GENERALIZED COFINITELY $\delta$-SEMIPERFECT MODULES

BY

FİĞEN YÜZBAŞI and ŞENOL EREN

Abstract. Let $R$ be a ring and $M$ be a left $R$-module. $M$ is called a cofinitely generalized (weak) $\delta$-supplemented module or briefly a $\delta$-CGS-module ($\delta$-CGWS-module) if every cofinite submodule of $M$ has a generalized (weak) $\delta$-supplement in $M$. In this paper, we give various properties of these modules. It is shown that (1) The class of cofinitely generalized (weak) $\delta$-supplemented modules are closed under taking homomorphic images, arbitrary sums, generalized $\delta$-covers and closed under extensions. (2) $M$ is a generalized cofinitely $\delta$-semiperfect module if and only if $M$ is a cofinitely generalized $\delta$-supplemented by generalized $\delta$-supplements which have generalized projective $\delta$-covers.

Mathematics Subject Classification 2010: 16E50, 16L30, 16D40.

Key words: cofinitely generalized $\delta$-supplemented module, cofinitely generalized weak $\delta$-supplemented module, generalized cofinitely $\delta$-semiperfect module, generalized projective $\delta$-cover.

1. Introduction

Throughout the paper, $R$ will be an associative ring with identity and all modules are unital left $R$-modules unless otherwise specified. Let $M$ be an $R$-module. By $N \leq M$ we mean that $N$ is a submodule of $M$. Recall that a submodule $N \leq M$ is called small, denoted by $N \ll M$, if $N + L \neq M$, for all proper submodules $L$ of $M$, and that $L \leq M$, is said to be essential in $M$, denoted by $L \leq M$, if $L \cap K \neq 0$ for each nonzero submodule $K \leq M$. A module $M$ is said to be singular if $M \cong \frac{N}{L}$ for some module $N$ and a submodule $L \leq N$ with $L \nleq N$. $\text{Rad}(M)$ will indicate Jacobson radical of $M$. $M$ is called supplemented, if every submodule $N$ of $M$ has a supplement in $M$, i.e. a submodule $K$ minimal with respect to $N + K = M$. $K$ is a
supplement of \( N \) in \( M \) if and only if \( N + K = M \) and \( N \cap K \ll K \) (see [8]). If \( N + K = M \) and \( N \cap K \ll M \), then \( K \) is called a \textit{weak supplement} of \( N \) in \( M \), (see [4], [11]), and clearly in this situation \( N \) is a weak supplement of \( K \), too. \( M \) is a \textit{weakly supplemented} module if every submodule of \( M \) has a weak supplement in \( M \).

By Zhou [10], a submodule \( L \) of \( M \) is called \( \delta \)-\textit{small} in \( M \) (denoted by \( L \ll_{\delta} M \)) if for any submodule \( N \) of \( M \) with \( \frac{M}{N} \) singular, \( M = N + L \) implies that \( M = N \). The sum of \( \delta \)-small submodules of a module \( M \) is denoted by \( \delta(M) \). It is easy to see that every small submodule of a module \( M \) is \( \delta \)-small in \( M \), so \( \text{Rad}(M) \subseteq \delta(M) \) and \( \text{Rad}(M) = \delta(M) \) if \( M \) is singular. Also any non-singular semisimple submodule of \( M \) is \( \delta \)-small in \( M \) and \( \delta \)-small submodules of a singular module are small submodules. For more detailed discussion on \( \delta \)-small submodules we refer to [10].

Let \( K, N \) be submodules of a module \( M \), then \( N \) is called a \( \delta \)-supplement of \( K \) in \( M \), if \( N + K = M \) and \( N \cap K \ll_{\delta} N \). \( N \) is called a weak \( \delta \)-supplement of \( K \) in \( M \), if \( N + K = M \) and \( N \cap K \ll_{\delta} M \). A module \( M \) is called \( \delta \)-\textit{supplemented} if every submodule of \( M \) has a \( \delta \)-supplement in \( M \). Also \( M \) is called \textit{weakly \( \delta \)-supplemented} if every submodule of \( M \) has a weak \( \delta \)-supplement in \( M \).

A submodule \( N \) of a module \( M \) is said to be \textit{cofinite} if \( \frac{M}{N} \) is finitely generated. An \( R \)-module \( M \) is called cofinitely (weak) \( \delta \)-supplemented, if each cofinite submodule of \( M \) has a (weak) \( \delta \)-supplement in \( M \).

Let \( M \) be an \( R \)-module and let \( N \) and \( K \) be any submodules of \( M \) with \( M = N + K \). If \( N \cap K \leq \delta(N) \) (\( N \cap K \leq \delta(M) \)) then \( N \) is called a generalized (weak) \( \delta \)-supplement of \( K \) in \( M \). Following [5], \( M \) is called a \textit{generalized \( \delta \)-supplemented} module (or briefly \( \delta \)-GS-module) if every submodule \( N \) of \( M \) has a generalized \( \delta \)-supplement \( K \) in \( M \). In [5], an \( R \)-module \( M \) is called \textit{generalized weakly \( \delta \)-supplemented} (or briefly \( \delta \)-GWS-module) (\( \delta \)-WGS-module in [5]) if every submodule \( K \) of \( M \) has a generalized weak \( \delta \)-supplement \( N \) in \( M \). Also in [5], \( M \) is called a \textit{generalized amply \( \delta \)-supplemented} module (or briefly \( \delta \)-GAS-module) if whenever \( M = N + K \) for submodules \( N, K \) of \( M \), then \( N \) contains a generalized \( \delta \)-supplement of \( K \) in \( M \). For characterizations of generalized (amply) \( \delta \)-supplemented and generalized weakly \( \delta \)-supplemented modules we refer to [5].

Let \( M \) be an \( R \)-module and let \( N \) and \( K \) be any submodules of \( M \) with \( M = N + K \). If \( N \cap K \leq \text{Rad}(K) \) (\( N \cap K \leq \text{Rad}(M) \)) then \( K \) is called a generalized (weak) supplement of \( N \) in \( M \). Following [6], \( M \) is called \textit{generalized supplemented} module (or briefly \( GS \)-module) if every submodule \( N \)
of $M$ has a generalized supplement $K$ in $M$. In [6], an $R$-module $M$ is called generalized weakly supplemented (or briefly GWS-module) (WGS-module in [6]) if every submodule $K$ of $M$ has a generalized weak supplement $N$ in $M$. For characterizations of generalized supplemented and generalized weakly supplemented modules we refer to [6] and [9].

$M$ is called a cofinitely (weak) supplemented module if every cofinite submodule of $M$ has a (weak) supplement in $M$ (see [1], [2]). Clearly supplemented modules are cofinitely supplemented and weakly supplemented modules are cofinitely weak supplemented. $M$ is called cofinitely generalized supplemented if every cofinite submodule of $M$ has a generalized supplement (see [3]). Since every submodule of a finitely generated module is cofinite, a finitely generated module is generalized supplemented if and only if it is cofinitely generalized supplemented.

A module $M$ is called cofinitely generalized weak supplemented (or briefly a CGWS-module) if every cofinite submodule of $M$ has a generalized weak supplement. Clearly generalized weakly supplemented modules and cofinitely weak supplemented modules are CGWS-modules.

In this paper, we define cofinitely generalized (weak) $\delta$-supplemented modules and investigate some properties of them.

2. Cofinitely generalized $\delta$-supplemented modules

Definition 2.1. An $R$-module $M$ is called cofinitely generalized $\delta$-supplemented, or briefly $\delta$-CGS-module if each cofinite submodule of $M$ has a generalized $\delta$-supplement in $M$.

It is clear from definitions, that a generalized $\delta$-supplemented module $M$ is a $\delta$-CGS-module, and if $M$ is finitely generated, then the converse also holds. Next we give some properties of $\delta$-CGS-modules.

Proposition 2.2. Let $M$ be a $\delta$-CGS-module, then any factor module of $M$ is a $\delta$-CGS-module.

Proof. Let $M$ be a $\delta$-CGS-module and $N$ be a submodule of $M$. Any cofinite submodule of $\frac{M}{N}$ is of the form $\frac{U}{N}$, where $U$ is a cofinite in $M$. So, there exists $K \leq M$, such that $M = U + K$ and $U \cap K \leq \delta(K)$. Thus $\frac{M}{N} = \frac{U}{N} + \frac{(K+N)}{N}$. Let $f : M \to \frac{M}{N}$ be a canonical epimorphism. Since
by Lemma 1.5 in [10]. Hence \((K + N) N\) is a generalized \(\delta\)-supplement of cofinite submodule \(U N\) in \(M N\). So \(\frac{M}{N}\) is a \(\delta\)-CGS-module. \(\square\)

**Corollary 2.3.** Any homomorphic image of a \(\delta\)-CGS-module is a \(\delta\)-CGS-module.

To show that an arbitrary sum of \(\delta\)-CGS-modules is a \(\delta\)-CGS-module, we use the following standard lemma.

**Lemma 2.4.** Let \(L, U\) be submodules of a module \(M\) such that \(L\) is a \(\delta\)-CGS-module, \(U\) is cofinite in \(M\) and \(L + U\) is a generalized \(\delta\)-supplement of \(K\) in \(M\). Then \(L \cap (K + U)\) has a generalized \(\delta\)-supplement \(X\) in \(L\). Moreover, \(K + X\) is a generalized \(\delta\)-supplement of \(U\) in \(M\).

**Proof.** Since \(K\) is a generalized \(\delta\)-supplement of \(L + U\) in \(M\), we have \(K + L + U = M\) and \(K \cap (L + U) \leq \delta(K)\). Now

\[
\frac{L}{L \cap (K + U)} \cong \frac{(K + (L + U))}{K + U} = \frac{M}{K + U} \cong \frac{(\frac{M}{N})}{(\frac{K + U}{N})},
\]

which is finitely generated, \(L \cap (K + U)\) is a cofinite submodule in \(L\). Since \(L\) is a \(\delta\)-CGS-module, there exists \(X \leq L\) a generalized \(\delta\)-supplement of \(L \cap (K + U)\) in \(L\). Thus \(L = [L \cap (K + U)] + X\) and \([L \cap (K + U)] \cap X = X \cap (K + U) \leq \delta(X)\). Since \(M = K + L + U = K + [L \cap (K + U)] + X + U = K + X + U\), \(X\) is a generalized \(\delta\)-supplement of \(K + U\) in \(M\). Therefore \(U \cap (K + X) \leq K \cap (L + U) + X \cap (K + U) \leq \delta(K) + \delta(X) \leq \delta(K + X)\). So \(K + X\) is a generalized \(\delta\)-supplement of \(U\) in \(M\). \(\square\)

**Proposition 2.5.** Any arbitrary sum of \(\delta\)-CGS-modules is a \(\delta\)-CGS-module.

**Proof.** Let \(M = \sum_{i \in I} M_i\) where each module \(M_i\) is a \(\delta\)-CGS-module and \(U\) be a cofinite submodule of \(M\). Then \(\frac{M}{U}\) is generated by some finite set \(\{x_1 + U, x_2 + U, \ldots, x_n + U\}\) and therefore \(M = Rx_1 + Rx_2 + \ldots +\)
$R_{x_n} + U$. Since each $x_i$ is contained in the sum $\sum_{j \in J} M_j$ for some finite subset $J = \{1_1, \ldots, 1_{s(1)}, \ldots, n_{s(n)}\}$ of $I$, $M = M_{1_1} + \sum_{j \in J \setminus \{1_1\}} M_j + U$ has a trivial generalized $\delta$-supplement 0 in $M$ and since $M_{1_1}$ is a $\delta$-CGS-module, $U + \sum_{j \in J \setminus \{1_1\}} M_j$ has a generalized $\delta$-supplement by Lemma 2.4. Continuing in this way we will obtain (after we have used Lemma 2.4 $\sum_{i=1}^n s(i)$ times) at last $U$ has a generalized $\delta$-supplement in $M$.

The following corollary follows from Corollary 2.3 and Proposition 2.5.

**Corollary 2.6.** If $M$ is a $\delta$-CGS-module, then any $M$-generated module is a $\delta$-CGS module.

**Proposition 2.7.** Let $M$ be a $\delta$-CGS-module. Then every cofinite submodule of $\frac{M}{\delta(M)}$ is a direct summand.

**Proof.** Assume $M$ is a $\delta$-CGS-module. Every cofinite submodule of $\frac{M}{\delta(M)}$ has the form $\frac{U}{\delta(M)}$, where $U$ is a cofinite submodule of $M$ and $\delta(M) \leq U$. By assumption, there exists $K \leq M$ such that $M = K + U$ and $K \cap U \leq \delta(K) \leq \delta(M)$. So $\frac{M}{\delta(M)} = (\frac{K}{\delta(M)}) \oplus (\frac{U}{\delta(M)})$.

**Lemma 2.8.** Let $M$ be a module with $\delta(M) \ll M$ and $U \leq M$. If $U$ has a generalized $\delta$-supplement that is a $\delta$-supplement in $M$, then $U$ has a $\delta$-supplement in $M$.

**Proof.** Let $V$ be a generalized $\delta$-supplement of $U$ in $M$. Note that $U \cap V \leq \delta(V) \leq \delta(M) \ll M$ and so $U \cap V \ll \delta M$. Since $V$ is a $\delta$-supplement in $M$, $U \cap V \ll \delta V$. Hence $V$ is a $\delta$-supplement of $U$ in $M$.

**Theorem 2.9.** Let $M$ be a module with $\delta(M) \ll M$. If $M$ is cofinitely generalized $\delta$-supplemented such that generalized $\delta$-supplements are $\delta$-supplements in $M$, then $M$ is cofinitely $\delta$-supplemented.

**Proof.** It can be seen from Lemma 2.8.

**Proposition 2.10.** Let $M$ be a module. If every cofinite submodule $U$ of $M$ has a generalized $\delta$-supplement $V$ in $M$ such that $U \cap V$ has a $\delta$-supplement in $V$, then $M$ is cofinitely $\delta$-supplemented.

**Proof.** Let $U$ be any cofinite submodule of $M$. By assumption, there is a submodule $V$ in $M$ such that $V$ is a generalized $\delta$-supplement of $U$ in $M$ and $U \cap V$ has a $\delta$-supplement $X$ in $V$. Then $U \cap V + X = V$, $(U \cap V) \cap X = U \cap X \ll X$. Now $M = U + V = U + U \cap V + X = U + X$. 
Hence $X$ is a $\delta$-supplement of $U$ in $M$ and it follows that $M$ is cofinitely $\delta$-supplemented.

Let $M$ and $N$ be $R$-modules. An epimorphism $f : M \to N$ is called a $\delta$-cover if $\text{Ker}(f) \leq \delta(M)$. Recall that an epimorphism $f : M \to N$ is called a generalized $\delta$-cover if $\text{Ker}(f) \leq \delta(M)$ and $M$ is called a generalized $\delta$-cover of $N$ with an epimorphism $f : M \to N$. A generalized $\delta$-cover $f : P \to N$ is called a generalized projective $\delta$-cover in case $P$ is a projective module. $\delta(M)$ is the sum of all $\delta$-small submodules of $M$, every $\delta$-cover is generalized $\delta$-cover. We have the following basic properties of (generalized) $\delta$-covers.

**Lemma 2.11.** If both $f : P \to M$ and $g : M \to N$ are (generalized) $\delta$-covers, then $gf : P \to N$ is a (generalized) $\delta$-cover.

**Proof.** If both $f$ and $g$ are $\delta$-covers, then $gf$ is a $\delta$-cover by Proposition 4.3 in [7].

Now let both $f$ and $g$ be generalized $\delta$-covers. To show $\text{Ker}(gf) \leq \delta(P)$, we let $p \in \text{Ker}(gf)$. Then $g(f(p)) = 0$ and $f(p) \in \text{Ker}(g) \leq \delta(M)$. Since $\text{Ker}(f) \leq \delta(P)$, it follows from [7, Proposition 4.2] that $f(\delta(P)) = \delta(M)$. Hence $f(p) = f(p_1)$ for some $p_1 \in \delta(P)$, so $p - p_1 \in \text{Ker}(f) \leq \delta(P)$. We obtain $p \in \delta(P)$.

**Lemma 2.12.** (1) If each $f_i : P_i \to M_i$, $(i = 1, 2, \ldots, n)$ is a $\delta$-cover, then $\oplus_{i=1}^{n} f_i : \oplus_{i=1}^{n} P_i \to \oplus_{i=1}^{n} M_i$ is a $\delta$-cover.

(2) If each $f_i : P_i \to M_i$, $(i = 1, 2, \ldots, n)$ is a generalized $\delta$-cover, then

$$\bigoplus_{i=1}^{n} f_i : \bigoplus_{i=1}^{n} P_i \to \bigoplus_{i=1}^{n} M_i$$

is a generalized $\delta$-cover.

**Proof.** (1) Lemma 4.4 in [7]

(2) Since each $\text{Ker}(f_i) \leq \delta(P_i)$ we have $\text{Ker}(\oplus_{i \in I} f_i) = \oplus_{i \in I} \text{Ker}(f_i) \leq \oplus_{i \in I} \delta(P_i) = \delta(\oplus_{i \in I} P_i)$ by Lemma 2.2 in [7]. So $\oplus_{i \in I} f_i$ is a generalized $\delta$-cover.

3. Cofinitely generalized weak $\delta$-supplemented modules

In this section we define and study cofinitely generalized weak $\delta$-supplemented modules.
Definition 3.1. A module $M$ is called cofinitely generalized weak $\delta$-supplemented, or briefly $\delta$-CGWS-module if every cofinite submodule of $M$ has a generalized weak $\delta$-supplement in $M$.

Proposition 3.2. Let $M$ be a $\delta$-CGWS-module, then every factor module of $M$ is a $\delta$-CGWS-module.

Proof. Let $M$ be a $\delta$-CGWS-module and $L$ be a submodule of $M$. Suppose that $\frac{M}{L}$ is a cofinite submodule of $\frac{M}{L}$. Note that $\frac{(\frac{M}{L})}{(\frac{M}{L})} \cong \frac{M}{L}$. Then $U$ is a cofinite submodule of $M$. Since $M$ is a $\delta$-CGWS-module, $U$ has a generalized weak $\delta$-supplement $V$ in $M$, i.e. $U + V = M$ and $U \cap V \leq \delta(M).$ Thus $\frac{M}{L} = \frac{U}{L} + \frac{(V + L)}{L}$. Let $f : M \rightarrow \frac{M}{L}$ be a canonical epimorphism. Since $U \cap V \leq \delta(M)$, $\frac{U}{L} \cap \frac{(V + L)}{L} = \frac{U \cap (V + L)}{L} = f(U \cap V) \leq f(\delta(M)) \leq \delta(\frac{M}{L})$ by Lemma 1.5 in [10]. This completes the proof. \qed

Corollary 3.3. Any homomorphic image of a $\delta$-CGWS-module is a $\delta$-CGWS-module.

To show that an arbitrary sum of $\delta$-CGWS-modules is a $\delta$-CGWS-module, we use the following standard lemma.

Lemma 3.4. Let $M$ be a module, $N$ and $U$ be submodules of $M$ with $\delta$-CGWS-module $N$ and cofinite $U$. If $N + U$ has a generalized weak $\delta$-supplement in $M$, then $U$ also has a generalized weak $\delta$-supplement in $M$.

Proof. Let $X$ be a generalized weak $\delta$-supplement of $N + U$ in $M$. Then we have

$$\frac{N}{[N \cap (X + U)]} \cong \frac{N + (X + U)}{X + U} = \frac{M}{X + U} \cong \frac{(\frac{M}{U})}{(\frac{(X + U)}{U})}.$$ 

Since $U$ is a cofinite submodule, $\frac{M}{U}$ is a finitely generated module. The last module is a finitely generated module hence $N \cap (X + U)$ has a generalized weak $\delta$-supplement $Y$ in $N$, i.e. $Y + [N \cap (X + U)] = N; Y \cap [N \cap (X + U)] = Y \cap (X + U) \leq \delta(N) \leq \delta(M)$. Since $M = U + X + N = U + X + Y + [N \cap (X + U)] = X + U + Y$, we get $Y$ is a generalized weak $\delta$-supplement of $X + U$ in $M$. Therefore $U \cap (X + Y) \leq [X \cap (Y + U)] + [Y \cap (X + U)] \leq \delta(M)$. So $X + Y$ is a generalized weak $\delta$-supplement of $U$ in $M$. \qed
Proposition 3.5. Any arbitrary sum of $\delta$-CGWS-modules is a $\delta$-CGWS-module.

Proof. Let $M = \sum_{i \in I} M_i$ where each module $M_i$ is a cofinitely generalized $\delta$-weak supplemented and $N$ be a cofinite submodule of $M$. Then $\frac{M}{N}$ is generated by some finite set $\{x_1 + N, x_2 + N, \ldots, x_n + N\}$ and therefore $M = Rx_1 + Rx_2 + \ldots + Rx_n + N$. Since each $x_i$ is contained in the sum $\sum_{j \in J} M_j$ for some finite subset $J = \{1, \ldots, 1_{s(1)}, \ldots, n_{s(n)}\}$ of $I$, $M = M_{1_1} + \sum_{j \in J - \{1_1\}} M_j + N$ has a trivial generalized weak $\delta$-supplement 0 in $M$ and since $M_{1_1}$ is a $\delta$-CGWS module, $N + \sum_{j \in J - \{1_1\}} M_j$ has a generalized weak $\delta$-supplement by Lemma 3.4. Continuing in this way we will obtain (after we have used Lemma 3.4 $\sum_{i=1}^n s(i)$ times) at last $N$ has a generalized weak $\delta$-supplement in $M$. \qed

Corollary 3.6. If $M$ is a $\delta$-CGWS module, then any $M$-generated module is a $\delta$-CGWS module.

Theorem 3.7. Let $M$ be a $\delta$-CGWS-module and $\delta(M) \leq N$. Then every cofinite submodule of $\frac{M}{N}$ is a direct summand.

Proof. Let $\frac{U}{N}$ be a cofinite submodule of $\frac{M}{N}$. Since $\left(\frac{\frac{M}{N}}{\frac{N}{N}}\right) \cong \frac{M}{U}$, $U$ is a cofinite submodule of $M$. Since $M$ is a $\delta$-CGWS-module, $U$ has a generalized weak $\delta$-supplement $V$, i.e. $U + V = M$ and $U \cap V \leq \delta(M)$. Let $f : M \to \frac{M}{N}$ be a canonical epimorphism. Then $\left(\frac{\frac{U}{N} + \frac{N}{N}}{N}\right)$ is a generalized weak $\delta$-supplement of $\frac{U}{N}$ in $\frac{M}{N}$ by Proposition 3.2 in [5]. Note that $U \cap V \leq \delta(M) \leq N$. So $\frac{U}{N}$ is a direct summand of $\frac{M}{N}$. \qed

Corollary 3.8. Let $M$ be a $\delta$-CGWS-module. Then every cofinite submodule of $\frac{M}{\delta(M)}$ is a direct summand.

Theorem 3.9. Let $M$ be a module and $N$ be a submodule with $N \leq \delta(M)$. If $\frac{M}{N}$ is a $\delta$-CGWS-module, then $M$ is a $\delta$-CGWS-module.

Proof. Let $U$ be any cofinite submodule of $M$. Note that $\left(\frac{M}{U+N}\right) \cong \left(\frac{\frac{M}{U}}{\frac{U+N}{N}}\right)$. Then $U+N$ is a cofinite submodule of $M$. Since $\frac{U+N}{N}$ is a cofinite submodule of $\frac{M}{N}$, there is a submodule $X$ of $\frac{M}{N}$ such that $\left(\frac{\frac{U+N}{N}}{N}\right) + X = \frac{M}{N}$.
and \((U + N) \cap \left( \frac{X}{N} \right) = \left( \frac{U \cap X + N}{N} \right) \leq \delta \left( \frac{M}{N} \right)\). So \(N \leq \delta(M)\), \(\delta \left( \frac{M}{N} \right) = \frac{\delta(M)}{N}\).

So \(U \cap X \leq \delta(M)\). Note that \(U + X = M\). Therefore, \(X\) is a generalized weak \(\delta\)-supplement of \(U\) in \(M\).

\[ \square \]

**Corollary 3.10.** A generalized \(\delta\)-cover of a \(\delta\)-CGWS-module is a \(\delta\)-CGWS-module.

**Theorem 3.11.** Let \(R\) be a ring. Then \(\frac{R}{\delta(H)}\) is semisimple if and only if every \(R\)-module is a \(\delta\)-CGWS-module.

**Proof.** Let \(M\) be an \(R\)-module. Suppose \(\frac{R}{\delta(H)}\) is semisimple. Then \(\frac{M}{\delta(M)}\) is an \(\frac{R}{\delta(H)}\)-module, hence semisimple. Therefore \(\frac{M}{\delta(M)}\) is semisimple and so \(M\) is a \(\delta\)-CGWS-module by Theorem 3.9.

The converse is by Corollary 3.8.

\[ \square \]

**Theorem 3.12.** Let \(0 \to L \to M \to N \to 0\) be a short exact sequence. If \(L\) and \(N\) are \(\delta\)-CGWS-modules and \(L\) has a generalized weak \(\delta\)-supplement in \(M\), then \(M\) is a \(\delta\)-CGWS-module.

**Proof.** Without restriction of generality, we will assume that \(L \leq M\). Let \(S\) be a generalized weak \(\delta\)-supplement of \(L\) in \(M\), i.e. \(L + S = M\) and \(L \cap S \ll_{\delta} M\). Then we have, \(\frac{M}{L + S} \cong \frac{L}{L + S} \oplus \frac{S}{L + S}\). \(\frac{L}{L + S}\) is cotinently generalized weak \(\delta\)-supplemented as a factor module of \(L\) which is cotinently generalized weak \(\delta\)-supplemented. On the other hand, \(\frac{S}{L + S} \cong \frac{M}{L} \cong N\) is cotinently generalized \(\delta\)-weak supplemented. Then \(\frac{M}{L + S}\) is cotinently generalized weak \(\delta\)-supplemented as a sum of cotinently generalized cotinently weak \(\delta\)-supplemented. Therefore \(M\) is a \(\delta\)-CGWS-module by Corollary 3.10.

\[ \square \]

**Theorem 3.13.** Suppose that \(M\) is a module with \(\delta(M) \ll_{\delta} M\). Then \(M\) is a cotinently generalized weak \(\delta\)-supplemented if and only if \(M\) is a cotinently weak \(\delta\)-supplemented.

**Proof.** Since \(\delta(M) = (\sum_{K \ll_{\delta} M} K) \ll_{\delta} M\), proof is obvious.

\[ \square \]

**Corollary 3.14.** For a finitely generated module \(M\), the following statements are equivalent:

(i) \(M\) is generalized weakly \(\delta\)-supplemented;

(ii) \(M\) is cotinently generalized weak \(\delta\)-supplemented;

(iii) \(M\) is cotinently weak \(\delta\)-supplemented;
(iv) $M$ is weakly $\delta$-supplemented.

**Proof.** (i) $\Rightarrow$ (ii) It is obvious.
(ii) $\Rightarrow$ (iii) Since $M$ is finitely generated, the result follows from Theorem 3.13.
(iii) $\Rightarrow$ (iv) Let $U$ be a submodule of $M$. Since $M$ is finitely generated, $U$ is a cofinite submodule of $M$. By the assumption, $M$ is weakly $\delta$-supplemented.
(iv) $\Rightarrow$ (i) It is an immediate result of $\delta(M) = \sum K \ll M K$.

4. Generalized cofinitely $\delta$-semiperfect modules

In this section, we characterize generalized cofinitely $\delta$-semiperfect modules via generalized projective $\delta$-covers of the generalized $\delta$-supplement submodules.

**Definition 4.1.** $M$ is called cofinitely generalized amply $\delta$-supplemented, or briefly $\delta$-CGAS-module if every cofinite submodule of $M$ has a generalized ample $\delta$-supplement in $M$.

**Definition 4.2.** A module $M$ is called generalized cofinitely $\delta$-semiperfect, or briefly $\text{gcof} \delta$-semiperfect, if every finitely generated factor module of $M$ has a generalized projective $\delta$-cover.

**Lemma 4.3.** Let $N$ be a submodule of the module and $f : M \to \frac{M}{N}$ be the canonical epimorphism. Also let $P$ be any module, $g : P \to \frac{M}{N}$ and $h : P \to M$ such that $g$ is composed with $f$. If the map $g$ is a generalized $\delta$-cover then $\text{Im}(h)$ is a generalized $\delta$-supplement of $N$ and $\text{Ker}(h) \leq \delta(P)$.

**Proof.** If $g$ is a generalized $\delta$-cover, then $N \cap \text{Im}(h) = \text{Ker}(f) \cap \text{Im}(h) = h(\text{Ker}(g)) \leq h(\delta(P)) \leq \delta(h(P)) = \delta(\text{Im}(h))$. It is clear that $M = \text{Im}(h) + \text{Ker}(f)$. This implies that $\text{Im}(h) = h(P)$ is a generalized $\delta$-supplement of $N = \text{Ker}(f)$ in $M$. Note that $\text{Ker}(h) \subseteq \text{Ker}(g)$. Therefore, we can obtain $\text{Ker}(h) \leq \delta(P)$. 

**Theorem 4.4.** For any module $M$, the following statements are equivalent:

(1) $M$ is a generalized cofinitely $\delta$-semiperfect module;

(2) $M$ is cofinitely generalized amply $\delta$-supplemented by generalized $\delta$-supplements which have generalized projective $\delta$-covers;
(3) \( M \) is cofinitely generalized \( \delta \)-supplemented by generalized \( \delta \)-supplements which have generalized projective \( \delta \)-covers.

**Proof.** (1)⇒(2): Let \( M = U + Y \), \( U \) is a cofinite submodule of \( M \). By assumption, there exists a generalized projective \( \delta \)-cover \( f : P \to \frac{M}{U} \). Let \( \pi : Y \to \frac{Y}{U \cap Y} \cong \frac{M}{U} \) be a canonical epimorphism. Since \( P \) is projective, there exists a homomorphism, \( g : P \to Y \), such that \( \pi g = f \). By Lemma 4.3, \( g(P) \) is a generalized \( \delta \)-supplement of \( \text{Ker}(\pi) = U \cap Y \) in \( Y \). Hence, \( M = U + Y = U + U \cap Y + g(P) = U + g(P) \). From this, it follows that \( g(P) \) is a generalized \( \delta \)-supplement of \( U \) in \( M \) and \( g(P) \subseteq Y \). It is clear that, \( g : P \to g(P) \) is a generalized projective \( \delta \)-cover of \( g(P) \).

(2)⇒(3): This is clear.

(3)⇒(1): Let \( U \) be a cofinite submodule of \( M \). By assumption, \( U \) has a generalized \( \delta \)-supplement \( K \) in \( M \) and \( K \) has a generalized projective \( \delta \)-cover \( f : P \to K \). Since \( U \cap K \leq \delta(K) \), then the canonical map \( \pi : K \to \frac{K}{U \cap Y} \cong \frac{M}{U} \) is a generalized \( \delta \)-cover of \( \frac{M}{U} \). The composition \( \alpha = \pi f : P \to \frac{M}{U} \) is a generalized projective \( \delta \)-cover of \( \frac{M}{U} \) by Lemma 2.11. Therefore \( M \) is a generalized cofinitely \( \delta \)-semiperfect module. \( \Box \)

**Acknowledgements.** The authors would like to thank the referee for carefully reading the manuscript and considerable remarks.

**REFERENCES**


Received: 18.V.2011
Revised: 8.II.2012
Accepted: 8.II.2012

Ondokuz Mayis University,
Faculty of Sciences and Arts,
Department of Mathematics,
55139 Kurupelit, Samsun,
TURKEY
figenyuzbas@gmail.com
seren@omu.edu.tr