Hyperchaos, adaptive control and synchronization of a novel 5-D hyperchaotic system with three positive Lyapunov exponents and its SPICE implementation

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In this research work, a twelve-term novel 5-D hyperchaotic Lorenz system with three quadratic nonlinearities has been derived by adding a feedback control to a ten-term 4-D hyperchaotic Lorenz system (Jia, 2007) with three quadratic nonlinearities. The 4-D hyperchaotic Lorenz system (Jia, 2007) has the Lyapunov exponents $L_1 = 0.3684$, $L_2 = 0.2174$, $L_3 = 0$ and $L_4 = -12.9513$, and the Kaplan-Yorke dimension of this 4-D system is found as $D_{KY} = 3.0452$. The 5-D novel hyperchaotic Lorenz system proposed in this work has the Lyapunov exponents $L_1 = 0.4195$, $L_2 = 0.2430$, $L_3 = 0.0145$, $L_4 = 0$ and $L_5 = -13.0405$, and the Kaplan-Yorke dimension of this 5-D system is found as $D_{KY} = 4.0159$. Thus, the novel 5-D hyperchaotic Lorenz system has a maximal Lyapunov exponent (MLE), which is greater than the maximal Lyapunov exponent (MLE) of the 4-D hyperchaotic Lorenz system. The 5-D novel hyperchaotic Lorenz system has a unique equilibrium point at the origin, which is a saddle-point and hence unstable. Next, an adaptive controller is designed to stabilize the novel 5-D hyperchaotic Lorenz system with unknown system parameters. Moreover, an adaptive controller is designed to achieve global hyperchaos synchronization of the identical novel 5-D hyperchaotic Lorenz systems with unknown system parameters. Finally, an electronic circuit realization of the novel 5-D hyperchaotic Lorenz system using SPICE is described in detail to confirm the feasibility of the theoretical model.

Key words: chaos, hyperchaos, control, synchronization, circuit realization

1. Introduction

Nonlinear dynamics occurs widely in engineering, physics, biology and many other scientific disciplines [1]. Poincaré was the first to observe the possibility of chaos, in which a deterministic system exhibits aperiodic behavior that depends on the initial conditions, thereby rendering long-term prediction impossible, since then it has received much attention [2, 3]. Interest in nonlinear dynamics and in particular chaotic dynamics

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has grown rapidly since 1963, when Lorenz published his numerical work on a simplified model of convection and discussed its implications for weather prediction [4].

Chaos theory describes the qualitative study of unstable aperiodic behavior in deterministic nonlinear dynamical systems. For the motion of a dynamical system to be chaotic, the system variables should contain nonlinear terms and it must satisfy three properties: boundedness, infinite recurrence and sensitive dependence on initial conditions [5].

The Lyapunov exponent of a dynamical system is a quantity that characterizes the rate of separation of infinitesimally close trajectories. The sensitive dependence on initial conditions of a dynamical system is characterized by the presence of a positive Lyapunov exponent. A positive Lyapunov exponent reflects a direction of stretching and folding and along with phase-space compactness indicates the presence of chaos in a dynamical system. An \( n \)-dimensional dynamical system has a spectrum of \( n \) Lyapunov exponents and the maximal Lyapunov exponent (MLE) of a chaotic system is defined as the largest positive Lyapunov exponent of the system.

Chaos has developed over time. For example, Ruelle and Takens [6] proposed a theory for the onset of turbulence in fluids, based on abstract considerations about strange attractors. Later, May [7] found examples of chaos in iterated mappings arising in population biology. Feigenbaum [8] discovered that there are certain universal laws governing the transition from regular to chaotic behaviours. That is, completely different systems can go chaotic in the same way, thus, linking chaos and phase transitions.

The first famous chaotic system was accidentally discovered by Lorenz, when he was designing a 3-D model for atmospheric convection in 1963 [9]. Subsequently, Rössler discovered a 3-D chaotic system in 1976 [10], which is algebraically simpler than the Lorenz system. Indeed, Lorenz’s system is a seven-term chaotic system with two quadratic nonlinearities, while Rössler’s system is a seven-term chaotic system with just one quadratic nonlinearity.

Some well-known paradigms of 3-D chaotic systems are Arneodo system [11], Sprott systems [12], Chen system [13], Lü-Chen system [14], Liu system [15], Cai system [16], T-system [17], etc. Many new chaotic systems have been also discovered like Li system [18], Sundarapandian systems [19, 20], Vaidyanathan systems [21, 22, 23, 24, 25, 26, 27, 28], Pehlivan system [29], Jafari system [30], Pham system [31], etc.

Chaos theory has applications in several fields of science and engineering such as oscillators [32, 33, 34], lasers [35, 36, 37], chemical reactions [38, 39, 40], biology [41, 42], ecology [43, 44, 45], neural networks [46, 47, 48, 49], robotics [50, 51, 52], fuzzy logic [53, 54], electrical circuits [55, 56, 57], etc.

A hyperchaotic system is generally defined as a chaotic system with at least two positive Lyapunov exponents [58]. Thus, the dynamics of a hyperchaotic system are expended in several different directions simultaneously. Thus, the hyperchaotic systems have more complex dynamical behaviour and hence they have miscellaneous applications in engineering such as secure communications [59, 60, 61], cryptosystems [62, 63, 64], encryption [65, 66, 67], electrical circuits [68, 69, 70, 71], etc.
The minimum dimension for an autonomous, continuous-time, hyperchaotic system is four. Since the discovery of a first 4-D hyperchaotic system by Rössler in 1979 [72], many 4-D hyperchaotic systems have been found in the literature such as hyperchaotic Lorenz system [73], hyperchaotic Lü system [74], hyperchaotic Chen system [75], hyperchaotic Wang system [76], hyperchaotic Newton-Leipnik system [77], hyperchaotic Vaidyanathan system [78], etc.

Recently, there is some considerable interest in finding novel 5-D hyperchaotic systems with three positive Lyapunov exponents and such 5-D hyperchaotic systems have been found in the literature such as hyperchaotic Hu system [79], [80], etc.

In this research work, a twelve-term novel 5-D hyperchaotic Lorenz system with three quadratic nonlinearities has been derived by adding a feedback control to a 4-D hyperchaotic Lorenz system with three quadratic nonlinearities [81]. The 4-D hyperchaotic Lorenz system [81] has the Lyapunov exponents $L_1 = 0.3684, L_2 = 0.2174, L_3 = 0$ and $L_4 = -12.9513$, and the Kaplan-Yorke dimension of this 4-D system is found as $D_{KY} = 3.0452$. The 5-D novel hyperchaotic Lorenz system proposed in this work has the Lyapunov exponents $L_1 = 0.4195, L_2 = 0.2430, L_3 = 0.0145, L_4 = 0$ and $L_5 = -13.0405$. The Kaplan-Yorke dimension of the 5-D novel hyperchaotic Lorenz system is found as $D_{KY} = 4.0159$. Thus, the novel 5-D novel hyperchaotic Lorenz system has a maximal Lyapunov exponent (MLE), which is greater than the maximal Lyapunov exponent (MLE) of the 4-D hyperchaotic Lorenz system.

In this work, adaptive control and synchronization schemes have been also developed for the novel 5-D hyperchaotic Lorenz system with three positive Lyapunov exponents.

The study of control of a chaotic system investigates methods for designing feedback control laws that globally or locally asymptotically stabilize or regulate the outputs of a chaotic system.

Many methods have been developed for the control and tracking of chaotic systems such as active control [82, 83, 84, 85], adaptive control [86, 87, 88, 89, 90, 91, 92], backstepping control [93, 94, 95], sliding mode control [96, 97], etc.

Chaos synchronization problem deals with the synchronization of a couple of systems called the master or drive system and the slave or response system. To solve this problem, control laws are designed so that the output of the slave system tracks the output of the master system asymptotically with time.

The study of chaos in the last decades had a tremendous impact on the foundations of science and engineering and one of the most recent exciting developments in this regard is the discovery of chaos synchronization, whose possibility was first reported by Fujisaka and Yamada [98] and later by Pecora and Carroll [99].

Because of the butterfly effect, the synchronization of chaotic systems is a challenging problem in the chaos literature even when the initial conditions of the master and slave systems are nearly identical because of the exponential divergence of the outputs of the two systems in the absence of any control.

Different types of synchronization such as complete synchronization [99], anti-synchronization [100, 101, 102], hybrid synchronization [103, 104], lag synchronization [105], phase synchronization [105, 106], anti-phase synchronization [107], gener-
alized synchronization [108], projective synchronization [109], generalized projective synchronization [110, 111, 112], etc. have been studied in the chaos literature.

Since the discovery of chaos synchronization, different approaches have been proposed to achieve it, such as PC method [99], active control method [113, 114, 115, 116], adaptive control method [117, 118, 119, 120], backstepping control method [121, 122, 123, 124, 125, 126], sliding mode control method [127, 128, 129, 130, 131], etc.

All the main adaptive results in this paper are proved using Lyapunov stability theory. MATLAB simulations are depicted to illustrate the phase portraits of the novel 5-D hyperchaotic Lorenz system with three positive Lyapunov exponents, adaptive stabilization and synchronization results for the novel 5-D hyperchaotic Lorenz system.

Finally, an electronic circuit realization of the novel 5-D hyperchaotic Lorenz system using SPICE is presented to confirm the feasibility of the theoretical model.

2. A 5-D novel hyperchaotic Lorenz system

In [81], Jia (2007) obtained a 4-D hyperchaotic Lorenz system by adding a feedback control to the famous Lorenz system [9], and this 4-D system is given by the dynamics

\[
\begin{align*}
\dot{x}_1 &= a(x_2 - x_1) + x_4 \\
\dot{x}_2 &= cx_1 - x_1 x_3 - x_2 \\
\dot{x}_3 &= x_1 x_2 - bx_3 \\
\dot{x}_4 &= -x_1 x_3 + px_4
\end{align*}
\]  

(1)

where \(x_1, x_2, x_3, x_4\) are the system parameters and \(a, b, c, p\) are positive, constant, parameters.

In [81], it was shown that the 4-D system (1) is hyperchaotic when the parameter values are taken as

\[a = 10, \quad b = 8/3, \quad c = 28, \quad p = 1.3\]  

(2)

Also, for these parameter values, the Lyapunov exponents of the 4-D hyperchaotic Lorenz system (1) are calculated as

\[L_1 = 0.3684, \quad L_2 = 0.2174, \quad L_3 = 0, \quad L_4 = -12.9513\]  

(3)

Thus, the maximal Lyapunov exponent (MLE) of the system (1) is obtained as \(L_1 = 0.3684\). The system (1) is dissipative, because \(\sum_{i=1}^{4} L_i = -12.3655 < 0\). Also, the Kaplan-Yorke dimension of the 4-D hyperchaotic Lorenz system (1) is found as

\[D_{KY} = 3 + \frac{L_1 + L_2 + L_3}{|L_4|} = 3.0452\]  

(4)
For numerical simulations, we take the initial state of the hyperchaotic system (1) as $x_1(0) = 1.2, x_2(0) = 0.8, x_3(0) = 1.6$ and $x_4(0) = 0.7$.

Figs. 1-2 depict the 3-D phase portraits of the 4-D hyperchaotic system (1) in $(x_1,x_2,x_3)$, and $(x_2,x_3,x_4)$ spaces, respectively.

Figure 1: 3-D projection of the 4-D hyperchaotic Lorenz system on $(x_1,x_2,x_3)$ space.

In this research work, we derive a twelve-term novel 5-D hyperchaotic system with three quadratic nonlinearities by adding a feedback control to the ten-term 4-D hyperchaotic Lorenz system (1) as follows:

\[
\begin{align*}
\dot{x}_1 &= a(x_2 - x_1) + x_4 + x_5 \\
\dot{x}_2 &= cx_1 - x_1x_3 - x_2 \\
\dot{x}_3 &= x_1x_2 - bx_3 \\
\dot{x}_4 &= -x_1x_3 + px_4 \\
\dot{x}_5 &= qx_1
\end{align*}
\]

where $x_1, x_2, x_3, x_4, x_5$ are the system parameters and $a, b, c, p, q$ are positive, constant, parameters.

The 5-D system (5) is hyperchaotic when the parameter values are taken as

\[ a = 10, \quad b = 8/3, \quad c = 28, \quad p = 1.3, \quad q = 2.5 \]
Also, for these parameter values, the Lyapunov exponents of the 5-D novel hyperchaotic system (5) are calculated as

\[ L_1 = 0.4195, \quad L_2 = 0.2430, \quad L_3 = 0.0145, \quad L_4 = 0, \quad L_5 = -13.0405 \]  

Thus, the 5-D novel hyperchaotic system (5) has three positive Lyapunov exponents. Also, the maximal Lyapunov exponent (MLE) of the system (5) is obtained as \( L_1 = 0.4195 \), which is greater than the maximal Lyapunov exponent (MLE) of the 4-D hyperchaotic Lorenz system (1). Since the sum of the Lyapunov exponents of the 5-D novel hyperchaotic system (5) is negative, it follows that the system is dissipative.

Also, the Kaplan-Yorke dimension of the 5-D novel hyperchaotic Lorenz system (5) is found as

\[ D_{KY} = 4 + \frac{L_1 + L_2 + L_3 + L_4}{|L_5|} = 4.0159 \]  

Since the 5-D hyperchaotic system (5) has three positive Lyapunov exponents, it has a very complex dynamics and its trajectories can expand in three directions simultaneously.

For numerical simulations, we take the initial state of the 5-D novel hyperchaotic system (5) as \( x_1(0) = 1.2, x_2(0) = 0.8, x_3(0) = 1.6, x_4(0) = 0.7 \) and \( x_5(0) = 2.3 \).

Figs. 3-6 depict the 3-D phase portraits of the 5-D novel hyperchaotic system (5) in \((x_1, x_2, x_3), (x_2, x_3, x_4), (x_1, x_2, x_5)\) and \((x_3, x_4, x_5)\) spaces, respectively.
3. Analysis of the 5-D novel hyperchaotic Lorenz system

3.1. Dissipativity

In vector notation, the novel 5-D hyperchaotic Lorenz system (5) can be expressed as

\[
\dot{x} = f(x) = \begin{bmatrix}
    f_1(x_1, x_2, x_3, x_4, x_5) \\
    f_2(x_1, x_2, x_3, x_4, x_5) \\
    f_3(x_1, x_2, x_3, x_4, x_5) \\
    f_4(x_1, x_2, x_3, x_4, x_5) \\
    f_5(x_1, x_2, x_3, x_4, x_5)
\end{bmatrix},
\]

where

\[
\begin{align*}
    f_1(x_1, x_2, x_3, x_4, x_5) &= a(x_2 - x_1) + x_4 + x_5 \\
    f_2(x_1, x_2, x_3, x_4, x_5) &= cx_1 - x_1x_3 - x_2 \\
    f_3(x_1, x_2, x_3, x_4, x_5) &= x_1x_2 - bx_3 \\
    f_4(x_1, x_2, x_3, x_4, x_5) &= -x_1x_3 + px_4 \\
    f_5(x_1, x_2, x_3, x_4, x_5) &= qx_1
\end{align*}
\]

Figure 3: 3-D projection of the 5-D novel hyperchaotic Lorenz system on \((x_1, x_2, x_3)\) space.
Let \( \Omega \) be any region in \( \mathbb{R}^5 \) with a smooth boundary and also, \( \Omega(t) = \Phi_t(\Omega) \), where \( \Phi_t \) is the flow of \( f \). Furthermore, let \( V(t) \) denote the volume of \( \Omega(t) \).

By Liouville’s theorem, we know that
\[
\dot{V}(t) = \int_{\Omega(t)} (\nabla \cdot f) \, dx_1 \, dx_2 \, dx_3 \, dx_4 \, dx_5
\]  
(11)

The divergence of the novel 5-D system (5) is found as:
\[
\nabla \cdot f = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + \frac{\partial f_3}{\partial x_3} + \frac{\partial f_4}{\partial x_4} + \frac{\partial f_5}{\partial x_5} = -a - 1 - b + p = -\mu
\]  
(12)

where \( \mu \) is defined as
\[
\mu = a + 1 + b - p
\]  
(13)

For the choice of parameter values given in (6), we find that \( \mu = 12.3667 > 0 \).

Inserting the value of \( \nabla \cdot f \) from (12) into (11), we get
\[
\dot{V}(t) = \int_{\Omega(t)} (-\mu) \, dx_1 \, dx_2 \, dx_3 \, dx_4 \, dx_5 = -\mu V(t)
\]  
(14)

Integrating the first order linear differential equation (14), we get
\[
V(t) = \exp(-\mu t) V(0)
\]  
(15)
Since $\mu > 0$, it follows from (15) that $V(t) \to 0$ exponentially as $t \to \infty$. This shows that the 5-D novel hyperchaotic Lorenz system (5) is dissipative. Hence, the system limit sets are ultimately confined into a specific limit set of zero volume, and the asymptotic motion of the 5-D novel hyperchaotic Lorenz system (5) settles onto a strange attractor of the system.

### 3.2. Equilibrium Points

The equilibrium points of the 5-D novel hyperchaotic system (5) are obtained by solving the equations

\[
\begin{align*}
    f_1(x_1, x_2, x_3, x_4, x_5) &= a(x_2 - x_1) + x_4 + x_5 = 0 \\
    f_2(x_1, x_2, x_3, x_4, x_5) &= c x_1 - x_1 x_3 - x_2 = 0 \\
    f_3(x_1, x_2, x_3, x_4, x_5) &= x_1 x_2 - b x_3 = 0 \\
    f_4(x_1, x_2, x_3, x_4, x_5) &= -x_1 x_3 + p x_4 = 0 \\
    f_5(x_1, x_2, x_3, x_4, x_5) &= q x_1 = 0 
\end{align*}
\]

We take the parameter values as in the equation (6). Since $x^* = 0$ is the unique solution of the system of equations (16), it is immediate that $x^* = 0$ is the unique equilibrium of the 5-D novel hyperchaotic Lorenz system (5).
Figure 6: 3-D projection of the 5-D novel hyperchaotic Lorenz system on \((x_3, x_4, x_5)\) space.

The Jacobian matrix of the 5-D hyperchaotic Lorenz system (5) at the equilibrium point \(x^* = 0\) is given by

\[
J(x^*) = \begin{bmatrix}
-c & -1 & 0 & 0 & 0 \\
0 & 0 & -b & 0 & 0 \\
0 & 0 & 0 & p & 0 \\
q & 0 & 0 & 0 & 0 \\
\end{bmatrix}
= \begin{bmatrix}
-10 & 10 & 0 & 1 & 1 \\
28 & -1 & 0 & 0 & 0 \\
0 & 0 & -8/3 & 0 & 0 \\
2.5 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

(17)

The eigenvalues of the matrix \(J(x^*)\) are numerically obtained as

\[
\lambda_1 = 1.3, \lambda_2 = 11.9057, \lambda_3 = -0.0092, \lambda_4 = -2.6667, \lambda_5 = -22.8966
\]

(18)

Thus, the equilibrium point \(x^* = 0\) is a saddle-point, which is unstable.
3.3. Rotation symmetry about the \( x_3 \)-axis

We define a new set of coordinates as

\[
\begin{align*}
\xi_1 &= -x_1 \\
\xi_2 &= -x_2 \\
\xi_3 &= x_3 \\
\xi_4 &= -x_4 \\
\xi_5 &= -x_5
\end{align*}
\] (19)

We find that

\[
\begin{align*}
\dot{\xi}_1 &= -a(x_2 - x_1) - x_4 - x_5 = a(\xi_2 - \xi_1) + \xi_4 + \xi_5 \\
\dot{\xi}_2 &= x_1x_3 - cx_1 + x_2 = -\xi_1\xi_3 + c\xi_1 - \xi_2 \\
\dot{\xi}_3 &= x_1x_2 - bx_3 = \xi_1\xi_2 - b\xi_3 \\
\dot{\xi}_4 &= x_1x_3 - px_4 = -\xi_1\xi_3 + p\xi_4 \\
\dot{\xi}_5 &= -qx_1 = q\xi_1
\end{align*}
\] (20)

This shows that the 5-D novel hyperchaotic Lorenz system (5) is invariant under the change of coordinates

\[
(x_1, x_2, x_3, x_4, x_5) \mapsto (-x_1, -x_2, x_3, -x_4, -x_5)
\] (21)

Since the transformation (21) persists for all values of the system parameters, it follows that the 5-D novel hyperchaotic Lorenz system (5) has rotation symmetry about the \( x_3 \)-axis and that any non-trivial trajectory must have a twin trajectory.

3.4. Invariance

It is easy to see that the \( x_3 \)-axis and \( x_4 \)-axis are invariant under the flow of the 5-D novel hyperchaotic Lorenz system (5). The invariant motion along the \( x_3 \)-axis is characterized by the scalar dynamics

\[
\dot{x}_3 = -bx_3, \quad (b > 0)
\] (22)

which is globally exponentially stable. The invariant motion along the \( x_4 \)-axis is characterized by the scalar dynamics

\[
\dot{x}_4 = px_4, \quad (p > 0)
\] (23)

which is unstable.
3.5. Lyapunov exponents and Kaplan-Yorke dimension

For the parameter values given in the equation (6), the Lyapunov exponents of the 5-D novel hyperchaotic Lorenz system (5) are calculated as

\[ L_1 = 0.4195, \quad L_2 = 0.2430, \quad L_3 = 0.0145, \quad L_4 = 0, \quad L_5 = -13.0405 \]  
(24)

Thus, the 5-D novel hyperchaotic Lorenz system (5) has three positive Lyapunov exponents. Also, the maximal Lyapunov exponent (MLE) of the system (5) is obtained as \( L_1 = 0.4195 \).

Also, the Kaplan-Yorke dimension of the novel hyperchaotic system (5) is obtained as

\[ D_{KY} = 4 + \frac{L_1 + L_2 + L_3 + L_4}{|L_5|} = 4.0159 \]  
(25)

which is fractional.

Since the 5-D hyperchaotic Lorenz system (5) has three positive Lyapunov exponents, it has a very complex dynamics and the system trajectories can expand in three different directions.

4. Adaptive control of the 5-D novel hyperchaotic Lorenz system with unknown parameters

In this section, we use adaptive control method to derive an adaptive feedback control law for globally stabilizing the 5-D novel hyperchaotic Lorenz system with unknown parameters.

Thus, we consider the 5-D novel hyperchaotic Lorenz system given by

\[
\begin{align*}
\dot{x}_1 &= a(x_2 - x_1) + x_4 + x_5 + u_1 \\
\dot{x}_2 &= cx_1 - x_1x_3 - x_2 + u_2 \\
\dot{x}_3 &= x_1x_2 - bx_3 + u_3 \\
\dot{x}_4 &= -x_1x_3 + px_4 + u_4 \\
\dot{x}_5 &= qx_1 + u_5
\end{align*}
\]  
(26)

In (26), \( x_i, (i = 1, \ldots, 5) \) are the states and \( u_i, (i = 1, \ldots, 5) \) are the adaptive controls to be determined using estimates \( \hat{a}(t), \hat{b}(t), \hat{c}(t), \hat{p}(t), \hat{q}(t) \) for the unknown parameters \( a, b, c, p, q \), respectively.
We consider the adaptive feedback control law

\[
\begin{align*}
    u_1 &= -\hat{a}(t)(x_2 - x_1) - x_4 - x_5 - k_1x_1 \\
    u_2 &= -\hat{c}(t)x_1 + x_1x_3 + x_2 - k_2x_2 \\
    u_3 &= -x_1x_2 + \hat{b}(t)x_3 - k_3x_3 \\
    u_4 &= x_1x_3 - \hat{p}(t)x_4 - k_4x_4 \\
    u_5 &= -\hat{q}(t)x_1 - k_5x_5
\end{align*}
\]

where \(k_i, (i = 1, \ldots, 5)\) are positive gain constants.

Substituting (27) into (26), we get the closed-loop plant dynamics as

\[
\begin{align*}
    \dot{x}_1 &= \left[a - \hat{a}(t)\right](x_2 - x_1) - k_1x_1 \\
    \dot{x}_2 &= \left[c - \hat{c}(t)\right]x_1 - k_2x_2 \\
    \dot{x}_3 &= -\left[b - \hat{b}(t)\right]x_3 - k_3x_3 \\
    \dot{x}_4 &= \left[p - \hat{p}(t)\right]x_4 - k_4x_4 \\
    \dot{x}_5 &= \left[q - \hat{q}(t)\right]x_1 - k_5x_5
\end{align*}
\]

The parameter estimation errors are defined as

\[
\begin{align*}
    e_a(t) &= a - \hat{a}(t) \\
    e_b(t) &= b - \hat{b}(t) \\
    e_c(t) &= c - \hat{c}(t) \\
    e_p(t) &= p - \hat{p}(t) \\
    e_q(t) &= q - \hat{q}(t)
\end{align*}
\]

In view of (29), we can simplify the plant dynamics (28) as

\[
\begin{align*}
    \dot{x}_1 &= e_a(x_2 - x_1) - k_1x_1 \\
    \dot{x}_2 &= e_cx_1 - k_2x_2 \\
    \dot{x}_3 &= -e_bx_3 - k_3x_3 \\
    \dot{x}_4 &= e_px_4 - k_4x_4 \\
    \dot{x}_5 &= e_qx_1 - k_5x_5
\end{align*}
\]
Differentiating (29) with respect to \( t \), we obtain

\[
\begin{align*}
\dot{e}_a(t) &= -\dot{\hat{a}}(t) \\
\dot{e}_b(t) &= -\dot{\hat{b}}(t) \\
\dot{e}_c(t) &= -\dot{\hat{c}}(t) \\
\dot{e}_p(t) &= -\dot{\hat{p}}(t) \\
\dot{e}_q(t) &= -\dot{\hat{q}}(t)
\end{align*}
\] (31)

We use adaptive control theory to find an update law for the parameter estimates. We consider the quadratic candidate Lyapunov function defined by

\[
V(x, e_a, e_b, e_c, e_p, e_q) = \frac{1}{2} \sum_{i=1}^{5} x_i^2 + \frac{1}{2} \left( e_a^2 + e_b^2 + e_c^2 + e_p^2 + e_q^2 \right)
\] (32)

Differentiating \( V \) along the trajectories of (30) and (31), we obtain

\[
\dot{V} = -k_1 x_1^2 - k_2 x_2^2 - k_3 x_3^2 - k_4 x_4^2 - k_5 x_5^2 + e_a [x_1 (x_2 - x_1) - \dot{a}] + e_b [-x_3^2 - \dot{\hat{b}}] + e_c [x_1 x_2 - \dot{\hat{c}}] + e_p [x_4^2 - \dot{\hat{p}}] + e_q [x_1 x_5 - \dot{\hat{q}}]
\] (33)

In view of (33), we take the parameter update law as

\[
\begin{align*}
\dot{\hat{a}}(t) &= x_1 (x_2 - x_1) \\
\dot{\hat{b}}(t) &= -x_3^2 \\
\dot{\hat{c}}(t) &= x_1 x_2 \\
\dot{\hat{p}}(t) &= x_4^2 \\
\dot{\hat{q}}(t) &= x_1 x_5
\end{align*}
\] (34)

Next, we state and prove the main result of this section.

**Theorem 1** The novel 5-D hyperchaotic Lorenz system (26) with unknown system parameters is globally and exponentially stabilized for all initial conditions by the adaptive control law (27) and the parameter update law (34), where \( k_1, k_2, k_3, k_4, k_5 \) are positive gain constants.

**Proof** We prove this result by applying Lyapunov stability theory.

We consider the quadratic Lyapunov function defined by (32), which is clearly a positive definite function on \( \mathbb{R}^{10} \).
By substituting the parameter update law (34) into (33), we obtain the time-derivative of $V$ as

$$
\dot{V} = -k_1 x_1^2 - k_2 x_2^2 - k_3 x_3^2 - k_4 x_4^2 - k_5 x_5^2
$$

(35)

From (35), it is clear that $\dot{V}$ is a negative semi-definite function on $\mathbb{R}^{10}$. Thus, we can conclude that the state vector $x(t)$ and the parameter estimation error are globally bounded, i.e.

$$
[ x_1(t) \cdots x_5(t) \ e_a(t) \ e_b(t) \ e_c(t) \ e_p(t) \ e_q(t) ]^T \in L_\infty.
$$

We define $k = \min\{k_1,k_2,k_3,k_4,k_5\}$. Then it follows from (35) that

$$
\dot{V} \leq -k \|x(t)\|^2
$$

(36)

Thus, we have

$$
k \|x(t)\|^2 \leq -\dot{V}
$$

(37)

Integrating the inequality (37) from 0 to $t$, we get

$$
k \int_0^t \|x(\tau)\|^2 \ d\tau \leq V(0) - V(t)
$$

(38)

From (38), it follows that $x \in L_2$. Using (30), we can conclude that $\dot{x} \in L_\infty$. Using Barbalat’s lemma, we conclude that $x(t) \to 0$ exponentially as $t \to \infty$ for all initial conditions $x(0) \in \mathbb{R}^5$.

This completes the proof.

For the numerical simulations, the classical fourth-order Runge-Kutta method with step size $h = 10^{-8}$ is used to solve the systems (26) and (34), when the adaptive control law (27) is applied.

The parameter values of the novel 5-D hyperchaotic Lorenz system (26) are taken as in the hyperchaotic case, viz. $a = 10, b = 8/3, c = 28, p = 1.3$ and $q = 2.5$. We take the positive gain constants as $k_i = 5$ for $i = 1, \ldots, 5$.

Furthermore, as initial conditions of the novel 5-D hyperchaotic Lorenz system (26), we take $x_1(0) = 5.2, x_2(0) = 3.8, x_3(0) = -11.2, x_4(0) = -4.5$ and $x_5(0) = 3.5$.

Also, as initial conditions of the parameter estimates $\hat{a}(t), \hat{b}(t), \hat{c}(t), \hat{p}(t), \hat{q}(t)$, we take $\hat{a}(0) = 4.2, \hat{b}(0) = 6.8, \hat{c}(0) = -3.5, \hat{p}(0) = 8.2$ and $\hat{q}(0) = 7.4$.

In Fig. 7, the exponential convergence of the controlled states of the novel 5-D hyperchaotic Lorenz system (26) is depicted.
Figure 7: Time-history of the controlled states $x_1(t), x_2(t), x_3(t), x_4(t), x_5(t)$.

5. Adaptive synchronization of the 5-D novel hyperchaotic Lorenz systems with unknown parameters

In this section, we use adaptive control method to derive an adaptive feedback control law for globally synchronizing identical 5-D novel hyperchaotic Lorenz systems with unknown parameters.

As the master system, we consider the 5-D novel hyperchaotic Lorenz system given by

$$
\begin{align*}
\dot{x}_1 &= a(x_2 - x_1) + x_4 + x_5 \\
\dot{x}_2 &= cx_1 - x_1x_3 - x_2 \\
\dot{x}_3 &= x_1x_2 - bx_3 \\
\dot{x}_4 &= -x_1x_3 + px_4 \\
\dot{x}_5 &= qx_1
\end{align*}
$$

In (39), $x_1, x_2, x_3, x_4, x_5$ are the states and $a, b, c, p, q$ are unknown system parameters.
As the slave system, we consider the 5-D novel hyperchaotic Lorenz system given by

\[
\begin{align*}
\dot{y}_1 &= a(y_2 - y_1) + y_4 + y_5 + u_1 \\
\dot{y}_2 &= cy_1 - y_1y_3 - y_2 + u_2 \\
\dot{y}_3 &= y_1y_2 - by_3 + u_3 \\
\dot{y}_4 &= -y_1y_3 + py_4 + u_4 \\
\dot{y}_5 &= qy_1 + u_5
\end{align*}
\] (40)

In (40), \(y_1, y_2, y_3, y_4, y_5\) are the states and \(u_1, u_2, u_3, u_4, u_5\) are the adaptive controls to be determined using estimates \(\hat{a}(t), \hat{b}(t), \hat{c}(t), \hat{p}(t), \hat{q}(t)\) for the unknown parameters \(a, b, c, p, q\), respectively.

The synchronization error between the novel 5-D hyperchaotic systems (39) and (40) is defined by

\[
e_i = y_i - x_i, \quad (i = 1, 2, \ldots, 5)
\] (41)

Then the synchronization error dynamics is obtained as

\[
\begin{align*}
\dot{e}_1 &= a(e_2 - e_1) + e_4 + e_5 + u_1 \\
\dot{e}_2 &= ce_1 - y_1y_3 + x_1x_3 - e_2 + u_2 \\
\dot{e}_3 &= y_1y_2 - x_1x_2 - be_3 + u_3 \\
\dot{e}_4 &= -y_1y_3 + x_1x_3 + pe_4 + u_4 \\
\dot{e}_5 &= qe_1 + u_5
\end{align*}
\] (42)

We consider the adaptive feedback control law

\[
\begin{align*}
u_1 &= -\hat{a}(t)(e_2 - e_1) - e_4 - e_5 - k_1e_1 \\
u_2 &= -\hat{c}(t)e_1 + y_1y_3 - x_1x_3 + e_2 - k_2e_2 \\
u_3 &= -y_1y_2 + x_1x_2 + \hat{b}(t)e_3 - k_3e_3 \\
u_4 &= y_1y_3 - x_1x_3 + \hat{p}(t)e_4 - k_4e_4 \\
u_5 &= -\hat{q}(t)e_1 - k_5e_5
\end{align*}
\] (43)

where \(k_i, (i = 1, \ldots, 5)\) are positive gain constants.
Substituting (43) into (42), we get the closed-loop error dynamics as

\[
\begin{align*}
\dot{e}_1 &= [a - \hat{a}(t)](e_2 - e_1) - k_1 e_1 \\
\dot{e}_2 &= [c - \hat{c}(t)]e_1 - k_2 e_2 \\
\dot{e}_3 &= -[b - \hat{b}(t)]e_3 - k_3 e_3 \\
\dot{e}_4 &= [p - \hat{p}(t)]e_4 - k_4 e_4 \\
\dot{e}_5 &= [q - \hat{q}(t)]e_1 - k_5 e_5 
\end{align*}
\]  

(44)

The parameter estimation errors are defined as

\[
\begin{align*}
e_a(t) &= a - \hat{a}(t) \\
e_b(t) &= b - \hat{b}(t) \\
e_c(t) &= c - \hat{c}(t) \\
e_p(t) &= p - \hat{p}(t) \\
e_q(t) &= q - \hat{q}(t)
\end{align*}
\]  

(45)

In view of (45), we can simplify the plant dynamics (44) as

\[
\begin{align*}
\dot{e}_1 &= e_a(e_2 - e_1) - k_1 e_1 \\
\dot{e}_2 &= e_c e_1 - k_2 e_2 \\
\dot{e}_3 &= -e_b e_3 - k_3 e_3 \\
\dot{e}_4 &= e_p e_4 - k_4 e_4 \\
\dot{e}_5 &= e_q e_1 - k_5 e_5 
\end{align*}
\]  

(46)

Differentiating (45) with respect to \( t \), we obtain

\[
\begin{align*}
\dot{e}_a(t) &= -\dot{\hat{a}}(t) \\
\dot{e}_b(t) &= -\dot{\hat{b}}(t) \\
\dot{e}_c(t) &= -\dot{\hat{c}}(t) \\
\dot{e}_p(t) &= -\dot{\hat{p}}(t) \\
\dot{e}_q(t) &= -\dot{\hat{q}}(t)
\end{align*}
\]  

(47)

We use adaptive control theory to find an update law for the parameter estimates. We consider the quadratic candidate Lyapunov function defined by

\[
V(e, e_a, e_b, e_c, e_p, e_q) = \frac{1}{2} \sum_{i=1}^{5} e_i^2 + \frac{1}{2} \left(e_a^2 + e_b^2 + e_c^2 + e_p^2 + e_q^2\right)
\]  

(48)
Differentiating $V$ along the trajectories of (46) and (47), we obtain

$$
\dot{V} = -k_1 e_1^2 - k_2 e_2^2 - k_3 e_3^2 - k_4 e_4^2 - k_5 e_5^2 \\
+ e_a [e_1 (e_2 - e_1) - \hat{a}] + e_b [-e_3^{-2} - \hat{b}] \\
+ e_c [e_1 e_2 - \hat{c}] + e_p [e_4^2 - \hat{p}] + e_q [e_1 e_5 - \hat{q}]
$$

(49)

In view of (49), we take the parameter update law as

$$
\begin{align*}
\dot{\hat{a}}(t) &= e_1 (e_2 - e_1) \\
\dot{\hat{b}}(t) &= -e_3^2 \\
\dot{\hat{c}}(t) &= e_1 e_2 \\
\dot{\hat{p}}(t) &= e_4^2 \\
\dot{\hat{q}}(t) &= e_1 e_5
\end{align*}
$$

(50)

Next, we state and prove the main result of this section.

**Theorem 2** The novel 5-D hyperchaotic Lorenz systems (39) and (40) with unknown system parameters are globally and exponentially synchronized for all initial conditions by the adaptive control law (43) and the parameter update law (50), where $k_1, k_2, k_3, k_4, k_5$ are positive gain constants.

**Proof** We prove this result by applying Lyapunov stability theory.

We consider the quadratic Lyapunov function defined by (48), which is clearly a positive definite function on $\mathbb{R}^{10}$.

By substituting the parameter update law (50) into (49), we obtain the time-derivative of $V$ as

$$
\dot{V} = -k_1 e_1^2 - k_2 e_2^2 - k_3 e_3^2 - k_4 e_4^2 - k_5 e_5^2
$$

(51)

From (51), it is clear that $\dot{V}$ is a negative semi-definite function on $\mathbb{R}^{10}$.

Thus, we can conclude that the state vector $e(t)$ and the parameter estimation error are globally bounded, i.e.

$$
[e_1(t) \quad \cdots \quad e_5(t) \quad e_a(t) \quad e_b(t) \quad e_c(t) \quad e_p(t) \quad e_q(t)]^T \in L_{\infty}.
$$

We define $k = \min\{k_1, k_2, k_3, k_4, k_5\}$.

Then it follows from (51) that

$$
\dot{V} \leq -k \|e(t)\|^2
$$

(52)

Thus, we have

$$
k \|e(t)\|^2 \leq -\dot{V}
$$

(53)
Integrating the inequality (53) from 0 to $t$, we get
\[
k \int_0^t \|e(\tau)\|^2 d\tau \leq V(0) - V(t) \tag{54}
\]

From (54), it follows that $e \in L_2$.
Using (46), we can conclude that $\dot{e} \in L_\infty$.
Using Barbalat’s lemma, we conclude that $e(t) \to 0$ exponentially as $t \to \infty$ for all initial conditions $e(0) \in \mathbb{R}^5$.
This completes the proof. \hfill \Box

For the numerical simulations, the classical fourth-order Runge-Kutta method with step size $h = 10^{-8}$ is used to solve the systems (39), (40) and (50), when the adaptive control law (43) is applied.
The parameter values of the novel 5-D hyperchaotic systems are taken as in the hyperchaotic case, viz. $a = 10, b = 8/3, c = 28, p = 1.3$ and $q = 2.5$. We take the positive gain constants as $k_i = 5$ for $i = 1, \ldots, 5$.
Furthermore, as initial conditions of the master system (39), we take
\[
x_1(0) = 3.1, \quad x_2(0) = -5.8, \quad x_3(0) = 7.3, \quad x_4(0) = 9.1, \quad x_5(0) = -2.6
\]
As initial conditions of the slave system (40), we take
\[
y_1(0) = -8.4, \quad y_2(0) = 3.5, \quad y_3(0) = 4.2, \quad y_4(0) = -5.4, \quad y_5(0) = 10.3
\]
Also, as initial conditions of the parameter estimates, we take
\[
\hat{a}(0) = 3.1, \quad \hat{b}(0) = 12.4, \quad \hat{c}(0) = 4.7, \quad \hat{p}(0) = -5.8, \quad \hat{q}(0) = 3.2
\]
Figs. 8-12 describe the complete synchronization of the 5-D novel hyperchaotic Lorenz systems (39) and (40), while Fig. 13 describes the time-history of the synchronization errors $e_1, e_2, e_3, e_4, e_5$. 
HYPERCHAOS, ADAPTIVE CONTROL AND SYNCHRONIZATION OF A NOVEL 5-D HYPERCHAOTIC SYSTEM WITH THREE POSITIVE LYAPUNOV EXPONENTS AND ITS SPICE IMPLEMENTATION

Figure 8: Synchronization of the states $x_1$ and $y_1$ of the 5-D novel hyperchaotic Lorenz systems.

Figure 9: Synchronization of the states $x_2$ and $y_2$ of the 5-D novel hyperchaotic Lorenz systems.
Figure 10: Synchronization of the states $x_3$ and $y_3$ of the 5-D novel hyperchaotic Lorenz systems.

Figure 11: Synchronization of the states $x_4$ and $y_4$ of the 5-D novel hyperchaotic Lorenz systems.
Figure 12: Synchronization of the states $x_5$ and $y_5$ of the 5-D novel hyperchaotic Lorenz systems.

Figure 13: Time-history of the synchronization errors $e_1, e_2, e_3, e_4, e_5$. 
6. Circuit realization of novel 5-D hyperchaotic Lorenz system

In order to illustrate the correction and feasibility of novel hyperchaotic system (5), an electronic circuit modeling new system (5) is designed. Due to the fact that the electronic circuit is designed following an approach based in operational amplifiers [19, 29, 31], the state variables of system (5) are scaled down to obtain attractors in the dynamical range of operational amplifiers. As a result, the new hyperchaotic system (5) can be rewritten as

\[
\begin{align*}
\dot{X}_1 &= -ax_1 + ax_2 + 4x_4 + x_5 \\
\dot{X}_2 &= cX_1 - 20x_1x_3 - x_2 \\
\dot{X}_3 &= 20x_1x_2 - bx_3 \\
\dot{X}_4 &= -5x_1x_3 + px_4 \\
\dot{X}_5 &= qx_1
\end{align*}
\]

(55)
in which \(x_1 = \frac{x_1}{20}, x_2 = \frac{x_2}{20}, x_3 = \frac{x_3}{20}, x_4 = \frac{x_4}{80}\), and \(x_5 = \frac{x_5}{20}\). The schematic of the designed circuit is presented in Fig. 14.

By applying Kirchhoff’s laws to the electronic circuit in Fig. 14, its circuital equations are derived in the following form

\[
\begin{align*}
\frac{dv_{C1}}{dt} &= -\frac{1}{R_1C_1}v_{C1} + \frac{1}{R_2C_1}v_{C2} + \frac{1}{R_3C_1}v_{C3} + \frac{1}{R_4C_1}v_{C5} \\
\frac{dv_{C2}}{dt} &= \frac{1}{R_5C_2}v_{C1} - \frac{1}{10R_6C_2}v_{C1}v_{C3} - \frac{1}{R_7C_2}v_{C2} \\
\frac{dv_{C3}}{dt} &= \frac{1}{10R_8C_3}v_{C1}v_{C2} - \frac{1}{R_9C_3}v_{C3} \\
\frac{dv_{C4}}{dt} &= -\frac{1}{10R_{10}C_4}v_{C1}v_{C3} + \frac{1}{R_{11}C_4}v_{C4} \\
\frac{dv_{C5}}{dt} &= \frac{1}{R_{12}C_5}v_{C1}
\end{align*}
\]

(56)

where \(v_{C1}, v_{C2}, v_{C3}, v_{C4}, v_{C5}\) are the voltages across the capacitors \(C_1, C_2, C_3, C_4\) and \(C_5\), respectively. It is noting that there are five operational amplifiers (\(U_1, U_2, U_3, U_4\) and \(U_5\)), which are connected as integrators in Fig. 14. Hence the state variables \(X_1, X_2, X_3, X_4, X_5\) of system (55) are the voltages \(v_{C1}, v_{C2}, v_{C3}, v_{C4}, v_{C5}\), respectively.

The values of the electronic components in Fig. 14 are chosen to match known parameters of system (5):

\[
\begin{align*}
R_1 &= R_2 = 20k\Omega, \quad R_3 = 50k\Omega, \quad R_4 = R_7 = R = 200k\Omega, \quad R_5 = 7.14k\Omega, \quad R_6 = R_8 = 1k\Omega, \\
R_9 &= 75k\Omega, \quad R_{10} = 4k\Omega, \quad R_{11} = 153.85k\Omega, \quad R_{12} = 80k\Omega, \quad C_1 = C_2 = C_3 = C_4 = C_5 = 1nF
\end{align*}
\]
Figure 14: Circuital schematic for realizing novel 5-D hyperchaotic Lorenz system.

The power supplies of all active devices are $\pm 15V_{DC}$.

The proposed circuit is implemented by using the electronic simulation package Cadence OrCAD. Figs. 15-18 show the obtained phase portraits in $(v_{C_1}, v_{C_2})$-plane, $(v_{C_1}, v_{C_3})$-plane, $(v_{C_1}, v_{C_4})$-plane, and $(v_{C_1}, v_{C_5})$-plane, respectively.
Figure 15: Phase portrait of the designed electronic circuit in \((v_{C1}, v_{C2})\)-plane obtained from Cadence OrCAD.

Figure 16: Phase portrait of the designed electronic circuit in \((v_{C1}, v_{C3})\)-plane obtained from Cadence OrCAD.
Figure 17: Phase portrait of the designed electronic circuit in \((v_{C1}, v_{C4})\)-plane obtained from Cadence OrCAD.

Figure 18: Phase portrait of the designed electronic circuit in \((v_{C1}, v_{C5})\)-plane obtained from Cadence OrCAD.
7. Conclusion

In the literature, 3-D chaotic systems and 4-D hyperchaotic systems as well as their control and synchronization problems were mainly investigated. However, a 5D system which can generate hyperchaos, especially with three positive Lyapunov exponents, is often rarely reported. In this paper, a twelve-term novel 5-D hyperchaotic Lorenz system with three quadratic nonlinearities has been proposed and its dynamics has been discovered. It is shown that the 5-D hyperchaotic system exhibits three positive Lyapunov exponents and possesses complex dynamical behaviour. In addition, global control and global hyperchaos synchronization of such identical novel 5-D hyperchaotic Lorenz systems with unknown system parameters can be achieved by using an adaptive controller. Moreover, SPICE results obtained from the electronic circuit realization of this novel 5-D system show the feasibility of the theoretical introduced model. It is well-known that hyperchaos is better than conventional chaos in a variety of applications. For example, hyperchaos increases the security of chaotic-based communication systems significantly, and in this context, the proposed 5-D hyperchaotic system will be very useful for secure communication systems and other applications as well.

References


