ON GENERALIZED ABSOLUTE CESÁRO SUMMABILITY

BY

HÜSEYİN BOR

Abstract. In this paper, a main theorem dealing with $|C,1|_k$ summability factors has been generalized under more weaker conditions for $|C,\alpha,\beta|_k$ summability factors. This theorem also includes some new results.

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1. Introduction

A positive sequence $(b_n)$ is said to be almost increasing if there exists a positive increasing sequence $c_n$ and two positive constants $A$ and $B$ such that $Ac_n \leq b_n \leq Bc_n$ (see [1]). Obviously every increasing sequence is almost increasing. However, the converse need not be true as can be seen by taking an example, say $b_n = ne^{-1}n^2$. A sequence $(d_n)$ of positive numbers is said to be $\delta$-quasi monotone, if $d_n > 0$ ultimately and $\Delta d_n = d_n - d_{n+1} \geq -\delta_n$, where $(\delta_n)$ is a sequence of positive numbers (see [2]). Let $\sum a_n$ be a given infinite series with partial sums $(s_n)$. We denote by $u_n^{\alpha,\beta}$ and $t_n^{\alpha,\beta}$ the $n$-th Cesàro means of order $(\alpha, \beta)$, with $\alpha + \beta > -1$, of the sequence $(s_n)$ and $(na_n)$, respectively, i.e., (see [5])

\begin{align*}
  u_n^{\alpha,\beta} &= \frac{1}{A_n^{\alpha+\beta}} \sum_{v=0}^{n} A_n^{\alpha-1} A_v^{\beta} s_v, \\
  t_n^{\alpha,\beta} &= \frac{1}{A_n^{\alpha+\beta}} \sum_{v=1}^{n} A_n^{\alpha-1} A_v^{\beta} a_v,
\end{align*}

(1) (2)
where

\[ A_{n+\beta}^\alpha = O(n^{\alpha+\beta}), \quad \alpha + \beta > -1, \quad A_0^{\alpha+\beta} = 1 \quad \text{and} \quad A_{-n}^{\alpha+\beta} = 0 \quad \text{for} \quad n > 0. \]

The series \( \sum a_n \) is said to be summable \( |C, \alpha, \beta|_k \) if (see [6])

\[ \sum_{n=1}^\infty n^{k-1} |u_n^\alpha - u_{n-1}^\alpha|^k < \infty. \]

Since \( t_n^\alpha = n(u_n^\alpha - u_{n-1}^\alpha) \) (see [6]), condition (4) can also be written as

\[ \sum_{n=1}^\infty \frac{1}{n} |t_n^\alpha|^k < \infty. \]

If we take \( \beta = 0 \), then \( |C, \alpha, \beta|_k \) summability reduces to \( |C, \alpha|_k \) (see [7]) summability. Also, if we take \( \beta = 0 \) and \( \alpha = 1 \), then we have \( |C, 1|_k \) summability. It should be noted that obviously \( (C, \alpha, 0) \) mean is the same as \( (C, \alpha) \) mean. Mazhar [8] has obtained the following theorem for \( |C, 1|_k \) summability factors of infinite series.

**Theorem A.** Let \( (X_n) \) be a positive non-decreasing sequence such that \( |\Delta X_n| = O(X_n/n) \) and \( \lambda_n \to 0 \) as \( n \to \infty \). Suppose that there exists a sequence of numbers \( (A_n) \) such that it is \( \delta \)-quasi-monotone with \( \sum n^\delta X_n < \infty \), \( \sum A_n X_n \) is convergent and \( |\Delta \lambda_n| \leq A_n \) for all \( n \). If

\[ \sum_{n=1}^m \frac{1}{n} |t_n^\alpha|^k = O(X_m) \quad \text{as} \quad m \to \infty, \]

then the series \( \sum a_n \lambda_n \) is summable \( |C, 1|_k, k \geq 1 \).

**2. The main result**

The aim of this paper is to generalize Theorem A under more weaker conditions for \( |C, \alpha, \beta|_k \) summability, by taking an almost increasing sequence instead of a positive non-decreasing sequence. We shall prove the following theorem.

**Theorem.** Let \( (X_n) \) be an almost increasing sequence such that \( |\Delta X_n| = O(X_n/n) \) and \( \lambda_n \to 0 \) as \( n \to \infty \). Suppose that there exists a sequence
of numbers \((A_n)\) such that it is \(\delta\)-quasi-monotone with \(\sum n\delta_n X_n < \infty\), \(\sum A_n X_n\) is convergent and \(|\Delta \lambda_n| \leq A_n\) for all \(n\). If the sequence \((\theta_n^{\alpha,\beta})\) defined by

\[\theta_n^{\alpha,\beta} = |t_n^{\alpha,\beta}|, \quad \alpha = 1, \beta > -1 \]

\[\theta_n^{\alpha,\beta} = \max_{1 \leq v \leq n} |t_v^{\alpha,\beta}|, \quad 0 < \alpha < 1, \beta > -1 \]

satisfies the condition

\[\sum_{n=1}^{m} \frac{1}{n} (\theta_n^{\alpha,\beta})^k = O(X_m) \quad as \quad m \to \infty, \]

then the series \(\sum a_n \lambda_n\) is summable \(|C, \alpha, \beta|_k\) for \(0 < \alpha \leq 1, \beta > -1, \alpha + \beta > 0\) and \(k \geq 1\).

It should be noted that if we take \((X_n)\) as a positive non-decreasing sequence, \(\alpha = 1\) and \(\beta = 0\), then we get Theorem A. In this case condition (9) reduces to condition (6).

We need the following lemmas for the proof of our theorem.

**Lemma 1** ([3]). Let \((X_n)\) be an almost increasing sequence such that \(n |\Delta X_n| = O(X_n)\). If \((A_n)\) is a \(\delta\)-quasi-monotone with \(\sum n\delta_n X_n < \infty\) and \(\sum A_n X_n\) is convergent, then

\[nA_n X_n = O(1) \quad as \quad n \to \infty, \]

\[\sum_{n=1}^{\infty} n X_n |\Delta A_n| < \infty. \]

**Lemma 2** ([4]). If \(0 < \alpha \leq 1, \beta > -1\) and \(1 \leq v \leq n\), then

\[|\sum_{p=0}^{v} A_{n-p}^{\alpha-1} A_p^{\beta} a_p| \leq \max_{1 \leq m \leq v} |\sum_{p=0}^{m} A_{n-p}^{\alpha-1} A_p^{\beta} a_p|. \]

3. **Proof of the theorem**

Let \((T_n^{\alpha,\beta})\) be the \(n\)-th \((C, \alpha, \beta)\) mean of the sequence \((n a_n \lambda_n)\). Then, by (2), we have

\[T_n^{\alpha,\beta} = \frac{1}{A_n^{\alpha+\beta}} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} A_v^{\beta} v a_v \lambda_v. \]
First, applying Abel’s transformation and then using Lemma 2, we have that

\[
T_n^{\alpha,\beta} = \frac{1}{A_n^{\alpha+\beta}} \sum_{v=1}^{n-1} \Delta \lambda_v \sum_{p=1}^{v} A_{n-p}^{\alpha-1} A_p^\beta \rho_p + \frac{\lambda_n}{A_n^{\alpha+\beta}} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} A_v^\beta \nu_v,
\]

\[
| T_n^{\alpha,\beta} | \leq \frac{1}{A_n^{\alpha+\beta}} \sum_{v=1}^{n-1} | \Delta \lambda_v || \sum_{p=1}^{v} A_{n-p}^{\alpha-1} A_p^\beta \rho_p | + \frac{\lambda_n}{A_n^{\alpha+\beta}} | \sum_{v=1}^{n} A_{n-v}^{\alpha-1} A_v^\beta \nu_v |
\]

\[
\leq \frac{1}{A_n^{\alpha+\beta}} \sum_{v=1}^{n-1} A_v^\alpha A_v^\beta \theta_v^{\alpha,\beta} | \Delta \lambda_v | + | \lambda_n | \theta_n^{\alpha,\beta}
\]

\[
= T_{n,1}^{\alpha,\beta} + T_{n,2}^{\alpha,\beta}, \text{ say.}
\]

Since \(| T_{n,1}^{\alpha,\beta} + T_{n,2}^{\alpha,\beta} |^k \leq 2^k (| T_{n,1}^{\alpha,\beta} |^k + | T_{n,2}^{\alpha,\beta} |^k)\), in order to complete the proof of the theorem, by (5), it is sufficient to show that

\[
\sum_{\infty}^{n=1} \frac{1}{n} | T_{n,r}^{\alpha,\beta} |^k < \infty \text{ for } r = 1, 2.
\]

Whenever \(k > 1\), we can apply Hölder’s inequality with indices \(k\) and \(k'\), where \(\frac{1}{k} + \frac{1}{k'} = 1\), we get that

\[
\sum_{n=2}^{m+1} \frac{1}{n} | T_{n}^{\alpha,\beta} |^k \leq \sum_{n=2}^{m+1} \frac{1}{n} A_n^{\alpha+\beta} \sum_{v=1}^{n-1} A_v^\alpha A_v^\beta \theta_v^{\alpha,\beta} \Delta \lambda_v |^k
\]

\[
= O(1) \sum_{v=1}^{m} v^{(\alpha+\beta)k} A_v (\theta_v^{\alpha,\beta})^k \sum_{n=n+1}^{m+1} \frac{1}{n^{1+(\alpha+\beta)k}}
\]

\[
= O(1) \sum_{v=1}^{m} v^{(\alpha+\beta)k} A_v (\theta_v^{\alpha,\beta})^k \int_{v}^{\infty} dx \frac{dx}{x^{1+(\alpha+\beta)k}}
\]

\[
= O(1) \sum_{v=1}^{m} A_v (\theta_v^{\alpha,\beta})^k = O(1) \sum_{v=1}^{m} v A_v \frac{1}{v} (\theta_v^{\alpha,\beta})^k
\]

\[
= O(1) \sum_{v=1}^{m} \Delta(v A_v) \sum_{p=1}^{v} \frac{1}{p} (\theta_p^{\alpha,\beta})^k + O(1) m A_m \sum_{v=1}^{m} \frac{1}{v} (\theta_v^{\alpha,\beta})^k
\]
\[ O(1) \sum_{v=1}^{m-1} v | \Delta A_v | X_v + O(1) \sum_{v=1}^{m-1} A_v X_v + O(1) m A_m X_m \]

in view of hypotheses of the theorem and Lemma 1. Similarly, we have that

\[ \sum_{n=1}^{m} \frac{1}{n} | T_{n,2}^{\alpha,\beta} |^k = O(1) \sum_{n=1}^{m} \frac{\lambda_n}{n} (\theta_n^{\alpha,\beta})^k \]

\[ = O(1) \sum_{n=1}^{m} \frac{1}{n} (\theta_n^{\alpha,\beta})^k \sum_{v=n}^{\infty} | \Delta \lambda_v | \]

\[ = O(1) \sum_{v=1}^{\infty} | \Delta \lambda_v | \sum_{n=1}^{v} \frac{1}{n} (\theta_n^{\alpha,\beta})^k \]

\[ = O(1) \sum_{v=1}^{\infty} | \Delta \lambda_v | X_v = O(1) \sum_{v=1}^{\infty} A_v X_v < \infty. \]

Therefore, we get that

\[ \sum_{n=1}^{\infty} \frac{1}{n} | T_{n,r}^{\alpha,\beta} |^k < \infty \quad \text{for} \quad r = 1, 2, \quad \text{by} \quad (5). \]

This completes the proof of the theorem. If we take \( \beta = 0 \), then we get a new result for \( | C, \alpha |_k \) summability factors. Also, if we take \( \beta = 0, \alpha = 1 \) and \( X_n = \log n \), then we obtain a result of Mazhar [8] dealing with \( | C, 1 |_k \) summability factors.

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P.O.Box 121, 06502 Bahçelievler, Ankara, TURKEY
hhbor33@gmail.com