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ON THE CLASSIFICATION OF LS-SEQUENCES

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ABSTRACT. This paper addresses the question whether the LS-sequences constructed in [Car12] yield indeed a new family of low-discrepancy sequences. While it is well known that the case $S = 0$ corresponds to van der Corput sequences, we prove here that the case $S = 1$ can be traced back to symmetrized Kronecker sequences and moreover that for $S \geq 2$ none of these two types occurs anymore. In addition, our approach allows for an improved discrepancy bound for $S = 1$ and L arbitrary.

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1. Introduction

There are essentially three classical families of low-discrepancy sequences, namely Kronecker sequences, digital sequences and Halton sequences (compare [Lar14], see also [Nie92]). In [Car12], Carbone constructed a class of one-dimen-sional low-discrepancy sequences, called LS-sequences with $L \in \mathbb{N}$ and $S \in \mathbb{N}_0$. The case $S = 0$ corresponds to the classical one dimensional Halton sequences, called van der Corput sequences. However, the question whether LS--sequences indeed yield a new family of low-discrepancy sequences for $S \geq 1$ or if it is just a different way to write down already known low-discrepancy sequences has not been answered yet. In this paper, we address this question and thereby derive improved discrepancy bounds for the case $S = 1$.

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Discrepancy. Let $S = (z_n)_{n\geq 0}$ be a sequence in $[0,1)^d$. Then the **discrepancy** of the first N points of the sequence is defined by

$$D_N(S) := \sup_{B\subset[0,1)^d} \left| \frac{A_N(B)}{N} - \lambda_d(B) \right|,$$

where the supremum is taken over all axis-parallel subintervals $B \subset [0,1)^d$ and $A_N(B) := \#\{n \mid 0 \leq n < N, z_n \in B\}$ and λ_d denotes the d-dimensional Lebesgue-measure. In the following we restrict to the case $d = 1$. If $D_N(S)$ satisfies

$$D_N(S) = O(N^{-1} \log N),$$

then S is called a **low-discrepancy sequence**. In dimension one this is indeed the best possible rate as was proved by Schmidt in [Sch72], that there exists a constant c with

$$D_N(S) \geq cN^{-1} \log N.$$

The precise value of the constant c is still unknown (see, e.g., [Lar14]). For a discussion of the situation in higher dimensions see, e.g., [Nie92, Chapter 3].

A theorem of Weyl and Koksma's inequality imply that a sequence of points $(z_n)_{n\geq 0}$ is uniformly distributed if and only if

$$\lim_{N\to\infty} D_N(z_n) = 0.$$

Thus, the only candidates for low-discrepancy sequences are uniformly distributed sequences. A specific way to construct uniformly distributed sequences goes back to the work of Kakutani [Kak76] and was later on generalized in [Vol11] in the following sense.

Definition 1.1. Let ρ denote a non-trivial partition of $[0,1)$. Then the **ρ-refinement** of a partition π of $[0,1)$, denoted by $\rho\pi$, is defined by subdividing all intervals of maximal length positively homothetically to ρ.

Successive application of a ρ-refinement results in a sequence which is denoted by $\{\rho^n\pi\}_{n\in\mathbb{N}}$. The special case of **Kakutani's α-refinement** is obtained by successive ρ-refinements, where $\rho = \{[0,\alpha),[\alpha,1)\}$. If π is the trivial partition $\pi = \{[0,1)\}$, then we obtain **Kakutani's-α-sequence**. In many articles Kakutani's α-sequence serves as a standard example and the general results derived therein may be applied to this case (see, e.g., [CV07], [DI12], [IZ17], [Vol11]). Another specific class of examples of ρ-refinement was introduced in [Car12].

Definition 1.2. Let $L \in \mathbb{N}, S \in \mathbb{N}_0$ and β be the solution of $L\beta + S\beta^2 = 1$. An **LS-sequence of partitions** $\left\{\rho^n_{L,S}\pi\right\}_{n\in\mathbb{N}}$ is the successive ρ-refinement of the trivial partition $\pi = \{[0,1)\}$ where $\rho_{L,S}$ consists of $L+S$ intervals such that the first L intervals have length β and the successive S intervals have length β^2.

The partition $\{\rho^n_{L,S}\pi\}$ consists of intervals only of length β^n and β^{n+1}. Its total number of intervals is denoted by t_n, the number of intervals of length β^n by l_n and the number of intervals of length β^{n+1} by s_n. In [Car12], Carbone derived the recurrence relations:

$$\begin{aligned} t_n &= Lt_{n-1} + St_{n-2}, \\ l_n &= Ll_{n-1} + Sl_{n-2}, \\ s_n &= Ls_{n-1} + Ss_{n-2} \end{aligned}$$

for $n \geq 2$ with initial conditions:

$$t_0 = 1, \quad t_1 = L + S, \quad l_0 = 1, \quad l_1 = L, \quad s_0 = 0 \quad \text{and} \quad s_1 = S.$$

Based on these relations, Carbone defined a possible ordering of the endpoints of the partition yielding the LS-sequence of points. One of the observations of this paper is that this ordering indeed yields a simple and easy-to-implement algorithm but also has a certain degree of arbitrariness.

Definition 1.3. Given an LS-sequence of partitions $\{\rho^n_{L,S}\pi\}_{n\in\mathbb{N}}$, the corresponding LS**-sequence of points** $(\xi^n)_{n\in\mathbb{N}}$ is defined as follows:
let $\Lambda^1_{L,S}$ be the first t_1 left endpoints of the partiton $\rho_{L,S}\pi$ ordered by magnitude. Given $\Lambda^n_{L,S} = \left\{\xi^{(n)}_1, \ldots, \xi^{(n)}_{t_n}\right\}$ an ordering of $\Lambda^{n+1}_{L,S}$ is then inductively defined as

$$\begin{aligned} \Lambda^{n+1}_{L,S} = \Big\{&\xi^{(n)}_1, \ldots, \xi^{(n)}_{t_n}, \\ &\psi^{(n+1)}_{1,0}\big(\xi^{(n)}_1\big), \ldots, \psi^{(n+1)}_{1,0}\big(\xi^{(n)}_{l_n}\big), \ldots, \psi^{(n+1)}_{L,0}\big(\xi^{(n)}_1\big), \ldots, \psi^{(n+1)}_{L,0}\big(\xi^{(n)}_{l_n}\big), \\ &\psi^{(n+1)}_{L,1}\big(\xi^{(n)}_1\big), \ldots, \psi^{(n+1)}_{L,1}\big(\xi^{(n)}_{l_n}\big), \ldots, \psi^{(n+1)}_{L,S-1}\big(\xi^{(n)}_1\big), \ldots, \psi^{(n+1)}_{L,S-1}\big(\xi^{(n)}_{l_n}\big)\Big\}, \end{aligned}$$

where

$$\psi^{(n)}_{i,j}(x) = x + i\beta^n + j\beta^{n+1}, \qquad x \in \mathbb{R}.$$

As the definition of LS-sequences might not be completely intuitive at first sight, we illustrate it by an explicit example.

Example 1.4. For $L = S = 1$ the LS-sequence coincides with the so-called **Kakutani-Fibonacci sequence** (see [CIV14]). We have

$$\begin{aligned} \Lambda^1_{1,1} &= \{0, \beta\}, \\ \Lambda^2_{1,1} &= \{0, \beta, \beta^2\}, \\ \Lambda^3_{1,1} &= \{0, \beta, \beta^2, \beta^3, \beta + \beta^3\}, \\ \Lambda^4_{1,1} &= \{0, \beta, \beta^2, \beta^3, \beta + \beta^3, \beta^4, \beta + \beta^4, \beta^2 + \beta^4\} \end{aligned}$$

and so on.

Theorem 1.5 (Carbone,[Car12]). *If $L \geq S$, then the corresponding LS-sequence has low-discrepancy.*

Carbone's proof is based on counting arguments but does not give explicit discrepancy bounds. These have been derived later by Iacò and Ziegler in [IZ17] using so-called generalized LS-sequences. A more general result implicating also the low-discrepancy of LS-sequences can be found in [AH13].

Theorem 1.6 (Iacò, Ziegler, [IZ17], Theorem 1, Section 3). *If $(\xi_n)_{n\in\mathbb{N}}$ is an LS-sequence with $L \geq S$, then*

$$D_N(\xi_n) \leq \frac{B\log(N)}{N|\log(\beta)|} + \frac{B+2}{N},$$

where

$$B = (2L+S-2)\Big(\tfrac{R}{1-S\beta}+1\Big), \quad \textit{with} \quad R = \max\{|\tau_1|, |\tau_1+(L+S-2)\lambda_1|\},$$

$$\tau_1 = \tfrac{-L-2S+\sqrt{L^2+4S}}{2\sqrt{L^2+4S}} \quad \textit{and} \quad \lambda_1 = \tfrac{-L+\sqrt{L^2+4S}}{2\sqrt{L^2+4S}}.$$

It has been pointed out that for parameters $S = 0$ and $L = b$, the corresponding LS-sequence conincides with the classical van der Corput sequence, see, e.g., [AHZ14].[1] However, for higher values of S it has been not been proved if LS-sequences indeed yield a new family of examples of low-discrepancy sequences or are just a new formulation of some of the well-known ones. We close this gap to a certain extent by showing the following main result:

Theorem 1.7. *For $S = 1$, the LS-sequences is a reordering of the symmetrized Kronecker sequences $(\{n\beta\})_{n\in\mathbb{Z}}$. For $S \geq 2$ the LS-construction neither yields a (re-)ordering of a van der Corput sequence nor of a (symmetrized) Kronecker sequence.*

Let us make the notion of symmetrized Kronecker sequences more precise: given $z \in \mathbb{R}$, let $\{z\} := z - \lfloor z \rfloor$ denote the fractional part of z. A (classical) **Kronecker sequence** is a sequence of the form $(z_n)_{n\geq 0} = (\{nz\})_{n\geq 0}$. If $z \notin \mathbb{Q}$ and z has bounded partial quotients in its continued fraction expansion (see Section 2), then (z_n) has low-discrepancy ([Nie92], Theorem 3.3). By a **symmtrized** Kronecker sequence we simply mean a sequence indexed over $\mathbb{Z}$ of the form $(\{nz\})_{n\in\mathbb{Z}}$ with ordering

$$0, \{z\}, \{-z\}, \{2z\}, \{-2z\}, \ldots$$

[1] If the reader is not familiar with the Definition of van der Coruput sequences, he may consult [Nie92], Section 3.1.

Note that it is still open, whether for $S \geq 2$ an LS-sequence is a reordering of some other well-known low-discprancy sequence such as a digital-sequence or if the LS-construction really yields a new class of examples.

Our approach does not only give a significantly shorter proof of low-discrepancy of LS-sequences for $L = 1$ but also improves the known discrepancy bounds by Iacó and Ziegler in this case.

Corollary 1.8. *For $S = 1$ the discrepancy of the LS-sequence $(\xi_n)_{n\in\mathbb{N}}$ is bounded by*

$$D_N(\xi_n) \leq \frac{3}{N} + \left(\frac{1}{\log(\alpha)} + \frac{L}{\log(L+1)}\right)\frac{\log(N)}{N}, \quad \textit{where } \alpha = (1+\sqrt{5})/2.$$

Corollary 1.8 indeed improves the discrepancy bounds for LS-sequences given in Theorem 1.6 in the specific case $S = 1$. Both results yield inequalities of the type

$$D_N(\xi_n) \leq \frac{\gamma}{N} + \frac{\delta\log(N)}{N}$$

For instance, if $L = S = 1$ then Corollary 1.8 implies $\gamma = 3$ and $\delta = 2.776$ while according to Theorem 1.6 the discrepancy can be bounded by $\gamma = 3.447$ and $\delta = 3.01$. The difference between the two results gets the more prominent the larger L is: If $L = 10$ and $S = 1$ we get $\gamma = 3$ and $\delta = 5.51$ while Theorem 1.6 only implies $\gamma = 22.87$ and $\delta = 9.03$.[2]

2. Proof of the main results

Continued fractions. Recall that every irrational number z has a uniquely determined infinite continued fraction expansion

$$z = a_0 + 1/\big(a_1 + 1/(a_2 + \ldots)\big) =: [a_0; a_1; a_2; \ldots],$$

where the a_i are integers with $a_0 = \lfloor z \rfloor$ and $a_i \geq 1$ for all $i \geq 1$. The sequence of **convergents** $(r_i)_{i\in\mathbb{N}}$ of z is defined by

$$r_i = [a_0; a_1; \ldots; a_i].$$

The convergents $r_i = p_i/q_i$ with $\gcd(p_i, q_i) = 1$ can also be calculated directly by the recurrence relation:

$$\begin{aligned} p_{-1} &= 0, \quad & p_0 &= 1, \quad & p_i &= a_i p_{i-1} + p_{i-2}, \quad & i &\geq 0; \\ q_{-1} &= 1, \quad & q_0 &= 0, \quad & q_i &= a_i q_{i-1} + q_{i-2}, \quad & i &\geq 0. \end{aligned}$$

[2] We obtain different numerical values than in [IZ17]. We checked our result on different computer algebra systems.

Remark 2.1. If $S = 1$, then $\beta^2 + L\beta - 1 = 0$ or equivalently,

$$\frac{1}{\beta} = L + \beta \quad \text{holds.}$$

Thus it follows that $a_i = L$ in the continued fraction expansion of β for all $i = 1, 2, \dots$

From now on the continued fraction expansion of β is studied and it is always tacitly assumed, that the q_i's are the denominators of the convergents of β. Although the proof of the following lemma is rather obvious we write it down here explictly because our proof of the main theorem is based on this arithmetic observation.

Lemma 2.2. *Let $n \in \mathbb{N}_0$. If $S = 1$, then we have*

(i) $\beta^{2n+1} + q_{2n} = q_{2n+1}\beta$.

(ii) $\beta^{2n} - q_{2n-1} = -q_{2n}\beta$.

Proof. We prove both claims by induction.

(i) The identity is trivial for $n = 0$. So we come to the induction step

$$\begin{aligned} \beta^{2n+1} + q_{2n} &= \beta^2\beta^{2n-1} + q_{2n}\left(\beta^2 + L\beta\right) \\ &= \beta^2\left(\beta^{2n-1} + q_{2n}\right) + Lq_{2n}\beta \\ &= \beta^2\left(q_{2n-1}\beta - q_{2n-2} + q_{2n}\right) + Lq_{2n}\beta \\ &= \beta^2\left(q_{2n-1}\beta + Lq_{2n-1}\right) + Lq_{2n}\beta \\ &= q_{2n-1}\beta(\beta^2 + L\beta) + Lq_{2n}\beta \\ &= q_{2n+1}\beta. \end{aligned}$$

(ii) The proof works analogously as in (i). We have $\beta^2 + 1 = -L\beta$ and

$$\begin{aligned} \beta^{2n} - q_{2n-1} &= \beta^2\beta^{2(n-1)} - q_{2n-1}\left(\beta^2 + L\beta\right) \\ &= \beta^2\left(\beta^{2(n-1)} - q_{2n-1}\right) - Lq_{2n-1}\beta \\ &= \beta^2\left(-q_{2n-2}\beta + q_{2n-3} - q_{2n-1}\right) - Lq_{2n-1}\beta \\ &= \beta^2\left(-q_{2n-2}\beta - Lq_{2n-2}\right) - Lq_{2n-1}\beta \\ &= -q_{2n-2}\beta\left(\beta^2 + L\beta\right) - Lq_{2n-1}\beta \\ &= -q_{2n}\beta. \end{aligned}$$

□

Example 2.3. Consider the Kakutani-Fibonacci sequence from Example 1.4. If we denote by $(f_n)_{n\geq 0}$ the Fibonacci sequence, i.e., the sequence inductively defined by

$$f_0 = 0, f_1 = 1 \quad \text{and} \quad f_n = f_{n-1} + f_{n-2} \quad \text{for } n \geq 2,$$

we have that $q_i = f_i$ for all $i = 1, 2, \ldots$

If $S = 1$, then we can furthermore deduce from Definition 1.3 that $t_{n+1} = t_n + Ll_n$ and that $q_{n-1} = l_n$. Starting from ξ_1 we split the LS-sequence into consecutive blocks where the first block B_1 is of length 1 and the n-th block B_n for $n \geq 2$ is of length $Ll_n = Lq_{n-1} = t_n - t_{n-1}$. We now study the blocks B_n,

$$\begin{aligned} B_n &= \psi^{(n)}_{1,0}(\xi_1), \ldots, \psi^{(n)}_{1,0}(\xi_{l_{n-1}}), \ldots, \psi^{(n)}_{L,0}(\xi_1), \ldots, \psi^{(n)}_{L,0}(\xi_{l_{n-1}}) \\ &= \xi_1 + \beta^{n-1}, \ldots, \xi_{l_{n-1}} + \beta^{n-1}, \ldots, \xi_1 + L\beta^{n-1}, \ldots, \xi_{l_{n-1}} + L\beta^{n-1}. \end{aligned}$$

Lemma 2.4. *Let $n \in \mathbb{N}$.*

(i) *If $n = 2k+1$ is odd, then B_n considered as a set consists of the $L \cdot q_{2k}$ elements $\{-q_{2k-1}\beta\}, \{-(q_{2k-1}+1)\beta\}, \ldots, \{-(q_{2k+1}-1)\beta\}$ (respectively, of the element 0 if $n = 1$).*

(ii) *If $n = 2k$ is even, then B_n considered as a set consists of the $L \cdot q_{2k-1}$ elements $\{(q_{2k-2}+1)\beta\}, \{(q_{2k-2}+2)\beta\}, \ldots, \{q_{2k}\beta\}$.*

Before going into the rather technical details of the proof, let us explain its idea for the example of the Kakutani-Fibonacci sequence ($L = S = 1$). This sequence of points is given by

$$\underbrace{0}_{B_0}, \underbrace{\beta}_{B_1}, \underbrace{\beta^2}_{B_2}, \underbrace{\beta^3, \beta + \beta^3}_{B_3}, \underbrace{\beta^4, \beta + \beta^4, \beta^2 + \beta^4}_{B_4}, \ldots$$

Using $\beta + \beta^2 = 1$ this can be easily re-written as

$$\underbrace{0}_{B_0}, \underbrace{\beta}_{B_1}, \underbrace{1 - \beta}_{B_2}, \underbrace{2\beta - 1, 3\beta - 1}_{B_3}, \underbrace{2 - 3\beta, 2 - 2\beta, 3 - 4\beta}_{B_4}, \ldots$$

Proof. The two assertions are proved simultaneously by induction on k. For $n = 1, 2$ the claim is obvious from definition, since $\xi_1 = 0$ and $\xi_2 = \beta, \ldots, \xi_L = L\beta$. Let $k \geq 2$ and $n = 2k+1$ be odd. If we denote by $\equiv$ equivalence modulo 1 we have for $m \in \{0, \ldots, l_{n-1}\}$ by Lemma 2.2 and induction hypothesis

$$\xi_m + j\beta^{2k+1-1} \equiv \xi_m - jq_{2k}\beta \equiv (r - jq_{2k})\beta$$
$$\text{with} \quad -q_{2k-1} \leq r \leq q_{2k} \quad \text{and} \quad 1 \leq j \leq L.$$

$$-q_{2k-1} + 1 - Lq_{2k} \leq r - jq_{2k} \leq q_{2k} - q_{2k} \Leftrightarrow -(q_{2k+1} - 1) \leq r - jq_{2k} \leq 0.$$

Since the sequence is injective, the claim follows for odd n. So let $n = 2k+2$ be even. Then we use again Lemma 2.2 and induction hypothesis to derive

$$\xi_m + j\beta^{2k+2-1} \equiv \xi_m + jq_{2k+1}\beta \equiv (r + jq_{2k+1})\beta, \qquad \text{with} \quad -q_{2k-1}+1 \leq r \leq q_{2k} \quad \text{and} \quad 1 \leq j \leq L.$$

This completes the induction since

$$-q_{2k-1} + 1 + q_{2k+1} \leq r + jq_{2k+1} \leq q_{2k} + Lq_{2k+1} \Leftrightarrow 1 \leq r + jq_{2k+1} \leq q_{2k+2}.$$

□

Proof of Theorem 1.7. If $S = 1$ the LS-sequence is indeed a reordering of the symmetrized Kronecker sequence by Lemma 2.4. So let $S \geq 2$ and $L \geq S$. Then β is irrational and the relation

$$\beta^2 = \frac{1 - L\beta}{S}. \tag{1}$$

holds. Hence the LS-sequence cannot be a reordering of a van der Corput sequence (which consists only of rational number).

Now assume that the LS-sequence is the reordering of a (possibly symmetrized) Kronecker sequence $\{n\alpha\}$ for some $\alpha \in \mathbb{R}$. Since α itself has to be an element of the LS-sequence, there exists an $n \in \mathbb{N}$ such that α can be uniquely written in the form

$$\alpha = \sum_{k=1}^{n} \alpha_k \beta^k$$

with $\alpha_k \in \{0, \ldots, L\}$ for $k = 1, \ldots, n$ and $\alpha_n \neq 0$. By (1) we have the equality $\beta^k = x_k\beta + y_k$ with $x_k,\ y_k \in \mathbb{Q}$ and $S^k x_k,\ S^k y_k \in \mathbb{Z}$. Thus, α itself can be rewritten as $\alpha = x_\alpha\beta + y_\alpha$ with $x_\alpha, y_\alpha \in \mathbb{Q}$ and $S^n x_\alpha, S^n y_\alpha \in \mathbb{Z}$. However, β^{n+1}, which is an element of the LS-sequence, cannot be an element of $\{n\alpha\}_n$ since $\beta^{n+1} = x_{n+1}\beta + y_{n+1}$, where at least one of x_{n+1} and y_{n+1} has denominator S^{n+1}. This is a contradiction. □

A main advantage of the approach via symmetrized Kronecker sequence is that it yields a possibility to calculate improved discrepancy bounds, namely Corollary 1.8.

Proof. (Proof of Corollary 1.8) We imitate the proofs in [Nie92], Theorem 3.3 and [KN74], Theorem 3.4 respectively and leave away here the technical details that are explained therein very nicely: The number N can be represented in the form

$$N = \sum_{i=0}^{l(N)} c_i q_i,$$

where $l(N)$ is the unique non-negative integer with $q_{l(N)} \leq N < q_{l(N)+1}$, and where the c_i are integers with $0 \leq c_i \leq L$. Let LS_N denote the set consisting of the first N numbers of the LS-sequence. We decompose LS_N into blocks of consecutive terms, namely c_i blocks of length q_i for all $0 \leq i \leq l(N)$. Consider a block of length q_i and denote the corresponding point set by A_i. If i is odd, A_i consists of the fractional parts $\{nz\}$ with $n = n_i, n_i + 1, \ldots, n_i + q_i - 1$ according to Lemma 2.4. As shown in the proof of [Nie92], Theorem 3.3., this point set has discrepancy

$$D_{q_i}(A_i) < \frac{1}{q_{i-1}} + \frac{1}{q_i}.$$

If i is even, A_i consists of the fractional parts $\{-nz\}$ with again

$$n = n_i, n_i + 1, \ldots, n_i + q_i - 1 \quad \text{by Lemma 2.4.}$$

Since z and $-z$ have the same continued fraction expansion up to signs, we also have

$$D_{q_i}(A_i) < \frac{1}{q_{i-1}} + \frac{1}{q_i}.$$

Analogous calculations as in [KN74] then yield the assertion. □

Asymptotically we deduce the following behaviour, again improving the more general result of [IZ17] in the special case $S = 1$.

Corollary 2.5. *If $S = 1$, then we obtain*

$$\lim_{N\to\infty} \frac{ND_N(\xi_n)}{\log N} \sim \frac{L}{\log(L)} \quad \textit{as} \quad L \to \infty.$$

Finally, we would like to point out the fact that it follows immediately from our approach that the Kakutani-Fibonacci sequence is the reordering of an orbit of an ergodic interval exchange transformation. In [CIV14], it was shown that a much more complicated interval exchange transformation is necessary in order to get the original ordering given in Definition 1.3.

Corollary 2.6. *For $L = 1$, the LS-sequence is always a reordering of an orbit of an ergodic interval exchange transformation.*

Proof. The map $R_\alpha : x \mapsto x + \alpha \pmod 1$, the rotation of the circle by α, is ergodic for $\alpha \notin \mathbb{Q}$, see, e.g., [EW11], Example 2.2. Moreover, it is an interval exchange transformation, compare e.g., [Via06]. □

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