Eigenvalue assignment in fractional descriptor discrete-time linear systems

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The problem of eigenvalue assignment in fractional descriptor discrete-time linear systems is considered. Necessary and sufficient conditions for the existence of a solution to the problem are established. A procedure for computation of the gain matrices is given and illustrated by a numerical example.

Key words: eigenvalue assignment, fractional, descriptor, discrete-time linear system, gain matrix.

1. Introduction

A dynamical system is called a fractional-order system if its state equations are given by fractional-order derivative of state vector. Mathematical fundamentals of the fractional calculus are given in the [23, 25, 26]. The standard and positive fractional linear systems have been investigated in [18, 24] and the positive fractional linear electrical circuits in [20]. Some recent interesting results in the fractional systems theory and its applications can be found in [8, 27, 28, 30].

Descriptor (singular) linear systems were considered in many papers and books [1-7, 9-11, 17, 18, 22, 29, 31]. The positive standard and descriptor systems and their stability have been analyzed in [13-16, 28]. Descriptor positive discrete-time and continuous-time nonlinear systems have been analyzed in [10] and the positivity and linearization of nonlinear discrete-time systems by state-feedbacks in [14]. New stability tests of positive standard and fractional linear systems have been investigated in [12]. The controllability of dynamical systems has been investigated in [21].

In this paper the eigenvalue assignment problem for fractional descriptor discrete-time linear systems will be investigated and procedure for computation of the state-feedback gain matrices will be presented.

The paper is organized as follows. In section 2 the problem of eigenvalue assignment in fractional descriptor discrete-time linear systems is formulated. In section 3 the
problem is solved and procedure for computation of the state-feedback gain matrices is presented. Concluding remarks are given in section 4.

The following notation will be used: \( \mathbb{R} \) — the set of real numbers, \( \mathbb{R}^{n \times m} \) — the set of \( n \times m \) real matrices and \( \mathbb{R}^n = \mathbb{R}^{n \times 1} \), \( I_n \) — the \( n \times n \) identity matrix, \( Z_+ \) — the set of nonnegative integers.

2. Problem formulation

Consider the descriptor discrete-time linear system

\[
E \Delta^\alpha x_{k+1} = Ax_k + Bu_k, \quad k \in Z_+ = \{0, 1, \ldots\}
\]

where \( x_k \in \mathbb{R}^n \), \( u_k \in \mathbb{R}^m \) are the state and input vectors and \( E, A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m} \). The fractional difference of the order \( \alpha \) is defined by

\[
\Delta^\alpha x_k = \sum_{i=0}^{k} (-1)^k \left( \begin{array}{c} \alpha \\ i \end{array} \right) x_{k-i}, \quad \left( \begin{array}{c} \alpha \\ i \end{array} \right) = \left\{ \begin{array}{ll} 1 & \text{for } i = 0, \\ \frac{\alpha(\alpha-1)\ldots(\alpha-i+1)}{i!} & \text{for } i = 1, 2, \ldots \end{array} \right. \]

Substituting (2) into (1) yields

\[
Ex_{k+1} = A_\alpha x_k + \sum_{i=1}^{k+1} c_i Ex_{k-i+1} + Bu_k
\]

where

\[
A_\alpha = A + \alpha E, \quad c_i = (-1)^i \left( \begin{array}{c} \alpha \\ i+1 \end{array} \right), \quad i = 1, 2, \ldots .
\]

It is assumed that \( \text{rank} E = r < n \) and \( \text{rank} B = m \). In practical problems it is also assumed that \( i \) is bounded by natural number \( h = k + 1 > n \). We may write the equation (3) in the form

\[
\bar{E} \bar{x}_{k+1} = \bar{A} \bar{x}_k + \bar{B} u_k,
\]
where

\[
\tilde{A} = \begin{bmatrix}
A_\alpha & c_1 E & c_2 E & \cdots & c_{h-1} E & c_h E \\
I_n & 0 & 0 & \cdots & 0 & 0 \\
0 & I_n & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & I_n & 0
\end{bmatrix} \in \mathbb{R}^{\bar{n} \times \bar{n}},
\tilde{B} = \begin{bmatrix}
B \\
0 \\
0 \\
\vdots \\
0
\end{bmatrix} \in \mathbb{R}^{\bar{n} \times m},
\] (6)

\[
\tilde{E} = \begin{bmatrix}
E & 0 & 0 & \cdots & 0 \\
0 & I_n & 0 & \cdots & 0 \\
0 & 0 & I_n & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & I_n
\end{bmatrix} \in \mathbb{R}^{\bar{n} \times \bar{n}},
\bar{x}_k = \begin{bmatrix}
x_k \\
x_{k-1} \\
x_{k-2} \\
\vdots \\
x_{k-h}
\end{bmatrix} \in \mathbb{R}^\bar{n},
\] (7)

Let us consider the system (1) with the state-feedback

\[
\tilde{u}_k = K_1 \bar{x}_{k+1} + K_2 \bar{x}_k
\] (8)

where \( \tilde{u}_k \in \mathbb{R}^m \) is a new input vector and \( K_1, K_2 \in \mathbb{R}^{m \times \bar{n}} \) are gain matrices. Substitution of (7) into (5) yields

\[
(\tilde{E} - \tilde{B}K_1)\bar{x}_{k+1} = (\tilde{A} + \tilde{B}K_2)\bar{x}_k.
\] (9)

The problem can be stated as follows. Given \( E, A, B, \alpha \in (0, 1) \) find \( K_1, K_2 \) such that the closed-loop system has desired eigenvalues \( z_1, z_2, \ldots, z_n \), \( |z_k| < 1, k = 1, \ldots, n \).

### 3. Problem solution

The problem will be solved by the use of the following two steps procedure.

**Step 1.** (Subproblem 1) Find \( K_1 \) such that \( \tilde{E} - \tilde{B}K_1 = I_\bar{n} \).

**Step 2.** (Subproblem 2) Find \( K_2 \) such that \( \tilde{A} + \tilde{B}K_2 \) has desired eigenvalues.

The first subproblem has a solution if and only if [3]

\[
\text{rank}[\begin{bmatrix} \tilde{E} & \tilde{B} \end{bmatrix}] = \bar{n}, \quad \text{rank} \tilde{B} = m.
\] (10)

**Theorem 8** If the conditions (9) are satisfied then the equation

\[
\tilde{E} - \tilde{B}K_1 = I_\bar{n}
\] (11)
has the solution

\[
K_1 = \{ [\bar{B}^T \bar{B}]^{-1} \bar{B}^T + K[I - \bar{B}][\bar{B}^T \bar{B}]^{-1} \bar{B}^T \} (\bar{E} - I),
\]

(11)

where \( K \) is an arbitrary matrix.

**Proof** From (10) we have

\[
\bar{B}K_1 = \bar{E} - I.
\]

(12)

If conditions (9) are met then there exists the left pseudoinverse of the matrix \( \bar{B} \) given by the formula [19]

\[
\bar{B}_L = [\bar{B}^T \bar{B}]^{-1} \bar{B}^T + K[I - \bar{B}][\bar{B}^T \bar{B}]^{-1} \bar{B}^T
\]

(13)

and

\[
K_1 = \bar{B}_L (\bar{E} - I) = \{ [\bar{B}^T \bar{B}]^{-1} \bar{B}^T + K[I - \bar{B}][\bar{B}^T \bar{B}]^{-1} \bar{B}^T \} (\bar{E} - I),
\]

(14)

which is equivalent to (11).

**Remark 1** In particular case when \( K = 0 \) we have

\[
K_1 = [\bar{B}^T \bar{B}]^{-1} \bar{B}^T (\bar{E} - I) = \begin{bmatrix} [B^T B]^{-1} B^T (E - I) & 0 & \cdots & 0 \end{bmatrix}
\]

(15)

and then

\[
K_1 \bar{x}_{k+1} = [B^T B]^{-1} B^T (E - I) \bar{x}_{k+1}.
\]

(16)

The second subproblem will be solved substituting (10) into (8). Thus we have

\[
\bar{x}_{k+1} = (\bar{A} + \bar{B}K_2) \bar{x}_k.
\]

**Theorem 9** There exists a matrix \( K_2 \) such that the matrix \( \bar{A} + \bar{B}K_2 \) has the desired eigenvalues \( \lambda_k, k = 1, \ldots, \bar{n} \) if and only if the pair \( (\bar{A}, \bar{B}) \) is controllable.

**Proof** The proof is given in [11].

To solve the problem one of the well-known methods [11] can be applied. To simplify the notation we consider the single-input system (17) with a controllable pair \( (\bar{A}, \bar{B}) \).

Following [11] there exists a matrix

\[
P = \begin{bmatrix} p_1 \\ p_1 \bar{A} \\ \vdots \\ p_1 \bar{A}^{\bar{n}-1} \end{bmatrix}
\]

(18)
that transforms every controllable pair \((\bar{A}, \bar{B})\) to the canonical form

\[
\begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1 \\
-\bar{a}_0 & -\bar{a}_1 & -\bar{a}_2 & \cdots & -\bar{a}_{\bar{n}-1}
\end{pmatrix},
\bar{B} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}
\]

The vector \(p_1\) in (18) is the \(\bar{n}\)-th row of the matrix

\[
\begin{pmatrix}
\bar{B} & \bar{AB} & \cdots & \bar{A}^{\bar{n}-1}\bar{B}
\end{pmatrix}^{-1}.
\]

The characteristic polynomial of the matrix \(\tilde{A}\) has the form

\[
\det[I_{\bar{n}}z - \tilde{A}] = z^{\bar{n}} + \bar{a}_{\bar{n}-1}z^{\bar{n}-1} + \cdots + \bar{a}_1z + \bar{a}_0
\]

and the characteristic polynomial of the closed-loop system matrix \(\tilde{A} + \tilde{B}K_2\) has the form

\[
\det[I_{\bar{n}}z - \tilde{A} - \tilde{B}K_2] = z^{\bar{n}} + \tilde{d}_{\bar{n}-1}z^{\bar{n}-1} + \cdots + \tilde{d}_1z + \tilde{d}_0.
\]

The matrix satisfying (22) is given by

\[
K_2 = \begin{pmatrix}
\tilde{d}_0 - \bar{a}_0 & \tilde{d}_1 - \bar{a}_1 & \cdots & \tilde{d}_{\bar{n}-1} - \bar{a}_{\bar{n}-1}
\end{pmatrix}.
\]

The considerations can be easily extended to multi-input systems [11].

From the above we have the following procedure.

**Procedure 1.**

**Step 1.** Knowing \(A, B, E, \alpha\) choose \(h > n\) and compute the matrices \(\bar{A}, \bar{B}, \bar{E}\) defined by (6).

**Step 2.** Check the conditions (9), then using \(\bar{E}\) and \(\bar{B}\) compute \(K_1\) defined by (11). In particular case when \(K = 0\) we can use matrices \(E\) and \(B\) (see (15)).

**Step 3.** Applying one of the well-known methods [11] and using \(\tilde{A}, \tilde{B}\) compute \(K_2\) such that the matrix \(\tilde{A} + \tilde{B}K_2\) has the desired eigenvalues \(\lambda_k, k = 1, \ldots, \bar{n}\), \(\text{Re}\lambda_k < 0\). The method for single-input systems presented above can be used.

**Example 1** Consider the fractional descriptor discrete-time linear system (1) with the matrices

\[
E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}
\]
and $\alpha = 0.5$. Find $K_1$ and $K_2$ such that the closed-loop system has the eigenvalues $\lambda_k = 0, k = 1, \ldots, 9$. Using the Procedure 1 we obtain the following.

**Step 1.** Step 1. We choose $h = 2$. From (6) we have

$$\bar{A} = \begin{bmatrix}
0.5 & 1 & 0 & 0.125 & 0 & 0 & 0.0625 & 0 & 0 \\
0 & 0.5 & 1 & 0 & 0.125 & 0 & 0 & 0 & 0.0625 & 0 \\
1 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix},$$

(25)

$$\bar{E} = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}, \quad \bar{B} = \begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
1 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0
\end{bmatrix}. $$

**Step 2.** The conditions (9) are satisfied. Using (25) with (11) for

$$K = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$
we obtain the first gain matrix

$$K_1 = \begin{bmatrix} 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. $$

(26)

It is easy to check that $\bar{E} - \bar{B}K_1 = I_9$.

**Step 3.** Step 3. Using the presented algorithm for single-input systems we compute the matrix
\[
\begin{bmatrix}
\bar{B} & \tilde{A}\bar{B} & \cdots & \tilde{A}^{n-1}\bar{B}
\end{bmatrix}^{-1} = \\
\begin{bmatrix}
0 & 0 & 1 & -1 & 0 & -0.5 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & -0.5 \\
-13.5 & 0.5 & 0 & 5.5 & 10.5 & -0.5 & 2.4 & 3.1 & 5.4 \\
54 & 52 & 0 & -48 & -96 & -52 & 3.3 & 22.7 & 9.2 \\
82 & 370 & 0 & -2 & -248 & -370 & -49.3 & -104.8 & -29.2 \\
-688 & -1656 & 0 & 376 & 1776 & 1656 & 114 & 84 & -174 \\
384 & 1056 & 0 & -160 & -984 & -1056 & -104 & -156 & 24 \\
-1024 & -2368 & 0 & 576 & 2560 & 2368 & 160 & 112 & 224 \\
640 & 1408 & 0 & -384 & -1600 & -1408 & -80 & -16 & 176
\end{bmatrix}
\]

The vector has the form
\[
p_1 = \begin{bmatrix}
640 & 1408 & 0 & -384 & -1600 & -1408 & -80 & -16 & 176
\end{bmatrix}. \tag{28}
\]

Using (18) we compute the matrix
\[
P = \begin{bmatrix}
640 & 1408 & 0 & -384 & -1600 & -1408 & -80 & -16 & 176 \\
-64 & -256 & 0 & 0 & 160 & 256 & 40 & -88 & 40 \\
-32 & -32 & 0 & 32 & 56 & 32 & -4 & -16 & -4 \\
16 & 8 & 0 & -8 & -20 & -8 & -2 & -2 & -2 \\
0 & 0 & 0 & 0 & -1 & 0 & 1 & 0.5 & 1 \\
0 & -1 & 0 & 1 & 0.5 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & -0.1 & 0 & 0 & -0.1 & 0 \\
0.5 & 0.9 & 0 & 0.1 & -0.1 & 0.1 & 0.1 & 0 & 0.1 \\
0.4 & 0.9 & 1 & 0.1 & 0.1 & 0.1 & 0 & 0.1 & 0
\end{bmatrix}, \tag{29}
\]

which transforms the pair \((\tilde{A}, \bar{B})\) to the canonical form (see (19))
\[
\tilde{A} = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & -0.002 & -0.0117 & -0.0234 & -0.0781 & 1.125 & -0.5 & 1.5
\end{bmatrix},
\]
\( \vec{B} = [0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 1]^T. \) (30)

Using (23) we have the second gain matrix

\[
K_2 = \begin{bmatrix} 0 & 0 & -0.002 & -0.0117 & -0.0234 & -0.0781 & 1.125 & -0.5 & 1.5 \end{bmatrix}. \quad (31)
\]

The closed-loop system matrix is given by

\[
\tilde{A} + \tilde{B}K_2 = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix} \quad (32)
\]

and has desired eigenvalues \( \lambda_k = 0, \ k = 1, \ldots, 9. \)

4. Concluding remarks

The problem of eigenvalue assignment in fractional descriptor discrete-time linear systems has been considered. Necessary and sufficient conditions for the existence of a solution to the problem have been established. A procedure for computation of the gain matrices has been given and illustrated by a numerical example.

The considerations can be extended to fractional descriptor continuous-time linear systems.

References


