EXISTENCE OF EQUILIBRIUM FOR GAMES WITH UPPER SEMICONTINUOUS CORRESPONDENCES, UNDER UNCERTAINTY*

BY

MONICA PATRICHE

Abstract. In this paper we prove the existence of equilibrium pairs for the new model of a Bayesian free abstract economy which extends Kim and Lee’s deterministic model of a free abstract economy (2006). Our existence theorems are proved for the case of upper semicontinuous correspondences. We also define a model of a general Bayesian abstract economy and introduce the new notion of equilibrium pair. We prove the existence of equilibrium pair and equilibrium for this type of abstract economy.

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Key words: Bayesian abstract economy, equilibrium pair, best proximity pair, Bayesian equilibrium pair, upper semicontinuous correspondences, fixed point theorem, maximal element.

1. Introduction

We define the model of a general Bayesian abstract economy, which extends the deterministic model of Yannelis and Prabhakar in [17] and the model of a Bayesian free abstract economy, which extends the deterministic model of Kim and Lee [8] in a Bayesian framework. The preference correspondences need not to be represented by utility functions. We introduce a new concept of equilibrium pair and prove the existence of equilibrium

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pairs for these models. The assumptions on correspondences refer to upper semicontinuity and measurability.

In the last years, many authors generalized the classical model of abstract economy proposed in their pioneering works by Shafer and Sonnenschein [13] or Yannelis and Praehakar [17]. In [8], Kim and Lee defined the free abstract economy and by using a fixed point theorem for Kakutani factorizable multifunctions, they proved existence theorems of best proximity pairs and equilibrium pairs. Their theorems for best proximity pairs generalize the previous results due to Srinivasan and Veeramani [14], [15], Sehgal and Singh [12], Reich [11]. Their existence theorems of equilibrium pairs refer to free abstract economies with upper semicontinuous constrained correspondences and preference correspondences with open lower sections. By using Park’s fixed point theorem for acyclic factorizable multifunctions, Kim [7] generalized Kim and Lee’s results. He deals with free abstract economies with $L^*$-majorized preference correspondences.

Our models capture the meaning of the idea of uncertainty or incomplet information. Uncertainty was introduced in equilibrium theory by Arrow and Debreu (1954), who realized that the classical equilibrium results on existence and optimality of the Walrasian equilibrium continue to hold if the agent’s characteristics become random (state dependent) and the exogenous uncertainty is described by a set which denotes the states of nature of the world. The Bayesian approach has new results (see [1], [6], [10]).

We also use some results on Bochner integration, measurable multifunctions and integration of correspondences. In the last years, many authors investigated the field of measure theory due to its applications in mathematical economics, statistics or theory of games. Many concepts and results of classical measure theory have been studied in the set-valued case. In the recent papers [2], [9] and [16] the authors studied some concepts related the measurability and generalized them to the case of set-valued set functions.

The paper is organized in the following way: Section 2 contains preliminaries and notation. Section 3 introduces our models and Section 4 states the equilibrium existence results. The Appendix presents the technical theorems used in our proofs.
2. Preliminaries and notation

$R_+^+$ denotes the set of strictly positive reals. $\text{co}A$ denotes the convex hull of the set $A$. $\text{clco} A$ denotes the closed convex hull of the set $A$. $2^A$ denotes the set of all non-empty subsets of the set $A$. If $A \subset X$, where $X$ is a topological space, $\text{cl} A (\text{int} A)$ denotes the closure (resp. the interior) of $A$. Let $X$ and $Y$ be sets. The graph of the correspondence $T : X \to 2^Y$ is the set $G_T = \{(x, y) \in X \times Y : y \in T(x)\}$.

Let $X$, $Y$ be topological spaces and $T : X \to 2^Y$ be a correspondence. $T$ is said to be upper semicontinuous if for each $x \in X$ and each open set $V$ in $Y$ with $T(x) \subset V$, there exists an open neighborhood $U$ of $x$ in $X$ such that $T(y) \subset V$ for each $y \in U$.

**Lemma 1** ([20]). Let $X$ and $Y$ be two topological spaces and let $A$ be an open subset of $X$. Suppose $T_1 : X \to 2^Y$, $T_2 : X \to 2^Y$ are upper semicontinuous such that $T_2(x) \subset T_1(x)$ for all $x \in A$. Then the correspondence $T : X \to 2^Y$ defined by

$$T(x) = \begin{cases} T_1(x), & \text{if } x \notin A, \\ T_2(x), & \text{if } x \in A \end{cases}$$

is also upper semicontinuous.

**Theorem 1** ([20]). Let $E$ be a locally convex topological vector space and $X$ be a nonempty compact and convex subset of $E$. Suppose that $T : X \to 2^X$ is upper semicontinuous with compact convex values. If $T$ is irreflexive (i.e., for each $x \in X$, $x \notin T(x)$), then there exists $x^* \in X$ such that $T(x^*) = \emptyset$.

**Definition 1.** Let $X$, $Y$ be topological spaces and $T : X \to 2^Y$ be a correspondence. An element $x \in X$ is named maximal element for $T$ if $T(x) = \emptyset$.

Theorem 2 is a maximal element theorem, consequence of Theorem 3.1 in [4].

**Theorem 2.** Let $I$ be any set of agents and for each $i \in I$:

(a) let $X_i$ be a nonempty convex subset of a locally convex Hausdorff topological vector space and $D_i$ be a nonempty, compact subset of $X_i$;

(b) let $T_i : X = \prod_{i \in I} X_i \to 2^{D_i}$ be an upper semicontinuous correspondence such that for each $x \in X$, $x_i \notin \text{clco} T_i(x)$;

(c) the set $W_i = \{x \in X : T_i(x) = \emptyset\}$ is open in $X$;
(d) there exists a nonempty closed convex subset $F_i$ of $D_i$ such that $F_i \cap T_i(x) \neq \emptyset$ for each $x \in W_i$.

Then there exists $x^* \in \text{co}D$ such that $T_i(x^*) = \emptyset$ for each $i \in I$.

**Notation.** Let $X$ and $Y$ be any two subsets of a normed space $E$ with a norm $\| \cdot \|$, and the metric $d(x, y)$ is induced by the norm. Then, we now recall the following notation:

$$\text{Prox}(X, Y) := \{(x, y) \in X \times Y : d(x, y) = d(X, Y) = \inf\{d(x, y) : x \in X, y \in Y\}\};$$

$$X_0 := \{x \in X : d(x, y) = d(X, Y) \text{ for some } y \in Y\};$$

$$Y_0 := \{y \in Y : d(x, y) = d(X, Y) \text{ for some } x \in X\}.$$

If $X$ and $Y$ are non-empty compact and convex subsets of a normed linear space, then it is easy to see that $X_0$ and $Y_0$ are both non-empty compact and convex.

Let $I$ be a finite (or an infinite) index set. For each $i \in I$, let $X$ and $Y_i$ be non-empty subsets of a normed space $E$ with a norm $\| \cdot \|$, and the metric $d(x, y)$ is induced by the norm. Then, we can use the following notation: for each $i \in I$:

$$X^0 := \{x \in X : \exists y_i \in Y_i \text{ such that } d(x, y_i) = d(X, Y_i)\};$$

$$Y^0_i := \{y \in Y_i : \exists x \in X \text{ such that } d(x, y) = d(X, Y_i)\}.$$

When $|I| = 1$, it is easy to see that $X_0 = X^0$ and $Y_0 = Y^0_i$.

To prove our equilibrium theorems we need the following results.

**Definition 2** ([8]). Let $X$ and $Y$ be two non-empty subsets of a normed linear space $E$, and let $T : X \to 2^Y$ be a correspondence. The pair $(x^*, T(x^*))$ is called the best proximity pair ([8]) for $T$ if $d(x^*, T(x^*)) = d(x^*, y^*) = d(X, Y)$ for some $y^* \in T(x^*)$. Then the best proximity pair theorem seeks an appropriate solution which is optimal. In fact, the best proximity pair theorem analyzes the conditions under which the problem of minimizing the real-valued function $x \to d(x, T(x))$ has a solution.

Kim and Lee [8] gave the following result of existence of best proximity pairs:

**Theorem 3.** For each $i \in I = \{1, \ldots, n\}$, let $X$ and $Y$ be non-empty compact and convex subsets of a normed linear space $E$, and let $T_i : X \to 2^{Y_i}$ be an upper semicontinuous correspondence in $X^0$ such that $T_i(x)$ is nonempty closed and convex subset of $Y_i$ for each $x \in X$. Assume that $T_i(x) \cap Y^0_i \neq \emptyset$ for each $x \in X^0$. 
Then there exists a system of best proximity pairs \((x^*, T_i(x^*)) \subseteq X \times Y_i, i.e., for each \(i \in I\), \(d(x^*, T_i(x^*)) = d(X, Y_i)\).

**Definition 3** ([7]). The set \(\mathcal{A}_x = \{y \in Y : y \in A(x) \text{ and } d(x, y) = d(X, Y)\}\) is named the best proximity set of the correspondence \(A : X \to 2^Y\) at \(x\).

In general, \(\mathcal{A}_x\) might be an empty set. If \((x, A(x))\) is a best proximity pair for \(A\) and \(A(x)\) is compact, then \(\mathcal{A}_x\) must be non-empty.

Let \((Ω, \mathcal{F}, \mu)\) be a measure space and \(Y\) be a Banach space. It is denoted by \(L_1(\mu, Y)\) the space of equivalence classes of \(Y\)-valued Bochner integrable functions \(x : Ω \to Y\) normed by \(\|x\| = \int_{Ω} \|x(ω)\|d\mu(ω)\). Also it is known (see [3], p.50) that \(L_1(\mu, Y)\) is a Banach space.

We denote by \(S_T\) the set of all selections of the correspondence \(T : Ω \to 2^Y\) that belong to the space \(L_1(\mu, Y)\), i.e. \(S_T = \{x \in L_1(\mu, Y) : x(ω) \in T(ω) \text{ } μ\text{-a.e.}\}\).

3. The models of general Bayesian abstract economies

We define the next model of the abstract economy.

Let \((Ω, \mathcal{F}, µ)\) be a complete finite measure space, where \(Ω\) denotes the set of states of nature of the world and the \(σ\)-algebra \(\mathcal{F}\) denotes the set of events. Let \(Y\) denote the strategy or commodity space, where \(Y\) is a separable Banach space.

Let \(I\) be a countable (or uncountable) set (the set of agents).

**Definition 4.** A general Bayesian abstract economy is a set \(G = \{Ω, \mathcal{F}, µ, (X_i, \mathcal{F}_i, A_i, B_i, P_i)_{i \in I}\}\), where:

(a) \(X_i : Ω \to 2^Y\) is the action (strategy) correspondence of agent \(i\) and let \(L_{X_i} = \{x \in S_{X_i} : x_i \text{ is } \mathcal{F}_i \text{ -measurable}\}\). Denote by \(L_X = \prod_{i \in I} L_{X_i}\) and by \(L_{X_{\bar{i}}}\) the set \(\prod_{j \neq i} L_{X_j}\). An element \(x_i\) of \(L_{X_i}\) is called a strategy for agent \(i\). The typical element of \(L_{X_i}\) is denoted by \(\bar{x}_i\) and that of \(X_i(ω)\) by \(x_i(ω)\) (or \(x_i\)). We assume that for each \(i \in I\), there exists a countable partition \(\prod_i\) of \(Ω\);

(b) \(\mathcal{F}_i\) is a sub \(σ\)-algebra of \(\mathcal{F}\) which denotes the private information of agent \(i\);

(c) for each \(ω \in Ω\), \(A_i(ω, \cdot), B_i(ω, \cdot) : L_X \to 2^Y\) are the random constraint correspondences of agent \(i\), where for all \((ω, x) \in Ω \times L_X\), \(A_i(ω, x) \subseteq X_i(ω)\) and \(B_i(ω, x) \subseteq X_i(ω)\).
(d) for each \( \omega \in \Omega \), \( P_i(\omega, \cdot) : L_X \to 2^Y \) is the random preference correspondences of agent \( i \), where for all \( (\omega, x) \in \Omega \times L_X \), \( P_i(\omega, x) \subset X_i(\omega) \).

**Definition 5.** A Bayesian equilibrium for \( G \) is a strategy profile \( \tilde{x}^* \in L_X \) such that for all \( i \in I \),

(a) \( \tilde{x}^*_i(\omega) \in \text{cl} B_i(\omega, \tilde{x}^*) \mu - a.e. \)

(b) \( A_i(\omega, \tilde{x}^*) \cap P_i(\omega, \tilde{x}^*) = \emptyset \mu - a.e. \)

**Remark 1.** Now we assume that for each \( i \in I \), \( X_i \) is a compact convex nonempty subset of \( Y \) and for each \( \omega \in \Omega \), we set \( X_i(\omega) = X_i \). Then we obtain the deterministic classical model of Yannelis and Prabhakar in [17] for an abstract economy with any set of players. This model also generalizes the Bayesian model of Yannelis in [18].

**Remark 2.** The interpretation of the preference correspondences \( P_i \) is the following: \( y_i \in P_i(\omega, x) \) means that at the state \( \omega \) of the nature, agent \( i \) strictly prefers \( y_i \) to \( x_i(\omega) \) if the given strategy of other agents is fixed. The preferences need not be representable by utility functions. However, it will be assumed that \( x_i(\omega) \notin P_i(\omega, x) \mu - a.e. \)

We now introduce the notion of equilibrium pair for our model of general Bayesian abstract economy.

**Definition 6.** A Bayesian equilibrium pair for \( G \) is a pair of strategy profiles \( (\tilde{x}^*, \tilde{y}^*) \in L_X \times L_X \) such that for all \( i \in I \),

(a) \( \tilde{x}^*_i(\omega) \in B_i(\omega, \tilde{x}^*) \mu - a.e., \tilde{y}^*_i(\omega) \in P_i(\omega, \tilde{x}^*) \mu - a.e; \)

(b) \( A_i(\omega, \tilde{x}^*) \cap P_i(\omega, \tilde{y}^*) = \emptyset \mu - a.e. \)

We also define the next model of the free abstract economy.

Let \( (\Omega, \mathcal{F}, \mu) \) be a complete finite measure space, where \( \Omega \) denotes the set of states of nature of the world and the \( \sigma \)-algebra \( \mathcal{F} \), denotes the set of events. Let \( Y \) denote the strategy or commodity space, where \( Y \) is a separable Banach space.

Let \( I \) be a finite set (the set of agents).

**Definition 7.** A free Bayesian abstract economy is a set \( G = (\Omega, \mathcal{F}, \mu), (X_i, Z_i, \mathcal{F}_i, A_i, P_i, B_i)_{i \in I}, \) where:

(a) \( X_i : \Omega \to 2^Y \) is the action (strategy) correspondence of agent \( i \) determining the manufacturing commodities and let \( L_{X_i} = \{ x \in S_{X_i} : x_i \) is \( \mathcal{F}_i \)-measurable \}. Denote by \( L_X = \prod_{i \in I} L_{X_i} \) and by \( L_{X^{-i}} \) the set \( \prod_{j \neq i} L_{X_j} \).
An element $x_i$ of $L_{X_i}$ is called a strategy for agent $i$. The typical element of $L_{X_i}$ is denoted by $\bar{x}_i$ and that of $X_i(\omega)$ by $x_i(\omega)$ (or $x_i$). We assume that for each $i \in I$, there exists a countable partition $\Pi_i$ of $\Omega$;

(b) $Z_i : \Omega \rightarrow 2^I$ is the action (strategy) correspondence of agent $i$ determining the selling commodities;

(c) $F_i$ is a sub $\sigma$–algebra of $F$ which denotes the private information of agent $i$;

(d) for each $\omega \in \Omega$, $A_i(\omega, \cdot), B_i(\omega, \cdot) : L_X \rightarrow 2^Y$ are the random constraint correspondences of agent $i$, where for all $(\omega, x) \in \Omega \times L_X$, $A_i(\omega, x) \subset X_i(\omega)$ and $B_i(\omega, x) \subset X_i(\omega)$;

(e) for each $\omega \in \Omega$, $P_i(\omega, \cdot) : L_Z \rightarrow 2^Y$ is the random preference correspondences of agent $i$, where for all $(\omega, x) \in \Omega \times L_Z$, $P_i(\omega, x) \subset Z_i(\omega)$.

Definition 8. A Bayesian equilibrium pair for $G$ is a strategy profiles pair $(\bar{x}^*, \bar{y}^*) \in L_X \times L_Z$ such that for all $i \in I$,

(a) $\bar{y}_i^* \in B_i(\bar{x}^*)$;
(b) $A_i(\omega, \bar{x}^*) \cap P_i(\omega, \bar{y}^*) = \emptyset$ $\mu$–a.e.;
(c) $d(\bar{x}^*, \bar{y}_i^*) = d(L_X, L_{Z_i})$.

The economic interpretation of a Bayesian equilibrium pair for $\Gamma$ is based on the requirement that for each $i \in I$, minimize the travelling cost $d(\bar{x}, \bar{y}_i)$, and also, maximize the preference $P_i(\omega, y)$ on the constraint set $A_i(\omega, x)$. Therefore, it is contemplated to find a pair of points $(x^*, y^*) \in L_X \times L_Z$ such that for each $i \in I$, $\bar{y}_i^* \in B_i(\bar{x}^*)$, $A_i(\omega, \bar{x}^*) \cap P_i(\omega, \bar{y}^*) = \emptyset$.

4. Existence of equilibrium

4.1. Existence of Bayesian equilibrium pairs for free Bayesian abstract economies

In this subsection we use the following notation: for each $i \in I$, $X_i^0 := \{x \in L_X : \text{for each } i \in I, \exists z_i \in L_{Z_i} \text{ such that } d(x, z_i) = d(L_{X_i}, L_{Z_i}) = \inf\{d(x, z_i) : x \in L_X, z_i \in L_{Z_i}\}\}$;

$Z_i^0 := \{y \in L_{Z_i} : \text{there exists } x \in L_X \text{ such that } d(x, y) = d(L_X, L_{Z_i})\}$.

This is our first theorem. The constraint and preference correspondences verify the assumptions of measurable graph and upper semicontinuity.

Theorem 4. Let $I$ be a finite set. Let $G = \{(X_i, F_i, A_i, P_i, B_i), i \in I\}$ be a general Bayesian abstract economy satisfying (a)-(h). Then there exists a Bayesian equilibrium pair for $G$.
For each $i \in I$:

(a) $X_i, Z_i : \Omega \to 2^Y$ are nonempty, convex, weakly compact-valued and integrably bounded correspondences;

(b) $X_i, Z_i : \Omega \to 2^Y$ are $\mathcal{F}_i$-lower measurable, i.e., for every open subset $V$ of $Y$, the set $\{ \omega \in \Omega : X_i(\omega) \cap V \neq \emptyset \}$ belongs to $\mathcal{F}_i$;

(c) For each $(\omega, x) \in \Omega \times L_X$, $A_i(\omega, x)$ has nonempty closed convex values included in $X(\omega)$;

(d) For each $(\omega, x) \in \Omega \times L_Z$, $P_i(\omega, x)$, has nonempty closed convex values included in $Z(\omega)$;

(e) $A_i$ and $P_i$ have measurable graph ($P_i$ has measurable graph in the sense that $\{(\omega, x, y) \in \Omega \times L_Z \times Y : y \in P_i(\omega, x)\} \in \mathcal{F} \otimes \beta_w(L_Z) \otimes \beta(Y)$ where $\beta_w(L_Z)$ is the Borel $\sigma$-algebra for the weak topology on $L_Z$ and $\beta(Y)$ is the Borel $\sigma$-algebra for the norm topology on $Y$);

(f) For each $\omega \in \Omega$, $P_i(\omega, \cdot) : L_Z \to 2^Y$ is upper semicontinuous ($P_i(\omega, \cdot)$ is u.s.c in the sense that the set $\{ x \in L_X : P_i(\omega, x) \subset V \}$ is weakly open in $L_X$ for every norm open subset $V$ of $Y$;

(g) $B_i : L_X \to 2^{L_Z}$, is upper semicontinuous in $X^0$, has closed convex values in $L_Z$, $B_i(x) \cap Z_i^0 \neq \emptyset$ for each $x \in X^0$;

(h) For each $x_i \in L_X$, for each $\omega \in \Omega$, $x_i(\omega) \notin P_i(\omega, x)$;

(i) For each $x' \in L_X$, the set $\{ x \in L_Z : there \ exists \ y_i \in L_X, such \ that \ y_i(\omega) \in A_i(\omega, x') \cap P_i(\omega, x) \mu - a.e. \}$ is weakly open in $L_Z$.

**Proof.** Firstly, we prove that $L_X$ and $L_Z$ are non-empty convex weakly compact subsets in $L_1(\mu, Y)$.

Since $(\Omega, \mathcal{F}, \mu)$ is a complete finite measure space, $Y$ is a separable Banach space and $X_i : \Omega \to Y$ has a measurable graph, by Aumann's selection theorem (see Appendix), it follows that there exists a $\mathcal{F}_i$-measurable function $f_i : \Omega \to Y$ such that $f_i(\omega) \in X_i(\omega) \mu - a.e$. Since $X_i$ is integrably bounded, we have that $f_i \in L_1(\mu, Y)$, hence $L_X$ is non-empty and $L_X = \prod_{i \in I} L_X$ is non-empty. Obviously $L_X$ is convex and $L_X$ is also convex. Since $X_i : \Omega \to Y$ is integrably bounded and has convex weakly compact values, by Diestel’s Theorem (see Appendix), it follows that $L_X$ is a weakly compact subset of $L_1(\mu, Y)$. Moreover, $L_X$ is weakly compact. $L_1(\mu, Y)$ equipped with the weak topology is a locally convex topological vector space. For $L_Z$ we use the same argument.

Since $B_i$ satisfies the whole assumption of Theorem 3 for each $i \in I$, there exists a point $x^* \in L_X$ satisfying the system of best proximity pairs, i.e., for each $i \in I$, $(x^* \times B_i(x^*)) \subseteq L_X \times L_Z$ such that $d(x^*, B_i(x^*)) = \ldots$
d(L_X, L_Z_i). Let \( B_i := \{ y_i \in B_i(x^*) / d(x^*, y_i) = d(L_X, L_Z_i) \} \) be the non-empty best proximity set of the correspondence \( B_i \). The set \( B_i \) is nonempty, closed and convex.

It remains to show that there exists \( y^* \in L_X \) such that \( y^*_i(\omega) \in A_i(\omega, x^*) \mu - a.e. \) and \( A_i(\omega, \tilde{x}^*) \cap P_i(\omega, \tilde{y}^*) = \emptyset \mu - a.e. \)

Define \( P'_i : L_Z \to 2^{L_Z_i} \), by \( P'_i(x) = \{ y_i \in L_Z_i : y_i(\omega) \in P_i(\omega, x) \mu - a.e. \} \).

Since for each \( x \in L_Z \), \( P'_i(\cdot, x) \) has a measurable graph and for each \( \omega \in \Omega \), \( P_i(\omega, \cdot) : L_Z \to 2^{L_Z_i} \) is upper semicontinuous and \( P_i(\omega, x) \subset Z_i(\omega) \) for each \( (\omega, x) \in \Omega \times L_Z \), by u. s. c. Lifting Theorem (see Appendix), it follows that \( P'_i \) is weakly upper semicontinuous. \( P'_i \) is convex valued since \( P_i \) is so.

\( P_i \) is nonempty valued and for each \( x \in L_Z \), \( P_i(\cdot, x) \) has a measurable graph. Hence, by the Aumann measurable selection theorem for each fixed \( x \in L_Z \), there exists an \( \mathcal{F}_i \)-measurable function \( y_i : \Omega \to Y \) such that \( y_i(\omega) \in P_i(\omega, x) \mu - a.e. \). Since for each \( (\omega, x) \in \Omega \times L_Z \), \( P_i(\omega, x) \) is contained in the integrably bounded correspondence \( X_i(\cdot) \), then \( y_i \in L_Z_i \) and we conclude that \( y_i \in P'_i(x) \) for each \( x \in L_Z \). Thus, \( P'_i \) is non-empty valued.

Define \( A'_i : L_X \to 2^{L_X_i} \), by \( A'_i(x) = \{ y_i \in L_X_i : y_i(\omega) \in A_i(\omega, x) \mu - a.e. \} \) and \( A' : L_X \to 2^{L_X} \) by \( A'(x) := \prod_{i \in I} A'_i(x) \) for each \( x \in L_X \). As \( P'_i, A'_i \) is nonempty valued. We have more that \( A'_i \) is closed convex valued.

Define \( G'_i : L_Z \to 2^{L_Z_i} \), by \( G'_i(x) = \begin{cases} A'_i(x^*) \cap P'_i(x), & \text{if } x_i \in \text{int} B_i \\ P'_i(x), & \text{if } x_i \notin \text{int} B_i \end{cases} \)

and \( G' : L_Z \to 2^{L_Z} \) by \( G'(x) := \prod_{i \in I} G'_i(x) \) for each \( x \in L_Z \). By Lemma 1, \( G'_i \) is an upper semicontinuous correspondence with respect to the weak topology of \( L_X \) and has convex closed values. \( G' \) is also an weakly upper semicontinuous correspondence, has non-empty convex closed values and \( x \notin G'(x) \) for each \( x \in L_X \). By (i), the set \( \{ x \in L_Z : G'_i(x) \neq \emptyset \} \) is weakly open in \( L_Z \). The set \( L_X \) is weakly compact and convex, and then, by Theorem 2, there exists \( y^* \in L_X \) such that for each \( i \in I \), \( G'_i(y^*) = \emptyset \).

For each \( x \notin \text{int} A_i \), \( P'_i(x) \) is a nonempty subset of \( X_i \). We have then \( A'_i(x^*) \cap P'_i(y^*) = \emptyset \) and \( y^*_i \in B_i \). It follows that \( A_i(\omega, x^*) \cap P_i(\omega, y^*) = \emptyset \mu - a.e. \), \( y^*_i \in B_i(x^*) \) and \( d(x^*, y^*) = d(L_X, L_Z_i) \).

\[ \square \]

4.2. Existence of equilibrium and equilibrium pairs for general Bayesian abstract economies

In the following theorem, the constraint and preference correspondences verify the assumptions of measurable graph and upper semicontinuity.
Theorem 5. Let $I$ be a countable (or uncountable) set. Let $G = \{ (X_i, F_i, A_i, P_i, B_i), i \in I \}$ be a general Bayesian abstract economy satisfying (a)-(f). Then there exists a Bayesian equilibrium pair for $G$.

For each $i \in I$:

(a) $X_i : \Omega \to 2^Y$ is a nonempty convex weakly compact-valued and integrably bounded correspondence;

(b) $X_i : \Omega \to 2^Y$ is $\mathcal{F}_i$-lower measurable, i.e., for every open subset $V$ of $Y$, the set $\{ \omega \in \Omega : X_i(\Omega) \cap V \neq \emptyset \}$ belongs to $\mathcal{F}_i$;

(c) For each $\omega, x \in \Omega \times L_X$, $A_i(\omega, x)$, $P_i(\omega, x)$ and $B_i(\omega, x)$ have nonempty closed convex values included in $X(\omega)$ and $A_i(\omega, x) \subseteq B_i(\omega, x)$;

(d) $B_i$ and $P_i$ have measurable graph ($B_i$ has measurable graph in the sense that $\{ (\omega, x, y) \in \Omega \times L_X \times Y : y \in B_i(\omega, x) \} \in \mathcal{F} \otimes \beta(L_X) \otimes \beta(Y)$ where $\beta(L_X)$ is the Borel $\sigma$-algebra for the weak topology on $L_X$ and $\beta(Y)$ is the Borel $\sigma$-algebra for the norm topology on $Y$;

(e) For each $\omega \in \Omega$, $B_i(\omega, \cdot)$ and $P_i(\omega, \cdot) : L_X \to 2^Y$ are upper semicontinuous ($B_i(\omega, \cdot)$ is u.s.c in the sense that the set $\{ x \in L_X : B_i(\omega, x) \subseteq V \}$ is weakly open in $L_X$ for every norm open subset $V$ of $Y$;

(f) For each $x_i \in L_{X_i}$, for each $\omega \in \Omega$, $x_i(\omega) \notin P_i(\omega, x)$.

Proof. Firstly, we prove that $L_X$ is a non-empty convex weakly compact subset in $L_1(\mu, Y)$.

Since $\Omega, F, \mu$ is a complete finite measure space, $Y$ is a separable Banach space and $X_i : \Omega \to 2^Y$ has a measurable graph, by Aumann’s selection theorem (see Appendix), it follows that there exists a $\mathcal{F}_i$-measurable function $f_i : \Omega \to Y$ such that $f_i(\omega) \in X_i(\omega) \ \mu-a.e.$ Since $X_i$ is integrably bounded, we have that $f_i \in L_1(\mu, Y)$, hence $L_X$ is non-empty and $L_X = \prod_{i \in I} L_{X_i}$ is non-empty. Obviously $L_{X_i}$ is convex and $L_X$ is also convex. Since $X_i : \Omega \to 2^Y$ is integrably bounded and has convex weakly compact values, by Diestel’s Theorem (see Appendix), it follows that $L_{X_i}$ is a weakly compact subset of $L_1(\mu, Y)$. More over, $L_X$ is weakly compact. $L_1(\mu, Y)$ equipped with the weak topology is a locally convex topological vector space.

Define $B'_i : L_X \to 2^{L_{X_i}}$, by $B'_i(x) = \{ y_i \in L_{X_i} : y_i(\omega) \in B_i(\omega, x) \}$ for each $x \in L_X$. $B'_i$ is convex valued since
$B_i$ is so.

$B_i$ is nonempty valued and for each $x \in L_X$, $B_i(\cdot, x)$ has a measurable graph. Hence, by the Aumann measurable selection theorem for each fixed $x \in L_X$, there exists an $\mathcal{F}_i$-measurable function $y_i : \Omega \to Y$ such that $y_i(\omega) \in B_i(\omega, x)$, $\mu$ - a.e. Since for each $(\omega, x) \in \Omega \times L_X$, $B_i(\omega, x)$ is contained in the integrably bounded correspondence $X_i(\cdot)$, then $y_i \in L_X$, and we conclude that $y_i \in B_i^*(x)$ for each $x \in L_X$. Thus, $B_i^*$ is non-empty valued.

$B'$ is an weakly upper semicontinuous correspondence and has also non-empty convex closed values.

The set $L_X$ is weakly compact and convex, and then, by Fan-Glicksberg’s fixed-point theorem in [5], there exists $x^* \in L_X$ such that $x^* \in B'(x^*)$, i.e., for each $i \in I$, $x_i^* \in B_i'(x^*)$. Then, $x_i^* \in L_X$, and $x_i^*(\omega) \in B_i(\omega, x_i^*)$, $\mu$ - a.e.

It remains to show that there exists $y^* \in L_X$ such that $y_i^*(\omega) \in P_i(\omega, x^*)$, $\mu$ - a.e. and $A_i(\omega, \tilde{x}^*) \cap P_i(\omega, \bar{y}^*) = \emptyset$, $\mu$ - a.e.

Define $A'_i : L_X \to 2^{L_X}$, by $A'_i(x) = \{ y_i \in L_X : y_i(\omega) \in A_i(\omega, x) \}$, $\mu$ - a.e. $A'_i$ is closed and convex valued since $A_i$ is so.

Define $P'_i : L_X \to 2^{L_X}$, by $P'_i(x) = \{ y_i \in L_X : y_i(\omega) \in P_i(\omega, x) \}$, $\mu$ - a.e. As $B'_i$, $P'_i$ is weakly upper semicontinuous and has nonempty convex closed values.

Define $G'_i : L_X \to 2^{L_X}$, by

$$G'_i(x) = \begin{cases} A'_i(x^*) \cap P'_i(x), & \text{if } x \in \text{int}P'_i(x^*) \\ P'_i(x), & \text{if } x \notin \text{int}P'_i(x^*) \end{cases}$$

and

$$G' : L_X \to 2^{L_X}$$

by $G'(x) := \prod_{i \in I} G'_i(x)$ for each $x \in L_X$. By Lemma 1, $G'_i$ is an upper semicontinuous correspondence with respect to the weakly topology on $L_X$ and has convex closed values. $G'$ is also an weakly upper semicontinuous correspondence, has non-empty convex closed values and $x \notin G'(x)$ for each $x \in L_X$. The set $L_X$ is weakly compact and convex, and then, by Theorem 1, there exists $y^* \in L_X$ such that $G'(y^*) = \emptyset$, i.e., for each $i \in I$, $G'_i(y^*) = \emptyset$.

For each $x \notin \text{int}P'_i(x^*)$, $P'_i(x)$ is a nonempty subset of $X_i$. We have then $A'_i(x^*) \cap P'_i(y^*) = \emptyset$ and $y^* \in P'_i(x^*)$. It follows that $A_i(\omega, x^*) \cap P_i(\omega, y^*) = \emptyset$, $\mu$ - a.e and $y_i^*(\omega) \in P_i(\omega, x^*)$, $\mu$ - a.e. \qed
5. Conclusion

This paper proposes models of Bayesian abstract economies and two notions of Bayesian equilibrium, capturing the meaning of uncertainty. Economic life is full of uncertainty, so that theories of mathematical economics must be open to create models which include this concept.

6. Appendix

The results below have been used in the proof of our theorems. For more details and further references see the paper quoted.

Theorem 6 (Projection theorem). Let \((Ω, F, μ)\) be a complete, finite measure space, and \(Y\) be a complete separable metric space. If \(H\) belongs to \(F \otimes β(Y)\), its projection \(\text{Proj}_Ω(H)\) belongs to \(F\).

Theorem 7 (Aumann measurable selection theorem [18]). Let \((Ω, F, μ)\) be a complete finite measure space, \(Y\) be a complete, separable metric space and \(T : Ω → 2^Y\) be a nonempty valued correspondence with a measurable graph, i.e., \(G_T ∈ F \otimes β(Y)\). Then there is a measurable function \(f : Ω → Y\) such that \(f(ω) ∈ T(ω)\) \(μ\)-a.e.

Theorem 8 (Diestel’s Theorem [18], Theorem 3.1). Let \((Ω, F, μ)\) be a complete finite measure space, \(Y\) be a separable Banach space and \(T : Ω → 2^Y\) be an integrably bounded, convex, weakly compact and nonempty valued correspondence. Then \(S_T = \{x ∈ L_1(μ, Y) : x(ω) ∈ T(ω) \ μ\text{-a.e.}\}\) is weakly compact in \(L_1(μ, Y)\).

Theorem 9 (U.s.c. Lifting Theorem, [18]). Let \(Y\) be a separable space, \((Ω, F, μ)\) be a complete finite measure space and \(X : Ω → 2^Y\) be an integrably bounded, nonempty, convex valued correspondence such that for all \(ω ∈ Ω, X(ω)\) is a weakly compact, convex subset of \(Y\). Denote by \(S_X\) the set \(\{x ∈ L_1(μ, Y) : x(ω) ∈ X(ω) \ μ\text{-a.e.}\}\). Let \(T : Ω × S_X → 2^Y\) be a nonempty, closed, convex valued correspondence such that \(T(ω, x) ⊂ X(ω)\) for all \((ω, x) ∈ Ω × S_X^1\). Assume that for each fixed \(x ∈ S_X, T(·, x)\) has a measurable graph and that for each fixed \(ω ∈ Ω, T(ω, ·) : S_X → 2^Y\) is u.s.c. in the sense that the set \(\{x ∈ S_X : T(ω, x) ⊂ V\}\) is weakly open in \(S_X\) for every norm open subset \(V\) of \(Y\). Define the correspondence \(Φ : S_X → 2^{S_X}\) by \(Φ(x) = \{y ∈ S_X : y(ω) ∈ T(ω, x) \ μ\text{-a.e.}\}\).

Then \(Φ\) is weakly u.s.c., i.e., the set \(\{x ∈ S_X : Φ(x) ⊂ V\}\) is weakly open in \(S_X\) for every weakly open subset \(V\) of \(S_X\).
REFERENCES


