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# LOG-LIKE FUNCTIONS AND UNIFORM DISTRIBUTION MODULO ONE

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ABSTRACT. For a function $f$ satisfying $f(x) = o((\log x)^K)$, $K > 0$, and a sequence of numbers $(q_n)_n$, we prove by assuming several conditions on $f$ that the sequence $(\alpha f(q_n))_{n\geq n_0}$ is uniformly distributed modulo one for any nonzero real number $\alpha$. This generalises some former results due to Too, Goto and Kano where instead of $(q_n)_n$ the sequence of primes was considered.

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## 1. Introduction and results

Let $p_n$ be the $n$th prime number in ascending order and $\alpha$ a nonzero real number. Then Goto and Kano [1], [2], as well as Too [7], proved by assuming several conditions on the function $f$ that the sequence $(\alpha f(p_n))_n$ is uniformly distributed modulo one. Recall that a sequence $(x_n)_{n\in\mathbb{N}}$ of real numbers is said to be *uniformly distributed modulo one* if for every pair $\alpha, \beta$ of real numbers with $0 \leq \alpha < \beta \leq 1$ the proportion of the fractional parts of the $x_n$ in the interval $[\alpha, \beta)$ tends to its length in the following sense:

$$\lim_{N\to\infty} \frac{\#\{1 \leq n \leq N : \{x_n\} \in [\alpha, \beta)\}}{N} = \beta - \alpha.$$

In fact Goto, Kano and Too proved their results by determining a bound for the discrepancy of the considered sequence.

**Definition 1.** Let $x_1, ..., x_N$ be a finite sequence of real numbers. The number

$$D_N := D_N(x_1, \ldots, x_N) := \sup_{0\leq\alpha<\beta\leq1} \left| \frac{\#\{1 \leq n \leq N : \{x_n\} \in [\alpha, \beta)\}}{N} - (\beta - \alpha) \right|$$

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is called the **discrepancy** of the given sequence. If $\omega = (x_n)$ is an infinite sequence (or a finite sequence containing at least $N$ terms), $D_N(\omega)$ is meant to be the discrepancy of the first $N$ terms of $\omega$.

It is well known that a sequence $\omega = (x_n)_{n\in\mathbb{N}}$ is uniformly distributed modulo one if and only if $\lim_{N\to\infty} D_N(\omega) = 0$. Instead of the sequence of primes, which was investigated in [1], [2] and [7], we consider a sequence of real numbers $(q_n)_{n\geq 1}$ satisfying $1 < q_1 < q_2 < \cdots$ with $q_n \to \infty$ as $n \to \infty$. Further, we assume that the sequence $(q_n)_{n\geq 1}$ satisfies

$$Q(x) - c\int_2^x \frac{dt}{\log t} \ll \frac{x}{(\log x)^k} \tag{1}$$

for every positive $k > 1$, where $Q(x) := \sum_{q_n\leq x} 1$ and $c > 0$ is some constant. Note that condition (1) holds for the sequence of primes (with $c = 1$), as well as for primes in arithmetic progressions (with $c = \varphi(q)^{-1}$, where $\varphi$ is Euler's function and $q$ is the modulus).
However, the sequence $(q_n)_{n\geq 1}$ we consider satisfies (1) and for such a sequence we prove the following theorems:

**Theorem 2.** *Let $a > 0$, $n_0 := \min\{n \in \mathbb{N} : q_n > a\}$ and let the function $f : [a,\infty) \to (0,\infty)$ satisfy the conditions*

(a.) *$f$ is twice differentiable with $f' > 0$,*
(b.) *$x^2 f''(x) \to -\infty$ as $x \to \infty$,*
(c.) *$(\log x)^2 f''(x)$ and $x(\log x)^2 f''(x)$ are nonincreasing for sufficiently large $x$,*
(d.) *$f(x) = o\left((\log x)^K\right)$ for some $K > 0$ as $x \to \infty$.*

*Then, for any nonzero real constant $\alpha$, the sequence $\left(\alpha f(q_n)\right)_{n\geq n_0}$ is uniformly distributed modulo one and*

$$D_N \ll \sqrt{\frac{f(q_N)}{(\log q_N)^K}} + \sqrt{\frac{1}{-q_N^2 f''(q_N)}} + \frac{1}{(\log q_N)(-q_N^2 f''(q_N))} \tag{2}$$

*as $N \to \infty$.*

**Theorem 3.** *Let $a > 0$, $n_0 := \min\{n \in \mathbb{N} : q_n > a\}$ and let the function $f : [a,\infty) \to (0,\infty)$ satisfy the conditions:*

(a.) *$f$ is twice differentiable with $f' > 0$,*
(b.) *$x^2 f''(x) \to \infty$ as $x \to \infty$,*
(c.) *$(\log x)^2 f''(x)$ is nonincreasing for sufficiently large $x$,*
(d.) *$f(x) = o\left((\log x)^K\right)$ for some $K > 0$ as $x \to \infty$.*

*Then, for any nonzero real constant $\alpha$, the sequence $\big(\alpha f(q_n)\big)_{n\ge n_0}$ is uniformly distributed modulo one and*

$$D_N \ll \sqrt{\frac{f(q_N)}{(\log q_N)^K}} + \sqrt{\frac{1}{q_N^2 f''(q_N)}} \tag{3}$$

*as $N \to \infty$.*

**Theorem 4.** *Let $a > 0$, $n_0 := \min\{n \in \mathbb{N} : q_n > a\}$ and let the function $f : [a,\infty) \to (0,\infty)$ satisfy the conditions:*

(a.) *$f$ is continuously differentiable,*

(b.) *$xf'(x) \to \infty$ as $x \to \infty$,*

(c.) *$(\log x)\, f'(x)$ is monotone for sufficiently large $x$,*

(d.) *$f(x) = o\big((\log x)^K\big)$ for some $K > 0$ as $x \to \infty$.*

*Then, for any nonzero real constant $\alpha$, the sequence $\big(\alpha f(q_n)\big)_{n\ge n_0}$ is uniformly distributed modulo one and*

$$D_N \ll \sqrt{\frac{f(q_N)}{(\log q_N)^K}} + \max\left\{\frac{1}{N}, \frac{1}{q_N f'(q_N)}\right\} \tag{4}$$

*as $N \to \infty$.*

In view of [1, Theorem 1] it should be remarked that in Theorem 4 a replacement of conditions (a.) and (b.) by:

(a'.) $f$ is continuously differentiable and $f(x) \to \infty$ as $x \to \infty$,

(b'.) $x\,|f'(x)| \to \infty$ as $x \to \infty$,

would lead to the same discrepancy estimate, where only $f'(q_N)$ has to be replaced by $|f'(q_N)|$. If we compare Theorem 2 with [7, Theorem 3], one notices that the "nondecreasing" condition is replaced by "nonincreasing". It was already remarked in [6] that this replacement is necessary.

Applying Theorem 2 to the function $f(x) = (\log x)^K$ with an arbitrary $K > 1$, we obtain that the sequence $\big((\log q_n)^K\big)_{n\ge 1}$ is uniformly distributed modulo one. This generalises the example in [7] on the uniform distribution modulo one of the sequence $\big((\log p_n)^K\big)_{n\ge 1}$.

Note that it was proved by Wintner [10] that the sequence $(\log p_n)_{n\ge 1}$ is not uniformly distributed modulo one. A shorter proof can also be found in [8, Exercise 5.19]. Similarly, the sequence $(\log q_n)_{n\ge 1}$ is not uniformly distributed modulo one if (1) is replaced by $Q(x) \sim x(\log x)^{-1}$ for $x \to \infty$. The proof is analogue to the one in [8]. An example of a sequence satisfying $Q(x) \sim x(\log x)^{-1}$ for $x \to \infty$ is a sequence where each $q_n$ fulfills $p_n \le q_n \le p_{n+1}$ (see [3], [5]).

## 2. Proofs

Except for the already pointed out change of the conditions "nondecreasing" and "nonincreasing" in Theorem 2, the theorems of this paper are generalisations of the theorems in [7]. Therefore it might not be surprising that the proofs are similar. However, for the sake of completeness we state the whole proofs and do not only point out changes in the reasoning.

In the proofs we will make use of a theorem due to Erdös and Turán to estimate the discrepancy, which was proved by them in 1940:

**Theorem 5.** *For any finite sequence $x_1, \dots, x_N$ of real numbers and any positive integer $m$, we have*

$$D_N \le C \cdot \left( \frac{1}{m} + \sum_{h=1}^{m} \frac{1}{h} \left| \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i h x_n} \right| \right),$$

*where $C$ is an absolute constant.*

Proof of Theorem 5. can be found in [4]. In addition, we need the following estimates

**Lemma 6.** *Let $F(x)$ and $G(x)$ be real functions, $\frac{G(x)}{F(x)}$ monotone and $\frac{F'(x)}{G(x)} \ge m > 0$, or $\frac{F'(x)}{G(x)} \le -m < 0$. Then*

$$\left| \int_a^b G(x) e^{iF(x)} dx \right| \le \frac{4}{m}.$$

**Lemma 7.** *Let $G(x)$ be a positive decreasing function.*

(a.) *If $F''(x) < 0$, $F'(x) \ge 0$ and $\frac{G'(x)}{F''(x)}$ is monotone, then*

$$\left| \int_a^b G(x) e^{2\pi i F(x)} dx \right| \le 4 \max_{a \le x \le b} \left\{ \frac{G(x)}{|F''(x)|^{\frac{1}{2}}} \right\} + \max_{a \le x \le b} \left\{ \left| \frac{G'(x)}{F''(x)} \right| \right\}.$$

(b.) *If $F''(x) > 0$ and $F'(x) \ge 0$, then*

$$\left| \int_a^b G(x) e^{2\pi i F(x)} dx \right| \le 4 \max_{a \le x \le b} \left\{ \frac{G(x)}{F''(x)^{\frac{1}{2}}} \right\}.$$

The first Lemma can be found in [9, Lemma 4.3] and the second one in [11, Lemma 10.2, Lemma 10.3, p.225]. □

P r o o f o f T h e o r e m 2. Note first that it is enough to prove (2), (3) and (4) for $\alpha > 0$. If we replace $f$ by $\frac{1}{\alpha} f$ we see that it is enough to prove these statements for $\alpha = 1$. In view of condition (b.) we can assume that $f''(x) < 0$ for sufficiently large $x$. Further, we may assume that for $x \geq a$ we have $f''(x) < 0$ and that both $(\log x)^2 f''(x)$ and $x(\log x)^2 f''(x)$ are nonincreasing for $x \geq a$. To prove that the sequence $(f(q_n))_{n\geq n_0}$ is uniformly distributed modulo one, it is enough to prove estimation (2). In view of conditions (b.) and (d.), the term on the right side in (2) surely tends to zero as $N$ tends to infinity. To obtain (2), our aim is to apply Theorem 5 in the form

$$D_N \ll \frac{1}{m} + \sum_{h=n_0}^{m} \frac{1}{h} \left| \frac{1}{N} \sum_{n=n_0}^{N} e^{2\pi i h f(q_n)} \right|, \tag{5}$$

where $m$ is an arbitrary positive integer to be specified later. The essential point is to estimate the exponential sum in (5). Therefore let $q_0 := \frac{q_{n_0}+a}{2}$ and $\chi(n)$ be the characteristic function of the sequence $(q_n)_{n\geq 1}$, i.e., $\chi(n) = 1$ if $n$ is a member of the sequence $(q_n)_{n\geq 1}$ and zero otherwise. Then $\sum_{m\leq x} \chi(n) = Q(x)$ and integration by parts yields

$$\begin{aligned}
E(n_0, N; q_n) := \sum_{n=n_0}^{N} e^{2\pi i h f(q_n)} &= \sum_{q_0 < m \leq q_N} e^{2\pi i h f(m)} \chi(m) \\
&= \int_{q_0}^{q_N} e^{2\pi i h f(x)} \,\mathrm{d}\, Q(x) \\
&= Q(q_N) e^{2\pi i h f(q_N)} - Q(q_0) e^{2\pi i h f(q_0)} \\
&\quad - \int_{q_0}^{q_N} \big(L^*(x) + R^*(x)\big) \,\mathrm{d}\, e^{2\pi i h f(x)},
\end{aligned}$$

where $R^*(x) := Q(x) - L^*(x)$, $L^*(x) := c \int_{q_0}^{x} \frac{dt}{\log t}$ for $c > 0$ constant, $x \geq q_0$ and $\int_a^b := \int_{(a,b]}$. Again by integration by parts,

$$\begin{aligned}
\int_{q_0}^{q_N} L^*(x) \,\mathrm{d}\, e^{2\pi i h f(x)} = L^*(q_N) e^{2\pi i h f(q_N)} &- L^*(q_0) e^{2\pi i h f(q_0)} \\
&- c \int_{q_0}^{q_N} (\log x)^{-1} e^{2\pi i h f(x)} dx
\end{aligned}$$

and

$$\int_{q_0}^{q_N} R^*(x) \,\mathrm{d}\, e^{2\pi i h f(x)} = 2\pi i h \int_{q_0}^{q_N} R^*(x) f'(x) e^{2\pi i h f(x)} dx.$$

Therefore,

$$\begin{aligned} E(n_0, N; q_n) &= \Big( R^*(q_N) e^{2\pi i h f(q_N)} - R^*(q_0) e^{2\pi i h f(q_0)} \Big) \\ &\quad + c \int_{q_0}^{q_N} (\log x)^{-1} e^{2\pi i h f(x)} dx \\ &\quad - 2\pi i h \int_{q_0}^{q_N} R^*(x) f'(x) e^{2\pi i h f(x)} dx \\ &=: I_1 + I_2 + I_3, \end{aligned}$$

where we used $R^*(x) = Q(x) - L^*(x)$. Now we estimate each $I_i$ $(i = 1, 2, 3)$ individually: In view of our assumption (1), we get

$$R^*(x) = Q(x) - c \int_{q_0}^{x} \frac{dt}{\log t} \ll \frac{x}{(\log x)^k} \tag{6}$$

for every $k > 1$, implying

$$I_1 \ll \frac{q_N}{(\log q_N)^{1+K}} \tag{7}$$

for $K > 0$ as $N \to \infty$. By estimation (6) and condition $(a)$,

$$I_3 \ll h \frac{q_N f(q_N)}{(\log q_N)^{1+K}}, \tag{8}$$

so it remains to estimate $I_2$. First, let us remark that (8) for $N \to \infty$ is also an estimation for (7), since $f > 0$ and $h$ is a positive integer. To estimate $I_2$ we apply Lemma 7 (a.) to get

$$\begin{aligned} I_2 &\ll \max_{q_0 \le x \le q_n} \left\{ \frac{4}{|h(\log x)^2 f''(x)|^{\frac{1}{2}}} + \left| \frac{1}{hx(\log x)^2 f''(x)} \right| \right\} \\ &\ll \frac{1}{(\log q_N)\,(-hf''(q_N))^{\frac{1}{2}}} + \frac{1}{q_N (\log q_N)^2\,(-hf''(q_N))} \end{aligned}$$

as $N \to \infty$. Note that for the last estimation condition (c.) is needed in terms of "nonincreasing". Putting our estimation for $I_2$ and (8) in (5) yields

$$D_N \ll \frac{1}{m} + \frac{1}{N(\log q_N)\,(-f''(q_N))^{\frac{1}{2}}} + \frac{1}{N q_N (\log q_N)^2\,(-f''(q_N))} + \frac{m q_N f(q_N)}{N(\log q_N)^{1+K}} \tag{9}$$

for $N \to \infty$. After comparing the first and the last term in (9), we choose

$$m = \left[ \left( N \cdot \frac{(\log q_N)^{1+K}}{q_N f(q_N)} \right)^{\frac{1}{2}} \right].$$

Using this together with $Q(q_N) = N$ and $Q(x) = \frac{cx}{\log x} + O\left(\frac{x}{(\log x)^2}\right)$, we end up with our desired estimation (2). $\square$

P r o o f o f T h e o r e m 3. In respect of condition (b.) we can assume that $f''(x) > 0$ for sufficiently large $x$. Further, we may assume that for $x \geq a$ we have $f''(x) > 0$ and that $(\log x)^2 f''(x)$ is nonincreasing for $x \geq a$. Like in the proof of the previous theorem, we will show that the discrepancy $D_N$ of the sequence $(f(q_n))_{n=n_0}^N$ tends to zero as $N \to \infty$. Since the estimations (7) and (8) for $I_1$ and $I_3$ still hold in the considered setting, it remains to estimate $I_2$. By applying Lemma 7 (b.) we get

$$I_2 \ll \max_{q_0 \leq x \leq q_N} \left\{ \frac{1}{(h(\log x)^2 f''(x))^{\frac{1}{2}}} \right\} = \frac{1}{(h(\log q_N)^2 f''(q_N))^{\frac{1}{2}}}.$$

Combining this with (7) and (8) we obtain in (5)

$$D_N \ll \frac{1}{m} + \frac{m q_N f(q_N)}{N(\log q_N)^{1+K}} + \frac{1}{(q_N^2 f''(q_N))^{\frac{1}{2}}},$$

where we used $Q(q_N) = N$ and $Q(x) = \frac{cx}{\log x} + O\left(\frac{x}{(\log x)^2}\right)$. Choosing

$$m = \left[ \left( N \cdot \frac{(\log q_N)^{1+K}}{q_N f(q_N)} \right)^{\frac{1}{2}} \right],$$

we get

$$D_N \ll \left( \frac{f(q_N)}{(\log q_N)^K} \right)^{\frac{1}{2}} + \frac{1}{(q_N^2 f''(q_N))},$$

which tends to zero as $N \to \infty$ in view of (b.) and (d.). $\square$

P r o o f o f T h e o r e m 4. With respect to condition (b.) we can assume that $f'(x) > 0$ for sufficiently large $x$. Further, we may assume that for $x \geq a$ we have $f'(x) > 0$ and that $(\log x) f'(x)$ is monotone for $x \geq a$. As in the previous proofs, we will show that the discrepancy $D_N$ of the sequence $(f(q_n))_{n=n_0}^N$ tends to zero as $N \to \infty$. Since the estimations (7) and (8) for $I_1$ and $I_3$ still hold in the considered setting, it remains to estimate $I_2$. Applying Lemma 6 yields

$$I_2 \ll \frac{1}{h} \cdot \max\left\{ 1, \frac{1}{[\log q_N f'(q_N)]} \right\}.$$

Thus, together with (7) and (8) in (5), we get

$$\begin{aligned} D_N &\ll \frac{1}{m} + \max\left\{\frac{1}{N}, \frac{1}{N\left[\log q_N f'(q_N)\right]}\right\} + \frac{m q_N f(q_N)}{N\left(\log q_N\right)^{1+K}} \\ &\ll \frac{1}{m} + \max\left\{\frac{1}{N}, \frac{1}{\left[q_N f'(q_N)\right]}\right\} + \frac{m q_N f(q_N)}{N\left(\log q_N\right)^{1+K}}, \end{aligned}$$

where we used in the second argument in the maximum that $Q(q_N) = N$ and $Q(x) = \frac{cx}{\log x} + O\left(\frac{x}{(\log x)^2}\right)$. After comparing the first and the last term in the discrepancy estimate, we choose

$$m = \left[\left(N \cdot \frac{(\log q_N)^{1+K}}{q_N f(q_N)}\right)^{\frac{1}{2}}\right],$$

and get

$$D_N \ll \left(\frac{f(q_N)}{(\log q_N)^K}\right)^{\frac{1}{2}} + \max\left\{\frac{1}{N}, \frac{1}{\left[q_N f'(q_N)\right]}\right\}.$$

By condition (b.) and (d.) this tends to zero as $N$ tends to infinity. □

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