) \subset \text{int}_q(f^{-1}(V)) \). Hence we have \( f^{-1}(V) = \text{int}_q(f^{-1}(V)) \).

(b)\(\Rightarrow\)(c) Let \( A \) be any subset of \( X \). Let \( x \in \text{cl}_q(A) \) and \( V \in m_Y \) containing \( f(x) \). Then \( x \in f^{-1}(V) = \text{int}_q(f^{-1}(V)) \). There exists \( U \in \tau_q \) such that \( x \in U \subset f^{-1}(V) \). Since \( x \in \text{cl}_q(A) \), \( U \cap A \neq \emptyset \) and \( \emptyset \neq f(U \cap A) \subset f(U) \cap f(A) \subset V \cap f(A) \). Since \( V \in m_Y \) containing \( f(x) \), \( f(x) \in m_Y-\text{Cl}(f(A)) \) and hence \( f(\text{cl}_q(A)) \subset m_Y-\text{Cl}(f(A)) \).

(c)\(\Rightarrow\)(d) Let \( B \) be any subset of \( Y \). Then, we have \( f(\text{cl}_q(f^{-1}(B))) \subset m_Y-\text{Cl}(f(f^{-1}(B))) \subset m_Y-\text{Cl}(B) \). Therefore we obtain \( \text{cl}_q(f^{-1}(B)) \subset f^{-1}(m_Y-\text{Cl}(B)) \).

(d)\(\Rightarrow\)(e) Let \( B \) be any subset of \( Y \). Then, we have \( X - \text{int}_q(f^{-1}(B)) \) \( = \text{cl}_q(f^{-1}(Y-B)) \subset f^{-1}(m_Y-\text{Cl}(Y-B)) = f^{-1}(Y - m_Y-\text{Int}(B)) = X - f^{-1}(m_Y-\text{Int}(B)) \). Therefore, we obtain \( f^{-1}(m_Y-\text{Int}(B)) \subset \text{int}_q(f^{-1}(B)) \).

(e)\(\Rightarrow\)(f) Let \( K \) be any subset of \( Y \) such that \( Y - K \in m_Y \). By (e), we have \( X - f^{-1}(K) = f^{-1}(m_Y - \text{int}(Y - K)) \subset \text{int}_q(f^{-1}(Y - K)) = \text{int}_q(X - f^{-1}(K)) = \text{cl}_q(f^{-1}(K)) \). Therefore, we have \( \text{cl}_q(f^{-1}(K)) \subset f^{-1}(K) \subset \text{cl}_q(f^{-1}(K)) \). Thus we obtain \( \text{cl}_q(f^{-1}(K)) = f^{-1}(K) \).

(f)\(\Rightarrow\)(a) Let \( x \in X \) and \( V \in m_Y \) containing \( f(x) \). By (f) we have \( X - f^{-1}(V) = f^{-1}(Y - V) = \text{cl}_q(f^{-1}(Y - V)) = \text{cl}_q(X - f^{-1}(V)) = X - \text{int}_q(f^{-1}(V)) \). Hence, we have \( x \in f^{-1}(V) = \text{int}_q(f^{-1}(V)) \). Therefore, there exists \( U \in \tau_q \) such that \( x \in U \subset f^{-1}(V) \). Therefore, \( x \in U \in \tau_q \) and \( f(U) \subset V \). This shows that \( f \) is \((\tau_q, m)\)-continuous.
Definition 11 ([28]). Let $X$ be a nonempty set with a minimal structure $m_X$ is said to be $m$-regular if for each $m_X$-closed set $F$ and each $x \notin F$, there exist disjoint $m_X$-open sets $U$ and $V$ such that $x \in U$ and $F \subset V$.

Lemma 6 ([28]). Let $X$ be a nonempty set with a minimal structure $m_X$ and $m_X$ satisfy property $B$. Then $(X, m_X)$ is said to be $m$-regular if and only if for each $x \in X$ and each $m_X$-open set $U$ containing $x$, there exists an $m_X$-open set $V$ such that $x \in V \subset m_X-\text{Cl}(V) \subset U$.

Theorem 2. Let $(Y, m_Y)$ be $m$-regular and satisfying property $(B)$. Then for a function $f : (X, \tau_q) \to (Y, m_Y)$ the following properties are equivalent:

(a) $f$ is $(\tau_q, m).c.$

(b) $f^{-1}(m\text{Cl}_\theta(B)) \subset cl_q(f^{-1}(m\text{Cl}_\theta(B)))$ for every subset $B$ of $Y$.

(c) $f$ is weakly $(\tau_q, m)$-continuous.

(d) $f^{-1}(F) = cl_q(f^{-1}(F))$ for every $m_\theta$-closed set $F$ of $Y$.

(e) $f^{-1}(V) \subset int_q(f^{-1}(V))$ for every $m_\theta$-open set $V$ of $Y$.

Proof. Consider a function $f : (X, m_X) \to (Y, m_Y)$, where $X$ and $Y$ are nonempty sets with minimal structures $m_X$ and $m_Y$, respectively, and let $(Y, m_Y)$ be $m$-regular and satisfying the property $(B)$. Then put $m_X = \tau_q$ in Theorem 4.2 of [28].

Theorem 3. Let $(Y, m_Y)$ satisfy the property $(B)$. For a function $f : (X, \tau_q) \to (Y, m_Y)$, the following are equivalent:

(a) $f$ is w.$(\tau_q, m).c.$

(b) For each $x \in X$ and each $m_Y$-open set $V$ of $Y$ containing $f(x)$, there exists a quasi-open set $U$ of $X$ containing $x$ such that $f(U) \subset m_Y-\text{Cl}(V)$.

(c) $f^{-1}(V) \subset int_q(f^{-1}(m_Y-\text{Cl}(V)))$ every $m_Y$-open set $V$ of $Y$.

(d) $cl_q(f^{-1}(m_Y-\text{Int}(m_Y-\text{Cl}(B))) \subset f^{-1}(m_Y-\text{Cl}(B))$ for every subset $B$ of $Y$.

(e) $cl_q(f^{-1}(m_Y-\text{Int}(F))) \subset f^{-1}(F)$ every $m_Y$-closed set $F$ of $Y$. 
(f) \( cl_q(f^{-1}(V)) \subseteq f^{-1}(m_Y-Cl(V)) \) every \( m_Y \)-open set \( V \) of \( Y \).

(g) \( f(cl_q(A)) \subseteq mCl_\theta(f(A)) \) for each subset \( A \) of \( X \).

(h) \( cl_q(f^{-1}(B)) \subseteq f^{-1}(mCl_\theta(B)) \) for each subset \( B \) of \( Y \).

**Proof.** (a)\(\Leftrightarrow\)(b): These implications are clear from the definition of quasi topology.

(b)\(\Rightarrow\)(c): Let \( V \) be a \( m_Y \)-open set of \( Y \) and \( x \in f^{-1}(V) \). Then \( f(x) \in V \) and by (b), there exists a quasi-open set \( U \) of \( X \) containing \( x \) such that \( f(U) \subseteq m_Y-Cl(V) \). Then \( x \in U \subseteq f^{-1}(m_Y-Cl(V)) \) and hence \( x \in \text{int}_q(f^{-1}(m_Y-Cl(V))) \).

(a)\(\Rightarrow\)(c): It follows from Theorem 3.2 of [28].

(a)\(\Rightarrow\)(d)\(\Rightarrow\)(e)\(\Rightarrow\)(a): It follows from Theorem 2.1 of [25].

(f)\(\Rightarrow\)(a): It follows from Theorem 3.4 of [28].

(a)\(\Rightarrow\)(g)\(\Rightarrow\)(h)\(\Rightarrow\)(a): It follows from Theorem 3.3 of [28].

The above proofs do not use the property (\( B \)), except for the case (f)\(\Leftrightarrow\)(a).

**Definition 12.** Let \( Y \) be a nonempty set and \( m_Y \) a minimal structure on \( Y \). A function \( f : X \to Y \), is said to be \( (\tau_q,m^*) \)-continuous if for each \( V \in m_Y \), \( f^{-1}(V) \) is clopen in \( X \).

**Remark 6.** Let \( Y \) be a nonempty set and \( m_Y \) a minimal structure on \( Y \). We put \( m_Y = \tau \) (resp. RO\( (Y) \), SR\( (Y) \)). Then, a \( (\tau_q,m^*) \)-continuous function \( f : (X,\tau_q) \to (Y,m_Y) \) is perfectly continuous ([23]) (resp. regular set connected ([10]), almost s-continuous)

**Proposition 2.** Let \( Y \) be a nonempty set and \( m_Y \) a minimal structure on \( Y \). Then, the following are equivalent:

(a) \( f : X \to Y \) is \( (\tau_q,m^*) \)-continuous.

(b) For each \( Y - F \in m_Y \), \( X - f^{-1}(F) \) is clopen in \( X \).

**Proof.** (a)\(\Rightarrow\)(b) Let \( Y - F \in m_Y \), since \( f : X \to Y \) is \( (\tau_q,m^*) \)-continuous \( f^{-1}(Y-F) = X - f^{-1}(F) \) is clopen in \( X \) and hence \( f^{-1}(F) \) is clopen in \( X \).

(b)\(\Rightarrow\)(a) Let \( U \in m_Y \) then \( Y - (Y-U) \in m_Y \) and by (b), \( f^{-1}(Y-(Y-U)) = X - f^{-1}(Y-U) = X - (X - f^{-1}(U)) = f^{-1}(U) \) is clopen in \( X \), hence \( f : X \to Y \) is \( (\tau_q,m^*) \)-continuous.

Note that property (\( mR \)) requires property (\( B \)).
Theorem 4. Let $Y$ be a nonempty set and $m_Y$ a minimal structure on $Y$ for which satisfying the property $(m_R)$. Then the following properties are equivalent for a function $f : (X, \tau_q) \to (Y, m_Y)$:

(a) $f$ is weakly $(\tau, m)$-continuous.

(b) For each $x \in X$ and each $V \in m_R(Y, f(x))$, there exists an open set $U$ containing $x$ such that $f(U) \subset V$.

(c) For each $x \in X$ and each $V \in m_R(Y, f(x))$, there exists an $\alpha$-open set $U$ containing $x$ such that $f(U) \subset V$.

(d) $f^{-1}(V)$ is $\alpha$-open in $X$ for every $V \in m_R(Y)$.

(e) $f^{-1}(V)$ is clopen in $X$ for every $V \in m_R(Y)$.

(f) $f$ is w.($\tau_q, m$)-c.

(g) For each $x \in X$ and each $V \in m_R(Y, f(x))$, there exists an clopen set $U$ containing $x$ such that $f(U) \subset V$.

(h) For each $x \in X$ and each $V \in m_R(Y, f(x))$, there exists a quasi-open set $U$ of $X$ containing $x$ such that $f(U) \subset V$.

(i) $f : (X, \tau_q) \to (Y, m_Y)$ is weakly $M$-continuous.

Proof. (a)$\Rightarrow$(b): Let $x \in X$ and $V \in m_R(Y, f(x))$. There exists an open set $U$ containing $x$ such that $f(U) \subset m_Y$-$\text{Cl}(V) = V$.

(b)$\Rightarrow$(c): This is clear.

(c)$\Rightarrow$(d): Let $V \in m_R(Y)$ and $x \in f^{-1}(V)$. Then $f(x) \in V \in m_Y$. There exists an $\alpha$-open set $U_x$ containing $x$ such that $f(U_x) \subset m_Y$-$\text{Cl}(V) = V$. Therefore, $x \in U_x \subset f^{-1}(V)$ and hence $\bigcup_{x \in f^{-1}(V)} U_x = f^{-1}(V)$ is $\alpha$-open in $X$.

(d)$\Rightarrow$(e): Let $V \in m_R(Y)$. Since $Y - V \in m_R(Y)$ by (d) $X - f^{-1}(V) = f^{-1}(Y - V)$ is $\alpha$-open. Therefore $f^{-1}(V)$ $\alpha$-closed and $\alpha$-open in $X$. Hence by Lemma 3.1 of [14], $f^{-1}(V)$ is clopen. Note that (e) can be rephrased as, $f : (X, \tau_q) \to (Y, m_Y^1)$ is $(\tau_q, m^1)$-continuous or $M^*$-continuous, where $m^1_Y = m_R(Y)$.

(e)$\Rightarrow$(f): Let $x \in X$ and $V$ be any $m_Y$-open set of $Y$ containing $f(x)$. By Lemma 4, $m_Y$-$\text{Cl}(V)$ is $m_Y$-clopen and hence $f^{-1}(m_Y$-$\text{Cl}(V))$ is clopen
in $X$. Put $U = f^{-1}(m_Y-Cl(V))$, then $U$ is clopen set containing $x$ and $f(U) \subset m_Y-Cl(V)$.

$(f) \Rightarrow (g)$: Let $x \in X$ and $V \in mR(Y, f(x))$. There exists a clopen set $U$ containing $x$ such that $f(U) \subset m_Y-Cl(V) = V$.

$(g) \Rightarrow (h)$: It follows from the definition of quasi topology.

$(h) \Rightarrow (i)$: Let $x \in X$ and $V \in mR(Y, f(x))$. Then by $(h)$, there exists a quasi-open set $U$ containing $x$ such that $f(U) \subset m_Y-Cl(V) = V$.

Since every $m_Y$-regular set is $m_Y$-open, $f$ is $(\tau_q, m)$-continuous. Then $f : (X, \tau_q) \rightarrow (Y, m_Y)$ is $M$-continuous, hence weakly $M$-continuous.

$(i) \Rightarrow (a)$: This is clear.□

**Remark 7.** Let $Y$ be a nonempty set and $m_Y$ a minimal structure on $Y$ if $m_Y \in \{SO(X), BO(X), \beta O(X)\}$ then $m_Y$ has property $(mR)$, so a weakly $(\tau_q, m)$-continuous function $f : (X, \tau_q) \rightarrow (Y, m_Y)$ is a generalization and unification of almost $s$-continuity (resp. $(\theta, b)$-continuity, weakly $(\tau, \beta)$-continuity).

**Remark 8.** We have the following implications for a function $f : X \rightarrow Y$:

$$(\tau, m)\text{-continuous} \Rightarrow \text{almost } (\tau, m)\text{-continuous} \Rightarrow \text{weakly } (\tau, m)\text{-continuous}$$

$$\uparrow \quad \uparrow \quad \uparrow$$

$$(\tau_q, m)\text{-continuous} \Rightarrow \text{almost } (\tau_q, m)\text{-continuous} \Rightarrow \text{weakly } (\tau_q, m)\text{-continuous}$$

$$\uparrow$$

$$(\tau_q, m^*)\text{-continuous}$$

Note that these implications cannot be reversed in general as the following examples shows:

**Example 1.** Let $X$ be the real numbers with the upper limit topology $\tau$ and $Y$ be the real numbers with the usual topology $\sigma$. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be the identity function. Then $f$ is $(\tau_q, \sigma)$-continuous function but not $(\tau_q, \sigma^*)$-continuous since $f^{-1}((0, 1))$ is not clopen in $(X, \tau)$.

**Example 2** ([11]). Let $X$ be the real numbers with the usual topology $\tau$ and $f : X \rightarrow X$ be the identity function. Then $f$ is an almost $(\tau, \tau)$-continuous function which is not almost $(\tau_q, \tau)$-continuous.

**Example 3** ([11]). Let $\mathbb{R}$ and $\mathbb{Q}$ be the real and rational numbers, respectively. Let $A = \{x \in \mathbb{R} : x \text{ is rational and } 0 < x < 1\}$. We define two topologies on $\mathbb{R}$ as $\tau = \{\mathbb{R}, \emptyset, A, \mathbb{R} - A\}$ and $\nu = \{\mathbb{R}, \emptyset, \{0\}\}$. Let $f : (\mathbb{R}, \tau) \rightarrow$
(\mathbb{R}, \nu) be a function which is defined by \( f(x) = 1 \) if \( x \in \mathbb{Q} \) and \( f(x) = 0 \) if \( x \notin \mathbb{Q} \). Then \( f \) is almost \((\tau_q, \nu)\)-continuous and weakly \((\tau_q, \nu)\)-continuous, but \( f \) is not \((\tau_q, \nu)\)-continuous since for \( f(x) = 0 \in \{0\} \in \nu(x \notin \mathbb{Q}) \), there is no clopen set \( U \) containing \( x \) such that \( f(U) \subset \{0\} \).

**Example 4** ([36]). Let \( X = \{a, b, c, d\} \), \( \tau = \{X, \emptyset, \{d\}, \{a, b, c\}\} \) and \( \sigma = \{X, \emptyset, \{c\}, \{d\}, \{a, c\}, \{c, d\}, \{a, c, d\}\} \). Then the identity function \( f : (X, \tau) \to (X, \sigma) \) is weakly \((\tau_q, \sigma)\)-continuous but not almost \((\tau, \sigma)\)-continuous (hence not almost \((\tau_q, \sigma)\)-continuous) since there exists a regular open set \( \{a, c\} \) of \((X, \sigma)\) such that \( f^{-1}(\{a, c\}) \) is not clopen in \((X, \tau)\).

**Example 5** ([36]). Let \( X \) be the real numbers and \( \tau \) be the usual topology on \( X \). Then the identity function \( f : (X, \tau) \to (X, \tau) \) is almost \((\tau, \tau)\)-continuous (hence weakly \((\tau, \tau)\)-continuous) but not weakly \((\tau_q, \tau)\)-continuous since the only clopen set of \( X \) is itself.

**Definition 13.** A filter base \( F \) is said to be:

(a) \( m_X\)-\(\theta\)-convergent ([21]) to a point \( x \) in \( X \), if for any \( m_X\)-open set \( U \) containing \( x \) there exist \( B \in F \) such that \( B \subset m_X-Cl(U) \).

(b) clopen convergent ([12]) to a point \( x \) in \( X \), if for any clopen set \( U \) containing \( x \), there exist \( B \in F \) such that \( B \subset U \).

**Definition 14.** A net \((x_\lambda)\) in a space \( X \), \(\theta\)-converges ([8]) (resp. clopen converges ([15]), \( m_X\)-\(\theta\)-converges ([21])) to \( x \) if and only if for each open (resp. clopen, \( m_X\)-open) set \( U \) containing \( x \), there exists a \( \lambda_0 \) such that \( x_\lambda \in cl(U) \) (resp. \( x_\lambda \in U \), \( x_\lambda \in m_X-Cl(U) \)) for all \( \lambda \geq \lambda_0 \).

**Lemma 7.** For a net \((x_\lambda)\) in a space \( X \):

(a) if \((x_\lambda)\) converges to \( x \), then \((x_\lambda)\) \(\theta\)-converges to \( x \) ([6]).

(b) if \((x_\lambda)\) converges or \(\theta\)-converges to \( x \), then \((x_\lambda)\) clopen converges to \( x \) ([15]).

**Theorem 5.** A function \( f : X \to Y \) is weakly \((\tau_q, m)\)-continuous if and only if for each point \( x \in X \) and each filter base \( F \) in \( X \) that clopen converging to \( x \) the filter base \( f(F) \) is \( m_Y\)-\(\theta\)-convergent to \( f(x) \).
Proof. Suppose that \( x \in X \) and \( \mathcal{F} \) is any filter base in \( X \) that clopen converges to \( x \). By hypothesis, for any \( m_Y \)-open set \( V \) containing \( f(x) \) there exists a clopen set \( U \) containing \( x \) in \( X \) such that \( f(U) \subseteq m_Y - \text{Cl}(V) \). Since \( \mathcal{F} \) is clopen convergent to \( x \) in \( X \) then there exists \( B \in \mathcal{F} \) such that \( B \subseteq U \). It follows that \( f(B) \subseteq m_Y - \text{Cl}(V) \). This means that \( f(\mathcal{F}) \) is \( m_Y \)-convergent to \( f(x) \).

Conversely, let \( x \) be a point in \( X \) and \( V \) be a \( m_Y \)-open set containing \( f(x) \). If we set \( \mathcal{F} = \{ U : U \text{ is clopen and } x \in U \} \), then \( \mathcal{F} \) will be a filter base which clopen converges to \( x \). So there exists \( U \in \mathcal{F} \) such that \( f(U) \subseteq m_Y - \text{Cl}(V) \). This completes the proof. \( \square \)

Theorem 6. The implications \((a) \Leftrightarrow (b) \Rightarrow (c) \Leftrightarrow (d) \) hold for the following properties of a function \( f : (X, \tau) \rightarrow (Y, m_Y) \):

(a) \( f \) is \( w(\tau_q, m).c. \)

(b) For each \( x \in X \) and each net \( \{ x_\lambda \} \) in \( X \) which clopen converges to \( x \), the net \( \{ f(x_\lambda) \} \) \( m_Y \)-converges to \( f(x) \).

(c) For each \( x \in X \) and each net \( \{ x_\lambda \} \) in \( X \) which \( \theta \)-converges to \( x \), the net \( \{ f(x_\lambda) \} \) \( m_Y \)-converges to \( f(x) \).

(d) For each \( x \in X \) and each net \( \{ x_\lambda \} \) in \( X \) which converges to \( x \), the net \( \{ f(x_\lambda) \} \) \( m_Y \)-converges to \( f(x) \).

(e) \( f \) is weakly \( (\tau, m) \)-continuous.

Proof. \((a) \Rightarrow (b)\): Let \( x \in X \) and let \( \{ x_\lambda \} \) be a net in \( X \) such that \( \{ x_\lambda \} \) clopen converges to \( x \). Let \( V \) be a \( m_Y \)-open set containing \( f(x) \). Since \( f \) is weakly \( (\tau_q, m) \)-continuous, there exists a clopen set \( U \) containing \( x \) such that \( f(U) \subseteq m_Y - \text{Cl}(V) \). Since \( \{ x_\lambda \} \) clopen converges to \( x \), there exists \( \lambda_0 \) such that \( x_\lambda \in U \) for all \( \lambda \geq \lambda_0 \). Hence \( f(x_\lambda) \in m_Y - \text{Cl}(V) \) for all \( \lambda \geq \lambda_0 \).

\((b) \Rightarrow (a)\): Suppose that \( f \) is not weakly \( (\tau_q, m) \)-continuous. Then there exists \( x \in X \) and a \( m_Y \) open set \( V \) containing \( f(x) \) such that \( f(U) \notin m_Y - \text{Cl}(V) \) for all clopen neighborhoods \( U \) of \( x \). Thus, for every clopen neighborhood \( U \) of \( x \) we can find \( x_U \in U \) such that \( f(x_U) \notin m_Y - \text{Cl}(V) \). Let \( \mathcal{N}(x) \) be the set of clopen neighborhoods of \( x \) in \( X \). The set \( \mathcal{N}(x) \) with the relation of inverse inclusion (that is \( U_1 \subseteq U_2 \) if and only if \( U_2 \subseteq U_1 \)) forms a directed set (Theorem 1.1 of [12]). Clearly the net \( \{ x_U : U \in \mathcal{N}(x) \} \)
clop is a clopen set \( U \) such that (\( \text{clp} \) said to be strongly

\( m \)-closed graph are generalizations of the the following notio ns.

Remark 9. If a function \( f : (X, m_X) \to (Y, m_Y) \) has the strongly

\( M \)-closed graph, then for the special case \( m_X = \tau_q \), \( G(f) \) has strongly
clop-
closed graph are generalizations of the the following notions.
Definition 17. A function \( f : (X, \tau) \to (Y, \sigma) \) has a strongly-closed (\([13]\) (strongly clp-closed = strongly clopen \([36]\))) graph if for each \((x, y) \notin G(f)\), there exists open sets \( U \in \tau \) (\( U \in \tau_q \)) and \( V \in \sigma \) containing \( x \) and \( y \), respectively, such that \( (U \times \text{cl}(V)) \cap G(f) = \emptyset \).

Note that for a graph \( G(f) \) strongly clp-closedness imply strongly closedness, but the reverse implication is not true in general as the following example shows.

Example 6 \([13]\). Let \( X = [0, 1] \) have the usual topology \( \tau_{R|X} (= m_X) \) and let \( Y = [0, 1] \) have the topology \( \sigma (= m_Y) \) generated by the usual open sets together with the set \( A = \{ r : r \in \mathbb{Q} \text{ and } \frac{1}{4} < r < \frac{3}{4} \} \) as subbase. The identity function \( i : (X, \tau_{R|X}) \to (Y, \sigma) \), has a strongly-closed graph \( G(i) \). But \( G(i) \) is not strongly clp-closed.

Theorem 7. The following properties are equivalent for a graph \( G(f) \) of a function:

(a) \( G(f) \) is strongly clp-\( m \)-closed.

(b) For each point \((x, y) \in (X \times Y) - G(f)\), there exists a clopen set \( U \) containing \( x \) in \( X \) and \( m_Y \)-open set \( V \) containing \( y \) such that \( f(U) \cap m_Y - \text{Cl}(V) = \emptyset \).

(c) For each point \((x, y) \in (X \times Y) - G(f)\), there exists a quasi-open set \( U \) containing \( x \) in \( X \) and \( m_Y \)-open set in \( Y \) containing \( y \) such that \( f(U) \cap m_Y - \text{Cl}(V) = \emptyset \).

Proof. (a)\( \Rightarrow \) (b) It follows from Lemma 8.

(b)\( \Rightarrow \) (c) It is clear since every clopen set is quasi-open.

(c)\( \Rightarrow \) (a) If (c) holds, then the set \( U \) in the statement of (c) is quasi open. Then, there exists a clopen set \( W \) such that \( W \subset U \) and we have \( f(W) \cap m_Y - \text{Cl}(V) \subset f(U) \cap m_Y - \text{Cl}(V) = \emptyset \). By Lemma 8 result follows. \( \square \)

Definition 18. A nonempty set \( X \) with a minimal structure \( m_X \), \((X, m_X)\), is said to be \( m \)-\( T_2 \) \([30]\) (resp. \( m \)-Urysohn \([28]\)) if for each distinct points \( x, y \in X \), there exist \( U, V \in m_X \) containing \( x \) and \( y \), respectively, such that \( U \cap V = \emptyset \) (resp. \( m_X - \text{Cl}(U) \cap m_X - \text{Cl}(V) = \emptyset \)).

See \([34]\) for a study on minimal structures and separation properties.
Theorem 8. If \( f : X \to Y \) is \((\tau_q, m).c.\) function and \( Y \) is \( m\text{-}T_2\), then \( G(f) \) is strongly clp\(\text{-}m\)-closed in \( X \times Y \).

Proof. If the condition holds and \((x, y) \in (X \times Y) - G(f)\) then, it is true that \( f(x) \neq y \) and there exists \( V, W \in m_Y \) containing \( y \) and \( f(x) \), respectively, such that \( V \cap W = \emptyset \). Then, by Lemma 2, \( m_X \cdot Cl(V) \cap W = \emptyset \).

Now, as \( f \) is \((\tau_q, m).c.\) there exists a clopen set \( U \) in \( X \) containing \( x \) such that \( f(U) \subset W \). Therefore \( f(U) \cap m_Y \cdot Cl(V) = \emptyset \) and \( G(f) \) is strongly clp\(\text{-}m\)-closed. \( \square \)

Theorem 9. If \( f : X \to Y \) is \( w.(\tau_q, m).c. \) function and \( Y \) is \( m\text{-}Urysohn \), then \( G(f) \) is strongly clp\(\text{-}m\)-closed in \( X \times Y \).

Proof. If the condition holds and \((x, y) \in (X \times Y) - G(f)\) then, it is true that \( f(x) \neq y \) and there exists \( V, W \in m_Y \) containing \( y \) and \( f(x) \), respectively, such that \( m_Y \cdot Cl(V) \cap m_Y \cdot Cl(W) = \emptyset \). Now, as \( f \) is \( w.(\tau_q, m).c. \) there exists a clopen set \( U \) in \( X \) containing \( x \) such that \( f(U) \subset m_Y \cdot Cl(W) \).

Therefore \( f(U) \cap m_Y \cdot Cl(V) = \emptyset \) and \( G(f) \) is strongly clp\(\text{-}m\)-closed. \( \square \)

Theorem 10 ([39]). If \( f : (X, \tau_q) \to (Y, m_Y) \) is a \( w.(\tau_q, m).c. \) and \((Y, m_Y)\) is \( m\text{-}T_2\), then \( f \) has quasi-closed point inverses in \( X \).

Theorem 11. If \( f, g : X \to Y \) is \((\tau_q, m).c. \) function and \( Y \) is \( m\text{-}Urysohn \) then \( A = \{ x \in X : f(x) = g(x) \} \) is quasi-closed in \( X \).

Proof. If \( x \in X - A \), then it follows that \( f(x) \neq g(x) \). Since \( Y \) is \( m\text{-}Urysohn \), there exists \( m_Y\)-open set \( U \) in \( Y \) containing \( f(x) \) and \( m_Y\)-open set \( V \) in \( Y \) containing \( g(x) \) such that \( m_Y \cdot Cl(U) \cap m_Y \cdot Cl(V) = \emptyset \). Since \( f \) and \( g \) are \((\tau_q, m).c. \) there exists clopen sets \( G \) and \( H \) with \( x \in G \) and \( x \in H \) such that \( f(G) \subset m_Y \cdot Cl(U) \) and \( g(H) \subset m_Y \cdot Cl(W) \), set \( O = G \cap H \).

Then \( O \) is clopen, \( f(O) \cap g(O) = \emptyset \) and \( A \cap O = \emptyset \). Thus every point of \( X - A \) has a clopen neighborhood disjoint from \( A \). Hence \( X - A \) is a union of clopen sets or equivalently \( A \) is quasi-closed. \( \square \)

Theorem 12. If \( f : (X, \tau_q) \to (Y, m_Y) \) is \((\tau_q, m).c. \) function and \( Y \) is \( m\text{-}Urysohn \), then \( A = \{(x, y) \in X \times X : f(x) = f(y)\} \) is quasi-closed in \( X \times X \).

Proof. Let \((x, y) \in (X \times X) - A\), then it follows that \( f(x) \neq f(y) \). Since \( Y \) is \( m\text{-}Urysohn \), there exist \( m_Y\)-open set \( U \) containing \( f(x) \) and \( m_Y\)-open set \( V \) containing \( f(y) \) such that \( m_Y \cdot Cl(U) \cap m_Y \cdot Cl(V) = \emptyset \). Since \( f \) is \((\tau_q, m).c. \), there exists clopen sets \( W \) and \( O \) with \( x \in O \) and \( y \in W \).
such that \( f(O) \subseteq m_Y\text{-}Cl(U) \) and \( f(W) \subseteq m_Y\text{-}Cl(V) \). Then, we have 
\((x, y) \in O \times W \subseteq f^{-1}(m_Y\text{-}Cl(U)) \times f^{-1}(m_Y\text{-}Cl(V)) \). Thus we have \( O \times W \) is a clopen set containing \((x, y)\) and \( O \times W \in (X \times X) - A \). Hence \((X \times X) - A \) is union of clopen sets or equivalently \( A \) is quasi-closed in \( X \times X \).

**Definition 19.** A space \( X \) is said to be **ultra Hausdorff** ([36]) if every two distinct points of \( X \) can be separated by disjoint clopen sets. Note that if a space \( X \) is ultra Hausdorff then it is totally disconnected.

**Theorem 13.** Let \( f : (X, \tau) \to (Y, m_Y) \) have a strongly clp-m-closed graph. Then the following properties hold:

(a) If \( f \) is injective then \( X \) is ultra Hausdorff.

(b) If \( f \) is surjective then \( Y \) is \( m\)-\( T_2 \).

**Proof.** (a) Suppose that \( x \) and \( y \) are any two distinct points of \( X \) by the injectivity of \( f \), \( (x, f(y)) \notin G(f) \). Since \( G(f) \) is strongly clp-m-closed, by Theorem 7, there exist a clopen set \( U \) containing \( x \) and \( m_Y\)-open set \( V \) containing \( f(y) \) such that \( f(U) \cap m_Y\text{-}Cl(V) = \emptyset \). We have \( U \cap f^{-1}(m_Y\text{-}Cl(V)) = \emptyset \). Therefore \( y \notin U \). Then \( U \) and \( X - U \) are disjoint clopen sets containing \( x \) and \( y \), respectively. Hence \( X \) is ultra Hausdorff.

(b) Let \( y_1 \) and \( y_2 \) be any two distinct points of \( Y \). Since \( f \) is surjective there exists a point \( x \in X \) such that \( f(x) = y_2 \). Since \( G(f) \) is strongly clp-m-closed and \( (x, y_1) \notin G(f) \) there exists a clopen set \( U \) containing \( x \) and \( m_Y\)-open set \( V \) in \( Y \) containing \( y_1 \) such that \( f(U) \cap m_Y\text{-}Cl(V) = \emptyset \). Therefore we have \( y_2 \in f(U) \subseteq Y - (m_Y\text{-}Cl(V)) \) and hence by Lemma 2, \( Y \) is \( m\)-\( T_2 \).

**Definition 20.** A function \( f : (X, m_X) \to (Y, m_Y) \) is said to be \( M\)-closed ([26]) if for each \( m_X\)-closed set \( F \), \( f(F) \) is \( m_Y\)-closed in \( Y \).

**Definition 21.** A space \( X \) is said to be:

(a) **ultraregular** ([37]) if for each closed set \( F \) and each \( x \notin F \), there exist disjoint clopen sets \( U \) and \( V \) such that \( x \in U \) and \( F \subseteq V \).

(b) **ultranormal** ([37]) if disjoint closed sets contained in disjoint clopen sets.

Note that if a space \( X \) is ultraregular then it has a basis consisting of clopen sets.

**Theorem 14.** Let \( Y \) be a nonempty set and \( m_Y \) a minimal structure on \( Y \) for which satisfying the property \((mR)\). If \( f : X \to Y \) is a \( w.(\tau, m).c. \) and \( M\)-closed injection and \( Y \) is \( m\)-regular, then \( X \) is ultraregular.
Proof. Let $F$ be any quasi-closed set of $X$ and $x \in X - F$. Since $f$ is $M$-closed, $f(F)$ is $m_Y$-closed and $f(x) \in Y - f(F)$. Since $(Y, m_Y)$ is $m$-regular, there exist disjoint $m_Y$-open sets $U$ and $V$ such that $f(x) \in U$ and $f(F) \subset V$. Since $U \cap V = \emptyset$ by Lemma 2, we have $m_Y$-Cl$(V) \cap U = \emptyset$. Since $m_Y$ has property (mR), $m_Y$-Cl$(V)$ is a $m$-regular set containing $f(F)$ and disjoint from $f(x)$. Since $f$ is w.($\tau_q, m$).c. by Theorem 4, the inverse image of $m_Y$-Cl$(V)$, under $f$ is a clopen subset of $X$ containing $F$ and disjoint from $x$. This shows $X$ is ultraregular. \hfill \Box

Definition 22. An $m$-space $(X, m_X)$ is said to be $m$-normal ([26]) if for each pair of disjoint $m$-closed sets $F_1, F_2$ of $X$, there exist $U_1, U_2 \in m_X$ such that $F_1 \subset U_1$, $F_2 \subset U_2$ and $U_1 \cap U_2 = \emptyset$.

Theorem 15. Let $Y$ be a nonempty set and $m_Y$ a minimal structure on $Y$ for which satisfying the property (mR). If $f : X \rightarrow Y$, is a w.$(\tau_q, m$).c. and $M$-closed injection and $Y$ is $m$-normal, then $X$ is ultranormal.

Proof. Let $A$ and $B$ be disjoint closed sets of $X$. Since $f$ is $M$-closed injection, $f(A)$ and $f(B)$ are disjoint $m_Y$-closed sets of $Y$. By the $m$-normality of $(Y, m_Y)$, there exist disjoint $m_Y$-open sets $U$ and $V$ such that $f(A) \subset U$ and $f(B) \subset V$. Since $U \cap V = \emptyset$ by Lemma 2, we have $m_Y$-Cl$(U) \cap V = \emptyset$. Since $m_Y$ has property (mR), $m_Y$-Cl$(U)$ is a $m$-regular set containing $f(A)$ and disjoint from $f(B)$. Since $f$ is w.($\tau_q, m$).c. by Theorem 4, the inverse image of $m_Y$-Cl$(U)$ under $f$ is a clopen subset of $X$ containing $A$ and disjoint from $B$. Thus $X$ is ultranormal. \hfill \Box

Definition 23. A subset $K$ of a space $X$ is said to be mildly compact ([37]), relative to $X$ if for every cover $\{V_\alpha : \alpha \in I\}$ of $K$ by clopen sets of $X$, there exists a finite subset $I_0$ of $I$ such that $K \subset \cup\{V_\alpha : \alpha \in I_0\}$.

Definition 24. A subset $K$ of a nonempty set $X$ with a minimal structure $m_X$ is said to be $m$-compact ([30]) ($m$-closed ([21])) relative to $(X, m_X)$ if any cover $\{U_i : i \in I\}$ of $K$ by $m_X$-open sets, there exists a finite subset $I_0$ of $I$ such that $K \subset \cup\{U_i : i \in I_0\} \cup \{m_X$-Cl$(U_i) : i \in I_0\}$.

It is clear that $(X, m_X)$ is $m$-closed if $X$ is $m$-closed relative to $(X, m_X)$. Let $(X, \tau)$ be a topological space. Note that, if $m_X = \tau$ (resp. $SO(X)$) the definition of $m$-closed sets gives the definitions of quasi $H$-closed ([31]) (resp. of $s$-closed ([9])) sets.

Theorem 16. Let $f : (X, \tau_q) \rightarrow (Y, m_Y)$ be a w.($\tau_q, m$).c. surjection. If $X$ is mildly compact, then $Y$ is $m$-closed.
Proof. Let \( \{V_\alpha : \alpha \in I\} \) be a cover of \( Y \) by \( m_Y \)-open sets of \( Y \). For each point \( x \in X \), there exists \( \alpha(x) \in I \) such that \( f(x) \in V_{\alpha(x)} \). Since \( f \) is \( w.(\tau_q, m).c. \), there exists a clopen set \( U_x \) of \( X \) containing \( x \) such that \( f(U_x) \subseteq m_X\text{-}cl(V_{\alpha(x)}) \). The family \( \{U_x : x \in X\} \) is a cover of \( X \) by clopen sets of \( X \) and hence there exists a finite subset \( X_0 \) of \( X \) such that \( X \subseteq \bigcup_{x \in X_0} U_x \). Therefore, we obtain \( Y = f(X) \subseteq \bigcup_{x \in X_0} m_X\text{-}cl(V_{\alpha(x)}) \). This shows that \( Y \) is \( m \)-closed.

Theorem 17 ([21]). Let \( f : (X, m_X) \to (Y, m_Y) \) be a function. Assume that \( m_X \) is a base for a topology. If the graph \( G(f) \) is strongly \( M \)-closed, then \( m_X\text{-}cl(f^{-1}(K)) = f^{-1}(K) \) whenever the set \( K \subseteq Y \) is \( m \)-closed relative to \( (Y, m_Y) \).

Corollary 1 ([39]). If a function \( f : (X, \tau_q) \to (Y, m_Y) \) has a strongly clp-\( m \)-closed graph, then \( f^{-1}(K) \) is quasi-closed in \( (X, \tau_q) \) for each set \( K \) which is \( m \)-closed relative to \( (Y, m_Y) \).

Theorem 18. If a function \( f : (X, \tau_q) \to (Y, m_Y) \) has a strongly clp-\( m \)-closed graph and \( Y \) is \( m \)-closed, \( m_Y \) has property \( (mR) \) then \( f \) is \( w.(\tau_q, m).c. \).

Proof. Let \( V \) be a \( m_Y \)-open set then by Lemma 4, \( m_Y\text{-}cl(V) \subseteq mR(Y) \) and \( Y-\{m_Y\text{-}cl(V)\} \subseteq mR(Y) \). By the \( m \)-closedness of \( Y \), \( Y-\{m_Y\text{-}cl(V)\} \) is \( m \)-closed. By Corollary 1, \( f^{-1}(Y-\{m_Y\text{-}cl(V)\}) = X-\{m_Y\text{-}cl(V)\} \) is \( f^{-1}(m_Y\text{-}cl(V)) \) is quasi-open. Then \( f^{-1}(V) \subseteq int_q(f^{-1}(m_Y\text{-}cl(V))) \) and by Theorem 3, \( f \) is \( w.(\tau_q, m).c. \). □

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Department of Mathematics,
Faculty of Science and Letters,
Marmara University,
34722 Göztepe-İstanbul,
TURKEY
usengul@marmara.edu.tr