ALMOST $\alpha$-COSYMPLECTIC $f$-MANIFOLDS

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Abstract. The purpose of this paper is to study a new class of framed manifolds. Such manifolds are called almost $\alpha$-cosymplectic $f$-manifolds. For some special cases of $\alpha$ and $s$, one obtains (almost) $\alpha$-cosymplectic, (almost) $C$-manifolds, and (almost) Kenmotsu $f$-manifolds. Moreover, several tensor conditions are studied. We conclude our results with a general example on $\alpha$-cosymplectic $f$-manifolds.

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1. Introduction and preliminaries

Let $M$ a real $(2n + s)$-dimensional smooth manifold. $M$ admits an $f$-structure ([1], [7]) if there exists a non null smooth $(1,1)$ tensor field $\phi$, of the tangent bundle $TM$, satisfying $\phi^3 + \phi = 0$, rank $\phi = 2n$. An $f$-structure is a generalization of almost complex ($s = 0$) and almost contact ($s = 1$) structure ([5], [7]). In the latter case, $M$ is orientable ([6]). Corresponding to two complementary projection operators $P$ and $Q$ applied to $TM$, defined by $P = -\phi^2$ and $Q = \phi^2 + I$, where $I$ is the identity operator, there exist two complementary distributions $D$ and $D^\perp$ such that $\dim(D) = 2n$ and $\dim(D^\perp) = s$. The following relations hold $\phi P = P\phi = \phi$, $\phi Q = Q\phi = 0$, $\phi^2 P = -P$, $\phi^2 Q = 0$. Thus, we have an almost complex distribution $(D, J = \phi|_D, J^2 = -I)$ and $\phi$ acts on $D^\perp$ as a null operator. It follows that $TM = D \oplus D^\perp$, $D \cap D^\perp = \{0\}$. Assume that $D^\perp_p$ is spanned by $s$ globally defined orthonormal vectors $\{\xi_i\}$ at each point $p \in M$, $(1 \leq i, j, \ldots \leq s)$,
with its dual set \( \{ \eta^i \} \). Then one obtains \( \phi^2 = -I + \sum_{i=1}^{s} \eta^i \otimes \xi_i \). In the above case, \( M \) is called a globally framed manifold (or simply an \( f \)-manifold) ([1], [4] and [5]) and we denote its framed structure by \( M(\phi, \xi_i) \). From the above conditions one has \( \phi \xi_i = 0, \eta^i \phi = 0, \eta^i(\xi_j) = \delta_j^i \). Now, we consider a Riemannian metric \( g \) on \( M \) that is compatible with an \( f \)-structure such that

\[
\begin{align*}
g(\phi X, Y) + g(X, \phi Y) &= 0, \\
g(\phi X, \phi Y) &= g(X, Y) - \sum_{i=1}^{s} \eta^i(X) \eta^i(Y), \\
g(X, \xi_i) &= \eta^i(X).
\end{align*}
\]

In the above case, we say that \( M \) is a metric \( f \)-manifold and its associated structure will be denoted by \( M(\phi, \xi_i, \eta^i, g) \).

A framed structure \( M(\phi, \xi_i) \) is said to be normal ([4]) if the torsion tensor \( N_\phi \) of \( \phi \) is zero i.e., if

\[
N_\phi = N + 2 \sum_{i=1}^{s} \eta^i \otimes \xi_i = 0,
\]

where \( N \) denotes the Nijenhuis tensor field of \( \phi \).

Define a 2-form \( \Omega \) on \( M \) by \( \Omega(X, Y) = g(\phi X, \phi Y) \), for any \( X, Y \in \Gamma(TM) \).

The Levi-Civita connection \( \nabla \) of a metric \( f \)-manifold satisfies the following formula ([1]):

\[
2g( (\nabla_X \phi) Y, Z) = 3d\Omega(X, \phi Y, \phi Z) - 3d\Omega(X, Y, Z) \\
+ g(N(Y, Z), \phi X) + N^j_i(Y, Z) \eta^j(X) \\
+ 2d\eta^i(\phi Y, \phi X) \eta^j(Z) - 2d\eta^j(\phi Z, X) \eta^i(Y),
\]

where the tensor field \( N^j_i \) is defined by

\[
N^j_i(X, Y) = (L_{\phi X} \eta^j) Y - (L_{\phi Y} \eta^j) X = 2d\eta^j(\phi X, Y) - 2d\eta^j(\phi Y, X), \text{ for each } j \in \{1, \ldots, s\}.
\]

Following the terminology introduced by BLAIR ([1]), we say that a normal metric \( f \)-manifold is a \( K \)-manifold if its 2-form \( \Omega \) closed (i.e., \( d\Omega = 0 \)). Since \( \eta^1 \wedge \ldots \wedge \eta^s \wedge \Omega^n \neq 0 \), a \( K \)-manifold is orientable. Furthermore, we say that a \( K \)-manifold is a \( C \)-manifold if each \( \eta^i \) is closed, an \( S \)-manifold if \( d\eta^1 = d\eta^2 = \ldots = d\eta^s = \Omega \).

Note that, if \( s = 1 \), namely if \( M \) is an almost contact metric manifold, the condition \( d\Omega = 0 \) means that \( M \) is quasi-Sasakian. \( M \) is said a \( K \)-contact manifold if \( d\eta = \Omega \) and \( \xi \) is Killing.

FALCITELLI and PASTORE [3] introduced and studied a class of manifolds which is called almost Kenmotsu \( f \)-manifold. Such manifolds admit an \( f \)-structure with \( s \)-dimensional parallelizable kernel. A metric \( f.pk \)-manifold of dimension \( 2n + s \), \( s \geq 1 \), with \( f.pk \)-structure \( (\phi, \xi_i, \eta^i, g) \), is said to be a almost Kenmotsu \( f.pk \)-manifold if the 1-forms \( \eta^i \)'s are closed and \( d\Omega = 0 \).
2\eta^1 \wedge \Omega. Several foliations canonically associated with an almost Kenmotsu \( f, pk \)-manifold are studied and locally conformal almost Kenmotsu \( f, pk \)-manifolds are characterized by Falcitelli and Pastore.

In this paper, we consider a wide subclass of \( f \)-manifolds called almost \( \alpha \)-cosymplectic \( f \)-manifolds. Firstly, we give the concept of almost \( \alpha \)-cosymplectic \( f \)-manifold and state general curvature properties. We derive several important formulas on almost \( \alpha \)-cosymplectic \( f \)-manifolds. These formulas enable us to find the geometrical properties of almost \( \alpha \)-cosymplectic \( f \)-manifolds with \( \eta \)-parallel tensors \( h_i \) and \( \varphi h_i \).

We also examine the tensor fields \( \tau_i \)’s which are defined by \( g(\tau_iX, Y) = (L_{\xi_i}g)(X, Y) \), for arbitrary vector fields \( X, Y \) on \( M \). Then we give some results on \( \eta \)-parallelity, cyclic parallelity, Codazzi condition. Finally, we give an explicit example of almost \( \alpha \)-cosymplectic \( f \)-manifold.

Throughout this paper we denote by \( \eta = \eta^1 + \eta^2 + ... + \eta^s \), \( \xi = \xi^1 + \xi^2 + ... + \xi^s \) and \( \delta_j^i = \delta_{j1}^i + \delta_{j2}^i + ... + \delta_{js}^i \).

### 2. Almost \( \alpha \)-cosymplectic \( f \)-manifolds

We introduce a notion of an almost \( \alpha \)-cosymplectic \( f \)-manifold for any real number \( \alpha \) which is defined as metric \( f \)-manifold with \( f \)-structure \((\varphi, \xi, \eta^i, g)\) satisfying the conditions \( d\eta^i = 0 \), \( d\Omega = 2\alpha \eta^1 \wedge \Omega \).

The manifold is called generalized almost Kenmotsu \( f \)-manifold for \( \alpha = 1 \).

Let \( M \) be an almost \( \alpha \)-cosymplectic \( f \)-manifold. Since the distribution \( D \) is integrable, we have \( L_{\xi_i}\eta_j = 0 \), \([\xi_i, \xi_j] \in D\) and \([X, \xi_j] \in D\) for any \( X \in \Gamma (D) \). Then the Levi-Civita connection is given by:

\[
2g \left( (\nabla_X \varphi) Y, Z \right) = 2\alpha g \left( \sum_{j=1}^{s} \left( g (\varphi X, Y) \xi_j - \eta^j(Y) \varphi X \right), Z \right) \\
+ g \left( N(Y, Z), \varphi X \right),
\]

for any \( X, Y, Z \in \Gamma (TM) \). Putting \( X = \xi_i \) we obtain \( \nabla_{\xi_i} \varphi = 0 \) which implies \( \nabla_{\xi_i} \xi_j \in D^\perp \) and then \( \nabla_{\xi_i} \xi_j = \nabla_{\xi_j} \xi_i \), since \([\xi_i, \xi_j] = 0\).

We put \( A_i X = -\nabla_X \xi_i \) and \( h_i = \frac{1}{2} (L_{\xi_i} \varphi) \), where \( L \) denotes the Lie derivative operator.

**Proposition 1.** For any \( i \in \{1, ..., s\} \) the tensor field \( A_i \) is a symmetric operator such that

1) \( A_i (\xi_j) = 0 \), for any \( j \in \{1, ..., s\} \)

2) \( A_i \circ \varphi + \varphi \circ A_i = -2\alpha \varphi \).

3) \( tr(A_i) = -2\alpha n \).
Proof. \( d\eta^i = 0 \) implies that \( A_i \) is symmetric.

1) For any \( i, j, k \in \{1, \ldots, s\} \) deriving \( g(\xi_i, \xi_j) = \delta^i_j \) with respect to \( \xi_k \), using \( \nabla_{\xi_j} \xi_j = \nabla_{\xi_j} \xi_i \), we get \( 2g(\xi_k, A_i(\xi_j)) = 0 \). Since \( \nabla_{\xi_j} \xi_j \in D^\perp \), we conclude \( A_i(\xi_j) = 0 \).

2) For any \( Z \in \Gamma(TM) \), we have \( \varphi(N(\xi_j, Z)) = (L_{\xi_i} \varphi) Z \) and, on the other hand, since \( \nabla_{\xi_i} \varphi = 0 \),

\[
(2.2) \quad L_{\xi_i} \varphi = A_i \circ \varphi - \varphi \circ A_i
\]

One can easily obtain from (2.2)

\[
(2.3) \quad -A_i X = -\alpha \varphi^2 X - \varphi h_i X
\]

Applying (2.1) with \( Y = \xi_i \), we have \( 2g(\varphi A_i X, Z) = -2\alpha g(\varphi X, Z) - g(\varphi N(\xi_i, Z), X) \), which implies the desired result.

3) Considering local adapted orthonormal frame \( \{X_1, \ldots, X_n, \varphi X_1, \ldots, \varphi X_n, \xi_1, \ldots, \xi_s\} \), by 1) and 2), one has

\[
\text{tr} A_i = \sum_{j=1}^n g(A_i X_j, X_j) + g(A_i \varphi X_j, \varphi X_j) = -2\alpha \sum_{j=1}^n g(\varphi X_j, \varphi X_j) = -2\alpha n.
\]

Proposition 2 ([1]). For any \( i \in \{1, \ldots, s\} \) the tensor field \( h_i \) is a symmetric operator and satisfies

1) \( h_i \xi_j = 0 \), for any \( j \in \{1, \ldots, s\} \)

2) \( h_i \circ \varphi + \varphi \circ h_i = 0 \)

3) \( \text{tr} h_i = 0 \)

4) \( \text{tr} \varphi h_i = 0 \)

Proposition 3. \( \nabla \varphi \) satisfies the following relation:

\[
(\nabla_X \varphi) Y + (\nabla_{\varphi X} \varphi) Y = \sum_{i=1}^s [\alpha(\eta^i(Y) \varphi X + 2g(X, \varphi Y) \xi_i) - \eta^i(Y) h_i X].
\]

Proof. By direct computations, we get \( \varphi N(X, Y) + N(\varphi X, Y) = 2\sum_{i=1}^s \eta^i(X) h_i Y, \) and \( \eta^i(N(\varphi X, Y)) = 0. \) From (2.1) and the equations above, the proof is completed.

Proposition 4. Let \( M \) be an almost \( \alpha \)-cosymplectic \( f \)-manifold. The integral manifolds of \( D \) are almost Kaehler manifolds with mean curvature vector field \( H = -\alpha \xi \).
Proof. Let \( \tilde{M} \) be an integral manifold of \( D \). We know that \( (D, J = \varphi|_D, J^2 = -I) \) is an almost complex distribution and the induced metric \( \tilde{g} \) on \( \tilde{M} \) is a Hermitian metric. Therefore, for any \( X, Y \in \Gamma(\tilde{M}) \), we have the induced 2-form on \( \tilde{M} \) such that \( \tilde{\Omega}(X, Y) = \tilde{g}(X, JY) = g(X, \varphi Y) = \Omega(X, Y) \) and \( d\tilde{\Omega} = 0 \) on \( \tilde{M} \). In this manner, \( \tilde{M} \) is an almost Kaehler manifold. Computing the second fundamental form \( B \), since, \( A_i \)'s are the Weingarten operators in the directions \( \xi_i \), we get,

\[
B(X, Y) = \sum_{i=1}^{s} g(A_i X, Y) \xi_i = \sum_{i=1}^{s} [-\alpha g(X, Y) \xi_i + g(\varphi h_i X, Y) \xi_i].
\]

Using the Proposition 2 and (2.3). Now, we choose a local orthonormal frame \( \{e_1, e_2, ..., e_{2n}\} \) such that \( e_{l+n} = \varphi e_l \), for \( l = 1, 2, ..., n \), in \( T\tilde{M} \). Taking \( X = Y = e_p \) in (2.4) and summing over \( p = 1, 2, ..., 2n \), we get

\[
H = \frac{1}{2n} \sum_{i=1}^{s} (tr A_i) \xi_i = -\alpha \xi.
\]

Proposition 5. Let \( M \) be an almost \( \alpha \)-cosymplectic \( f \)-manifold and \( \tilde{M} \) be an integral manifold of \( D \). Then

1) when \( \alpha = 0 \), \( \tilde{M} \) is totally geodesic if and only if all the operators \( h_i \) vanish;

2) when \( \alpha \neq 0 \), \( \tilde{M} \) is totally umbilic if and only if all the operators \( h_i \) vanish.

Proof. The proof is obvious through (2.4). \( \square \)

Proposition 6. Under the same situation as in Proposition 5, \( M \) is \( \alpha \)-cosymplectic \( f \)-manifold with structure \( f \)-structure \( (\varphi, \xi_i, \eta^i, g) \) if and only if the integral manifolds of \( D \) are tangentially Kaehler and all the operators \( h_i \) vanish.

Proof. If the structure is normal, for any \( X \in \Gamma(TM) \), one obtains that

\[
0 = N(X, \xi_j) = N_\varphi(X, \xi_j) + 2 \sum_{i=1}^{s} d\eta^i(X, \xi_j) \xi_i = -\varphi [\varphi X, \xi_j] + \varphi^2 [X, \xi_j] + 2 \sum_{i=1}^{s} d\eta^i(X, \xi_j) \xi_i = 2 \varphi h_j X.
\]
Hence, all the operators $h_i$ vanish. On the other hand, for each $X, Y \in \Gamma(D)$ we have

\begin{equation}
N_{\varphi}(X, Y) = [\varphi X, \varphi Y] - \varphi[X, Y] = N_{J = \varphi|D}(X, Y).
\end{equation}

It is obvious that $N_J = 0$ if and only if almost complex structure $J$ is integrable. Therefore, the proof is completed by (2.5) and (2.6).

**Theorem 1** ([1]). A $C$-manifold $M^{2n+s}$ is a locally decomposable Riemannian manifold which is locally the product of a Kaehler manifold $M^{2n}_1$ and an Abelian Lie group $M^s_2$.

3. Curvature properties

**Proposition 7.** Let $M$ be an almost $\alpha$-cosymplectic $f$-manifold. Then we have

\begin{equation}
R(X, Y)\xi_i = \alpha^2 \sum_{k=1}^s \left( \eta^k(Y)\varphi^2 X - \eta^k(X)\varphi^2 Y \right) - \alpha \sum_{k=1}^s \left( \eta^k(X)\varphi h_k Y - \eta^k(Y)\varphi h_k X \right) + (\nabla_Y \varphi h_i) X - (\nabla_X \varphi h_i) Y.
\end{equation}

**Proof.** Using the Riemannian curvature tensor and (2.3), we obtain (3.1).

Using (2.3) and (3.1), by simple computations, we have the following proposition.

**Proposition 8.** For an almost $\alpha$-cosymplectic $f$-manifold with the $f$-structure $(\varphi, \xi_i, \eta^i, g)$, the following relations hold

\begin{equation}
R(X, \xi_j)\xi_i = \sum_{k=1}^s \delta^k_j \left[ \alpha^2 \varphi^2 X + \alpha \varphi h_k X \right] + \alpha \varphi h_i X - h_i h_j X + \varphi \left( \nabla_{\xi_j} h_i \right) X
\end{equation}

\begin{equation}
R(\xi_j, X)\xi_i - \varphi R(\xi_j, \varphi X)\xi_i = 2 \left[ -\alpha^2 \varphi^2 X + h_i h_j X \right],
\end{equation}

\begin{equation}
(\nabla_{\xi_j} h_i) X = -\varphi R(X, \xi_j)\xi_i + \sum_{k=1}^s \delta^k_j \left[ -\alpha^2 \varphi X - \alpha h_k X \right]
\end{equation}

\begin{equation}
- \alpha h_i X - \varphi h_i h_j X,
\end{equation}
\textbf{Corollary 1.} Let $M$ be an almost $\alpha$-cosymplectic $f$-manifold. The Ricci tensor satisfies the following conditions:

1) $S(\xi_i, \xi_i)$ always takes negative value when $\alpha \neq 0$,
2) If all the values $S(\xi_i, \xi_i)$ vanish then any leaf of $D$ is totally geodesic.
3) If all the values $S(\xi_i, \xi_i)$ vanish and $M$ is normal then $M$ is locally the product of a Kaehler manifold $M_1^{2n}$ and an Abelian Lie group $M_2^s$.

\textbf{Proof.} The proof is clear through (3.6). \hfill $\square$

The tensor $\tau$ was introduced by Chern and Hamilton \cite{2} and is defined by $g(\tau X, Y) = (L_\xi g)(X, Y)$ for arbitrary vector fields $X, Y$ on a contact metric manifold. Now, we define and examine this tensor field for an almost $\alpha$-cosymplectic $f$-manifold

\textbf{Proposition 9.} An almost $\alpha$-cosymplectic $f$-manifold with $f$-structure $(\varphi, \xi_i, \eta^i, g)$ has tensor fields $\tau_i$ such that $\tau_i X = 2\nabla_X \xi_i$, where $\tau_i$’s are defined by $g(\tau_i X, Y) = (L_\xi g)(X, Y)$ for arbitrary vector fields $X, Y$ on $M$.

\textbf{Proof.} Using the definition of the tensor fields $\tau_i$, we get

\begin{equation}
(L_\xi g)(X, Y) = g(\nabla_X \xi_i, Y) + g(X, \nabla_Y \xi_i)
\end{equation}

\begin{equation}
= 2g(-\alpha \varphi^2 X - \varphi h, X, Y)
\end{equation}

for arbitrary vector fields $X, Y$ on $M$. Applying the formula (2.3), the proof is completed. \hfill $\square$

\textbf{Proposition 10.} Let $M$ be a locally symmetric almost $\alpha$-cosymplectic $f$-manifold. Then, $\nabla_{\xi_i} h_i = 0$, for any $\gamma \in \{1, \ldots, s\}$.

\textbf{Proof.} Notice that (3.3) can be written as \(\frac{1}{2} (R(\xi_j, \cdot) \xi_i - \varphi R(\xi_j, \varphi) \xi_i) = -\alpha^2 \varphi^2 + h_i h_j\) and since the operator $R(\xi_j, \cdot) \xi_i$ is parallel with respect to $\xi_k$, we get $\nabla_{\xi_i} h_i h_j = 0$. Applying $\nabla_{\xi_i}$ to (3.4), we obtain $\nabla_{\xi_i} (\nabla_{\xi_j} h_i) = -\alpha \nabla_{\xi_i} h_i - \alpha \nabla_{\xi_j} h_i$. Moreover, $\nabla_{\xi_i} h_i h_j = 0$ implies that $(\nabla_{\xi_k} h_i) h_j + h_i (\nabla_{\xi_k} h_j) = 0$, and applying $\nabla_{\xi_i}$ to this equation, we get $\nabla_{\xi_i} h_i = 0$. \hfill $\square$
Theorem 2. Let \( M \) be a locally symmetric generalized almost Kenmotsu \( f \)-manifold. Then the following conditions are equivalent:

1) \( M \) is a generalized \( \alpha \)-Kenmotsu \( f \)-manifold;

2) all the operators \( h_i \) vanish.

Moreover, if any of the conditions above holds, then \( M \) cannot have constant sectional curvature.

Proof. Assuming that \( M \) is a generalized \( \alpha \)-Kenmotsu \( f \)-manifold, we have \( \nabla \xi_i = -\alpha \varphi^2 \) and, by (2.3), all the operators \( h_i \) vanish. Now, supposing all the operators \( h_i \) vanish, it follows that \( \nabla \xi_i = -\alpha \varphi^2 \) and \( \nabla \eta_i = \alpha \left( g - \sum_{k=1}^{s} \eta_k \otimes \eta_k \right) \) and by (3.1), \( R(X,Y)\xi_i = \alpha^2 \sum_{k=1}^{s} \eta_k(Y)\varphi^2 X - \eta_k(X)\varphi^2 Y \). So, \( M \) is a generalized \( \alpha \)-Kenmotsu \( f \)-manifold. Moreover, the sectional curvature of any 2-plane spanned by \( \{ Y, \xi_i \} \) is \( K(Y, \xi_i) = -\alpha^2 \| \varphi Y \|^2 \), for all vector fields \( Y \) on \( M \). So, the sectional curvature of any 2-plane spanned by \( \{ \xi_i, \xi_j \} \), for any \( i, j \in \{1, 2, ..., s\} \), vanishes and one gets that the sectional curvature of any plane spanned by \( Y \in D \) and \( \xi_i \) is equal to \( -\alpha^2 \).

4. Some tensor conditions

For any vector field \( X \) on \( M \), we can take \( X = X^T + \sum_{i=1}^{s} \eta^i(X)\xi_i \) where \( X^T \) is the tangential part of \( X \) and \( \sum_{i=1}^{s} \eta^i(X)\xi_i \) is the normal part of \( X \). We can rewrite \( \eta \)-parallel condition for a given almost \( \alpha \)-cosymplectic \( f \)-manifold. We say that any \( (1,1) \)-type tensor field \( B \) is \( \eta \)-parallel if and only if \( g((\nabla X^T B)Y^T, Z^T) = 0 \), for \( X^T, Y^T, Z^T \in D \).

The starting point of the investigation of almost \( \alpha \)-cosymplectic \( f \)-manifolds with \( \eta \)-parallel tensors \( h_i \) and \( \varphi h_i \) is the following propositions:

Proposition 11. Let \( M \) be an almost \( \alpha \)-cosymplectic \( f \)-manifold and \( h_i \)’s are \( (1,1) \)-type tensor fields. If the tensor fields \( h_i \)’s are \( \eta \)-parallel, then

\[
(\nabla X h_i)Y = -\sum_{k=1}^{s} \eta^k(X) \left[ \varphi l_k Y + \sum_{\gamma=1}^{s} \delta^\gamma_k [\alpha^2 \varphi Y + \alpha h_i Y] + \varphi h_i h_k Y + \alpha h_i Y \right]
\]

(4.1) \[-\sum_{k=1}^{s} \eta^k(Y)[\alpha h_i X + \varphi h_i h_k X] - \sum_{k=1}^{s} g(\alpha h_i X + \varphi h_i h_k X, Y)\xi_k, \]

for all vector fields \( X, Y \) on \( M \), where the tensor \( l_k = R(., \xi_k)\xi_i \) is the Jacobi operator with respect to the characteristic vector fields and \( h_i \)’s are \( (1,1) \)-type tensor fields.
Proof. Suppose that each $h_i$ is $\eta$-parallel. Denoting the component of $X$ orthogonal to $\xi$ by $X_T$, we obtain

$$0 = g((\nabla_{X_T} h_i) Y^T, Z^T)$$

$$= g\left(\nabla_X - \sum_{k=1}^s \eta^k(X)\xi_k h_i\right) (Y - \sum_{k=1}^s \eta^k(Y)\xi_k), Z - \sum_{k=1}^s \eta^k(Z)\xi_k$$

$$= g((\nabla_X h_i) Y, Z) - \sum_{k=1}^s \eta^k(Y)g((\nabla_{\xi_k} h_i) Y, Z) - \sum_{k=1}^s \eta^k(Y)g((\nabla_X h_i) \xi_k, Z)$$

$$- \sum_{k=1}^s \eta^k(Z)g((\nabla_X h_i) Y, \xi_k) = g((\nabla_X h_i) Y, -\varphi^2 Z)$$

$$- \sum_{k=1}^s \eta^k(X)g((\nabla_{\xi_k} h_i) Y, Z) - \sum_{k=1}^s \eta^k(Y)g((\nabla_X h_i) \xi_k, Z),$$

for all vector fields $X, Y, Z$ on $M$. Using (2.3) and (3.4), the proof is completed.

**Proposition 12.** Let $M$ be an almost $\alpha$-cosymplectic $f$-manifold. If the tensor fields $\varphi h_i$’s are $\eta$-parallel, then

$$(\nabla_X \varphi h_i) Y = \sum_{k=1}^s \eta^k(X) \left[ l_k Y - \sum_{\gamma=1}^s \delta^k_\gamma \left[ \alpha^2 \varphi^2 Y + \alpha \varphi h_i Y + h_i h_k Y - \alpha \varphi h_i Y \right] \right]$$

$$- \sum_{k=1}^s \eta^k(Y) [\alpha \varphi h_i X - h_i h_k X] - \sum_{k=1}^s g(\alpha \varphi h_i X - h_i h_k X, Y) \xi_k.$$  \hspace{1cm} (4.2)

**Proof.** We consider that $\varphi h_i$ is $\eta$-parallel. Thus,

$$0 = g((\nabla_{X_T} \varphi h_i) Y^T, Z^T)$$

$$= g\left(\nabla_X - \sum_{k=1}^s \eta^k(X)\xi_k \varphi h_i\right) (Y - \sum_{k=1}^s \eta^k(Y)\xi_k), Z - \sum_{k=1}^s \eta^k(Z)\xi_k$$

$$= g((\nabla_X \varphi h_i) Y, Z) - \sum_{k=1}^s \eta^k(X)g((\nabla_{\xi_k} \varphi h_i) Y, Z)$$

$$- \sum_{k=1}^s \eta^k(Y)g((\nabla_X \varphi h_i) \xi_k, Z) - \sum_{k=1}^s \eta^k(Z)g((\nabla_X \varphi h_i) Y, \xi_k)$$
for all vector fields $X, Y$ on $M$. If we simplify the equation above, then

$$g((\nabla_X \varphi h_i) Y, Z) = \sum_{k=1}^{s} \eta^k(X) g((\nabla_{\xi_k} \varphi h_i) Y, Z) + \sum_{k=1}^{s} \eta^k(Y) g((\nabla_X \varphi h_i) \xi_k, Z) + \sum_{k=1}^{s} \eta^k(Z) g((\nabla_X \varphi h_i) Y, \xi_k).$$

Using (2.3) and $(\nabla_{\xi_k} \varphi h_i) Y = \varphi(\nabla_{\xi_k} h_i) Y$, the proof is completed. □

**Theorem 3.** An almost $\alpha$-cosymplectic $f$-manifold with the $\eta$-parallel tensor fields $\varphi h_i$’s satisfy the following relation:

$$R(X, Y)\xi_i = \sum_{k=1}^{s} \eta^k(Y) l_{ki} X - \eta^k(X) l_{ki} Y,$$

where $l_{ki} = R(., \xi_k)\xi_i$ is the Jacobi operator with respect to the characteristic vector fields $\xi_k$ and $\xi_i$.

**Proof.** Using (3.1) and (4.2), we get

$$R(X, Y)\xi_i = \alpha^2 \sum_{k=1}^{s} \left[ \eta^k(Y) \varphi^2 X - \eta^k(X) \varphi^2 Y \right]$$

$$- \alpha \sum_{k=1}^{s} \left[ \eta^k(X) \varphi h_k Y - \eta^k(Y) \varphi h_k X \right]$$

$$+ \sum_{k=1}^{s} \eta^k(Y) \left[ l_{ki} X - \sum_{\gamma=1}^{s} \delta^\gamma_k \left[ \alpha^2 \varphi^2 X + \alpha \varphi h_\gamma X \right] + h_i h_k X - \alpha \varphi h_i X \right]$$

$$- \sum_{k=1}^{s} \eta^k(X) \left[ \alpha \varphi h_i Y - h_i h_k Y \right] - \sum_{k=1}^{s} g(\alpha \varphi h_i Y - h_i h_k Y, X) \xi_k$$

$$- \sum_{k=1}^{s} \eta^k(X) \left[ l_{ki} Y - \sum_{\gamma=1}^{s} \delta^\gamma_k \left[ \alpha^2 \varphi^2 Y + \alpha \varphi h_\gamma Y \right] + h_i h_k Y - \alpha \varphi h_i Y \right]$$

$$+ \sum_{k=1}^{s} \eta^k(Y) \left[ \alpha \varphi h_i X - h_i h_k X \right] + \sum_{k=1}^{s} g(\alpha \varphi h_i X - h_i h_k X, Y) \xi_k.$$

Then, we can easily write (4.3) by simplifying the equation above. □
Theorem 4. An almost \( \alpha \)-cosymplectic \( f \)-manifold has negative pointwise constant \( \xi_i \)-sectional curvature.

Proof. Let \( M \) be an almost \( \alpha \)-cosymplectic \( f \)-manifold with a pointwise constant \( \xi_i \)-sectional curvature \( K(p), p \in M \). It means that \( g(R(X^T, \xi_i)\xi_i, X^T) = K_i(p)g(X^T, X^T) \) for all tangent vectors \( X^T \) orthogonal to \( \xi_i \) at the point \( p \in M \), i.e., \( X^T \in D \). Putting \( X^T = X - \sum_{k=1}^{s} \eta_k(X)\xi_k \) and using the symmetries of curvature tensor \( R \), we see that the equation above is equivalent to \( \varphi_{\xi i}X = K_i\varphi X \), for any vector field \( X \), where \( K_i \) is a smooth function in \( M \).

From the equation (3.4), we get

\[
(\nabla_{\xi_i}h_i)X = -K_i\varphi X + \sum_{k=1}^{s} \delta^k_i \left[ -\alpha^2 \varphi X - \alpha h_kX \right] - \alpha h_iX - \varphi h_i^2X
\]

Separating the equation above to symmetric and skew-symmetric parts, we obtain

\[
(\nabla_{\xi_i}h_i)X = -\alpha \left[ \sum_{k=1}^{s} \delta^k_i h_kX + h_iX \right]
\]

and

\[
(4.4)\quad -K_i\varphi X - \alpha^2 \varphi X - \varphi h_i^2X = 0.
\]

Let \( \{E_1, E_2, ..., E_{2n}, \xi_1, ..., \xi_s\} \) be an orthonormal basis of the tangent space at any point. Firstly, we apply inner product with \( \varphi X \) both two sides in (4.4). Then, the sum for \( 1 \leq j \leq 2n \) of the relation (4.4) with \( X = E_j \) yields \( K_i = -\left( \alpha^2 + \frac{||h_i||^2}{2n} \right) \).

Remark 1. The conditions "\( h_i \) is a Codazzi tensor" and "\( \varphi h_i \) is a Codazzi tensor" are equivalent.

Proposition 13. Let \( M \) be an almost \( \alpha \)-cosymplectic \( f \)-manifold. If the tensor field \( \varphi h_i \)'s (or \( h_i \)'s) are Codazzi, then the following conditions hold:

1) If \( \alpha = 0 \) then the integral manifolds of \( D \) are totally geodesic.

2) If \( \alpha = 0 \) and \( M \) is normal then \( M \) is a locally decomposable Riemannian manifold which is locally the product of a Kaehler manifold \( M^2_{1n} \) and an Abelian Lie group \( M^s_2 \).

3) The integral manifolds of \( D \) are totally umbilic when \( \alpha \neq 0 \).
Proof. Let the tensor field \( \varphi h_i \) be Codazzi. Taking \( X = \xi_j, Y \in D \), we get \( (\nabla_{\xi_j} h_i) Y - (\nabla_Y h_i) \xi_j = 0 \). By using (3.4), we obtain \(-\varphi l_j Y = \alpha^2 \varphi Y + \alpha h_j Y \). By (3.3), we have \( h_i h_j Y = 0 \), for any \( i, j \), so \( h_i = 0 \), for any \( i \), and the statement follows by Proposition 5.

Theorem 5. Let \( M \) be an almost \( \alpha \)-cosymplectic \( f \)-manifold. If the tensors \( \tau_i \)'s are parallel and \( M \) is normal then \( M \) is a locally decomposable Riemannian manifold which is locally the product of a Kaehler manifold \( M_1^{2n} \) and an Abelian Lie group \( M_2^s \).

Proof. Let the tensor fields \( \tau_i \)'s are the parallel tensor field. It means that \( (\nabla_X \tau_i) Y = 0 \), for all \( i \in \{1, 2, ..., s\} \) and \( X, Y \in \Gamma(TM) \). Putting \( Y = \xi_j \) for any \( j \in \{1, 2, ..., s\} \) and contracting the equation above with respect to \( X \), we get \(-2\alpha \eta^2 + \alpha \text{trace} (\varphi h_i) Y + \alpha \text{trace} (\varphi h_i) - \text{trace} (h_i h_j) = 0 \). If we examine the last equation for all values of \( i \) and \( j \), we see that suffices \( \alpha = 0 \) and \( h_\zeta = 0 \) for all \( \zeta \in \{1, 2, ..., s\} \). Hence, the proof is obvious by Theorem 1.

Proposition 14. Let \( M \) be an almost \( \alpha \)-cosymplectic \( f \)-manifold. If the tensor fields \( \tau_i \)'s are \( \eta \)-parallel, then

\[
(\nabla_X \varphi h_i) Y = \sum_{k=1}^s \left[ \eta^k(X) (\nabla_{\xi_k} \varphi h_i) Y - \eta^k(Y) \varphi h_i \nabla_X \xi_k 
\right. \\
\left. + g ((\nabla_X \varphi h_i) \xi_k, Y) \xi_k \right].
\]

Proof. Suppose that \( \tau_i \) is \( \eta \)-parallel. It satisfies equation \( g((\nabla_X \tau_i) Y^T, Z^T) = 0 \) for any vector fields \( X^T, Y^T, Z^T \) on \( D \). By simple computations, we get

\[
(\nabla_X \tau_i) Y = \sum_{k=1}^s \left[ g ((\nabla_X \tau_i) Y, \xi_k) \xi_k 
\right. \\
\left. + \eta^k(Y) (\nabla_X \tau_i) \xi_k + \eta^k(X) (\nabla_{\xi_k} \tau_i) Y \right].
\]

On the other hand, one can easily obtain that

\[
(\nabla_X \tau_i) Y = \sum_{\nu=1}^s [-2\alpha \eta^\nu(Y) \nabla_X \xi_\nu - 2\alpha g(\nabla_X \xi_\nu, Y) \xi_\nu] - 2(\nabla_X \varphi h_i) Y.
\]

From (4.6) and (4.7) we have the desired result. \( \square \)
Theorem 6. Let $M$ be an almost $\alpha$-cosymplectic $f$-manifold. If the tensor fields $\tau_i$'s are $\eta$-parallel, then $R(X,Y)\xi_i = \sum_{k=1}^{s} \eta^k(Y)l_{ki}X - \eta^k(X)l_{ki}Y$.

Proof. Using equation (4.5), we obtain the following difference:

$$(\nabla_Y \phi h_i)X - (\nabla_X \phi h_i)Y = \sum_{k=1}^{s} \eta^k(Y)(\nabla_{\xi_k} \phi h_i)X - \sum_{k=1}^{s} \eta^k(X)(\nabla_{\xi_k} \phi h_i)Y$$

$+ \sum_{k=1}^{s} \eta^k(Y)\phi h_i \nabla X \xi_k - \sum_{k=1}^{s} \eta^k(X)\phi h_i \nabla Y \xi_k.$

(4.8)

Using (3.4) and (4.8), we get,

$$R(X,Y)\xi_i = \sum_{k=1}^{s} \eta^k(Y)l_{ki}X - \eta^k(X)l_{ki}Y.$$

Hence, the proof is completed. \qed

Proposition 15. Let $M$ be an almost $\alpha$-cosymplectic $f$-manifold. If the tensor field $\phi h_i$'s are cyclically parallel, then the following conditions hold:

1) If $\alpha = 0$ then the integral manifolds of $D$ are totally geodesic.

2) If $\alpha = 0$ and $M$ is normal then $M$ is a locally decomposable Riemannian manifold which is locally the product of a Kaehler manifold $M_{2n}^{2n}$ and an Abelian Lie group $M_2^s$.

3) The integral manifolds of $D$ are totally umbilic when $\alpha \neq 0$.

Proof. The hypothesis can be written

$$g((\nabla_X \phi h_i)Y, \xi_j) + g((\nabla_Y \phi h_i) \xi_j, X) + g((\nabla_{\xi_j} \phi h_i)X, Y) = 0$$

for all vector fields $X,Y$ on $M$. From this equation, we get the following equation $(\nabla_{\xi_j} h_i)X = 2\alpha h_iX + \phi(h_i \circ h_j + h_j \circ h_i)X$. Making use of (3.2), we obtain $R(X,\xi_j)\xi_i = \sum_{k=1}^{s} \delta^k_i [\alpha^2 \phi^2 X + \alpha \phi h_k X] + 3\alpha \phi h_iX - 3h_i^2 X$ Applying $\phi$ to the last equation, substituting $\phi X$ for $X$ and using (3.3), we get $h_i^2 = 0$. So, we obtain trace$(h_i^2) = 0$, for any $i$, and apply Proposition 5. \qed

Theorem 7. Let $M$ be an almost $\alpha$-cosymplectic $f$-manifold. If the tensors $\tau_i$'s are cyclically parallel, then the following conditions hold:

1) The integral manifolds of $D$ are totally geodesic.

2) If $M$ is normal then $M$ is a locally decomposable Riemannian manifold which is locally the product of a Kaehler manifold $M_{2n}^{2n}$ and an Abelian Lie group $M_2^s$.

Proof. As $\tau_iX = -2\alpha \phi^2 X - 2\phi h_i X$, the hypothesis can be written

$$g((\nabla_X \tau_i)Y, Z) + g((\nabla_Y \tau_i)Z, X) + g((\nabla_Z \tau_i)X, Y) = 0,$$

for arbitrary vector.
fields \(X, Y, Z\) on \(M\). Using (2.3) and replacing \(Z\) by \(\xi_j\), we reduce the following relation:

\[
(4.9) \quad \varphi (\nabla_{\xi_j} h_i) X = 2\alpha^2 \varphi^2 X + 2\alpha \varphi h_j X + 2\alpha \varphi X - h_i h_j X - h_j h_i X.
\]

Substitution of the (4.9) into (3.2), we get

\[
(4.10) \quad l_{ji} X - \varphi l_{ji} \varphi X = 6\alpha^2 \varphi^2 X - 4h_i h_j X - 2h_j h_i X.
\]

From equality of (3.3) and (4.10), we have \(2\alpha^2 \varphi^2 X - h_j h_i X - h_i h_j X = 0\). Hence, the proof is clear.

**Example 1.** Let, \(n = 1\) and \(s = 2\). We consider the 4-dimensional manifold \(M = \{(x, y, z_1, z_2) \in \mathbb{R}^4\}\), where \((x, y, z_1, z_2)\) are the standard coordinates in \(\mathbb{R}^4\). The vector fields \(e_1 = f_1(z_1, z_2) \frac{\partial}{\partial z_1} + f_2(z_1, z_2) \frac{\partial}{\partial z_2}, e_2 = -f_2(z_1, z_2) \frac{\partial}{\partial z_1} + f_1(z_1, z_2) \frac{\partial}{\partial z_2}, e_3 = \frac{\partial}{\partial x}, e_4 = \frac{\partial}{\partial y}\), where \(f_1\) and \(f_2\) are given by

\[
\begin{align*}
    f_1(z_1, z_2) &= c_2 e^{-\alpha (z_1+z_2)} \cos(z_1 + z_2) - c_1 e^{-\alpha (z_1+z_2)} \sin(z_1 + z_2), \\
    f_2(z_1, z_2) &= c_1 e^{-\alpha (z_1+z_2)} \cos(z_1 + z_2) + c_2 e^{-\alpha (z_1+z_2)} \sin(z_1 + z_2)
\end{align*}
\]

for constant \(c_1, c_2, \alpha \in \mathbb{R}\). It is obvious that \(\{e_1, e_2, e_3, e_4\}\) are linearly independent at each point of \(M\). Let \(g\) be the Riemannian metric defined by

\[
g(e_i, e_j) = \begin{cases} 
1, & \text{for } i = j \\
0, & \text{for } i \neq j
\end{cases}
\]

for all \(i, j \in \{1, 2, 3, 4\}\) and given by the tensor product \(g = \frac{1}{f_1^2 + f_2^2} (dx \otimes dx + dy \otimes dy) + dz_1 \otimes dz_1 + dz_2 \otimes dz_2\). Let \(\eta^1\) and \(\eta^2\) be the 1-form defined by \(\eta^1(X) = g(X, e_3)\) and \(\eta^2(X) = g(X, e_4)\), respectively, for any vector field \(X\) on \(M\) and \(\varphi\) be the \((1, 1)\) tensor field defined by \(\varphi(e_1) = e_2, \varphi(e_2) = -e_1, \varphi(e_3) = \xi_1, \varphi(e_4) = \xi_2\). Also, let \(h_i\)'s be the \((1, 1)\) tensor fields defined by \(h_i(e_1) = -e_1, h_i(e_2) = e_2, h_i(e_3) = 0\) and \(h_i(e_4) = 0\). Then using linearity of \(g\) and \(\varphi\), we have

\[
\begin{align*}
\varphi^2 X &= -X + \eta^1(X)e_3 + \eta^2(X)e_4 \\
g(\varphi X, \varphi Y) &= g(X, Y) - \eta^1(X)\eta^1(Y) - \eta^2(X)\eta^2(Y) \\
\eta^1(e_3) &= 1 \text{ and } \eta^2(e_4) = 1
\end{align*}
\]
for any vector fields on $M$.

It remains to prove that $d\Omega = 2\eta \wedge \Omega$ and Nijenhuis torsion tensor of $\varphi$ is zero. It follows that $\Omega(e_i, e_2) = -1$ and otherwise $\Omega(e_i, e_j) = 0$ for $i \leq j$. Therefore, the essential non-zero component of $\Omega$ is $\Omega(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}) = -\frac{1}{f_1^2 + f_2^2}$, and hence

\begin{equation}
\Omega = -\frac{2e^{2\alpha(z_1+z_2)}}{c_1^2 + c_2^2}dx \wedge dy.
\end{equation}

Consequently, the exterior derivative $d\Omega$ is given by

\begin{equation}
\begin{split}
d\Omega &= -4\alpha e^{2\alpha(z_1+z_2)}dx \wedge dy \wedge (dz_1 + dz_2). 
\end{split}
\end{equation}

Since $\eta^1 = dz_1$ and $\eta^2 = dz_2$, by (4.11) and (4.12), we find $d\Omega = 2\alpha(\eta^1 + \eta^2) \wedge \Omega$. Let $\nabla$ be the Levi-Civita connection with respect to the metric $g$. Then, we obtain $[e_1, e_3] = [e_1, e_4] = \alpha e_1 - e_2$, $[e_2, e_3] = [e_2, e_4] = e_1 + \alpha e_2$, $[e_1, e_2] = 0$, $[e_3, e_4] = 0$. In conclusion, it can be noted that Nijenhuis torsion tensor of $\varphi$ is zero. Thus, the manifold is an $\alpha$-cosymplectic $f$-manifold.

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