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SETS OF BOUNDED REMAINDER FOR THE BILLIARD ON A SQUARE

Ida Aichinger[1] — Gerhard Larcher[2]

[1]CERN, European Organization for Nuclear Research, Meyrin, SWITZERLAND

[2]University Linz, Linz, AUSTRIA

ABSTRACT. We study sets of bounded remainder for the billiard on the unit square. In particular, we note that every convex set S whose boundary is twice continuously differentiable with positive curvature at every point, is a bounded remainder set for almost all starting angles α and every starting point $\mathbf{x}$. We show that this assertion for a large class of sets does not hold for *all* irrational starting angles α.

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1. Introduction and statement of results

In this paper we will be concerned with bounded remainder sets for the two-dimensional billiard on the unit-square $I^2 = [0,1)^2$.

Definition 1. Let $\mathbf{x} = (x_1, x_2) \in I^2$ and let $\alpha \in \mathbb{R}\backslash\mathbb{Q}$. We say that the function $Y : [0,\infty) \to I^2$ defined by

$$Y(t) = \left(2 \cdot \left\|\frac{x_1 + t}{2}\right\|,\ 2 \cdot \left\|\frac{x_2 + \alpha t}{2}\right\|\right) \qquad 0 \le t < \infty, \tag{1}$$

where $\|z\| := \min_{a\in\mathbb{Z}} |z-a|$, is the two-dimensional billiard with starting slope α and starting point $\mathbf{x}$.

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It is easily checked that this definition indeed coincides with our image of a real billiard-path in the unit square.

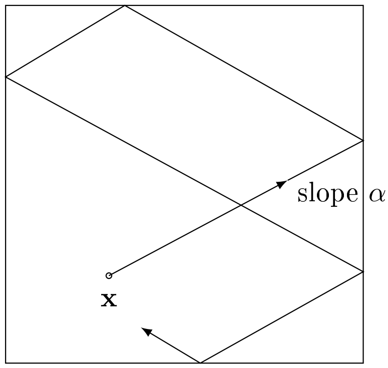

FIGURE 1.

DEFINITION 2. Let $S \subset I^2$ be an arbitrary measurable subset of the unit square with Lebesgue measure $\lambda(S)$. We say that S is a bounded remainder set for the two-dimensional billiard with starting slope α and starting point $\mathbf{x} = (x_1, x_2) \in I^2$ if the distribution error

$$\Delta_T^Y(S, \alpha, \mathbf{x}) = \int_0^T \chi_S\big(Y(t)\big)\,\mathrm{d}t - T\lambda(S) \tag{2}$$

is uniformly bounded for all $T > 0$. Here, χ_S denotes the characteristic function for the set S.

Distribution properties for continuous motions in an s-dimensional unit cube were studied, for example, by Drmota in [3] (see also [4] or [7]) and quite recently by Beck [1]. Beck ([1], [2]), especially studied continuous irrational rotations and billiard paths. For the two-dimensional billiard path, for example, he showed the following surprising result:

THEOREM (Beck). *Let $S \subseteq I^2$ be an arbitrary Lebesgue measurable set in the unit square with positive measure. Then for every $\epsilon > 0$, almost all $\alpha > 0$ and every starting point $\mathbf{x} = (x_1, x_2) \in I^2$ we have*

$$\Delta_T^Y(S, \alpha, \mathbf{x}) = \mathcal{O}\Big((\log T)^{3+\epsilon}\Big). \tag{3}$$

As pointed out by Beck, the poly-logarithmic error term is shockingly small compared to the linear term $T\lambda(S)$. Moreover, it holds for *all* measurable sets S. It is thus natural to ask if imposing certain regularity conditions on S could give an even lower bound on the error term.

We demonstrate in the following that the estimate of Beck indeed can be significantly improved for a large collection of sets S. We show:

THEOREM 1.

a) *For almost all $\alpha > 0$ and every $\mathbf{x} \in I^2$, every polygon $S \subset I^2$ with no edge of slope α or $-\alpha$ is a bounded remainder set for the two-dimensional billiard with starting slope α and starting point $\mathbf{x}$.*

b) *For almost all $\alpha > 0$ and every $\mathbf{x} = (x_1, x_2) \in I^2$, every convex set $S \subset I^2$ whose boundary ∂S is a twice continuously differentiable curve with positive curvature at every point is a bounded remainder set for the two-dimensional billiard with starting slope α and starting point $\mathbf{x}$.*

We will see in the proofs of these results that this Theorem easily follows from an analogous result shown in [5] for the continuous irrational rotation $X(t) := (\{x_1 + t\}, \{x_2 + \alpha t\})_{t \geq 0}$, and by the "unfolding-technique" suggested by Beck in [2]. It is obvious that the results given in Theorem 1 do not hold for a rational slope α. However, one could ask whether the results can be improved first by omitting the condition on the slopes of the edges of the polygon S in part a) of the Theorem and, second, whether both results maybe are valid even for *all* irrational slopes α. We will give an easy argument that indeed the condition on the slopes of the edges cannot be omitted in general. Moreover, we will prove - and this will be the main effort in this paper - that the results of Theorem 1a and 1b in general do not hold for *all* irrational α.

I.e., we will show:

THEOREM 2.

a) *For every $\alpha > 0$ there is a polygon S with an edge of slope α or $-\alpha$ such that S is not a bounded remainder set for the billiard with starting-slope α and for any starting point $\mathbf{x}$.*

b) *For every $\boldsymbol{m} \in [0, 1)^2$ there are uncountably many radii r, dense in an interval of positive length, such that there is a slope α and a starting point $\boldsymbol{x}$ such that the disk with midpoint $\boldsymbol{m}$ and radius r is not a set of bounded remainder with respect to the billiard with starting slope α and starting point $\boldsymbol{x}$.*

In Chapter 2 we prove Theorem 1 and Theorem 2a. In Chapter 3 we carry out the main work, namely the proof of Theorem 2 b.

2. Proofs of Theorem 1 and of Theorem 2a

The proofs of these two results can be traced back to the results given in [5] via the technique of unfolding.

As was pointed out in detail, for example, by Beck in [2] the technique of "unfolding" a billiard path (see Figure 2) shows that the problem of uniformity

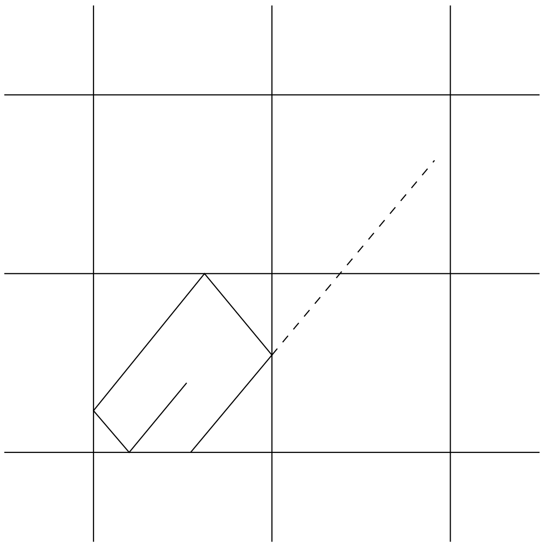

FIGURE 2.

of a billiard path in the unit square with respect to a given test set S is equivalent to the problem of uniformity of the corresponding continuous rotation in the 2×2 square, where each one of the four unit-subsquares contains a reflected copy of the given test set (see S_1, S_2, S_3, S_4 in Figure 3) Of course, this again can be reduced to the problem of studying continuous irrational rotation in $[0,1)^2$ with respect to a factor $1/2$ reduced versions of S_1, S_2, S_3, S_4.

So in all the following, when studying the distribution error $\Delta_T^Y(S, \alpha, \mathbf{x})$ for the two-dimensional billiard this task can be traced back to the investigation of the distribution error $\Delta_T^X(\tilde{S}, \alpha, \mathbf{x})$ for the continuous irrational rotation where $\tilde{S}$ consists of four mirrored and by a factor $1/2$ reduced copies of S lying symmetrical to $(1/2, 1/2)$.

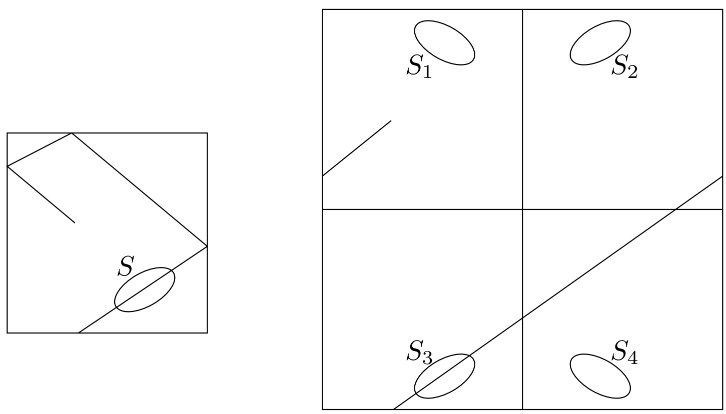

FIGURE 3.

P r o o f o f T h e o r e m 1. Theorem 1 follows immediately from the above considerations on the unfolding technique and from Theorem 1 and Theorem 2 in [5]. □

P r o o f o f T h e o r e m 2a. Let $\alpha > 0$ be given. Consider first a triangle S (see Figure 4) with corners in $(a, 1)$, $(1, 1 - a\alpha)$, $(1, 1)$, where a is such that $0 < a\alpha < 1$ and such that $a \neq \frac{1}{\alpha}\{k\alpha\}$ for all $k \in \mathbb{N}$.

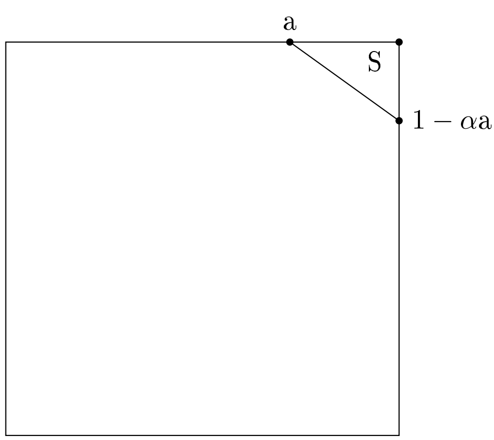

FIGURE 4.

So one side of S has slope $-\alpha$. Unfolding leads to the investigation of the continuous irrational rotation with slope α with respect to the parallelogram $\tilde{S}$ (see Figure 5) with corners in

$$\left(\frac{1}{2}\,,\,\frac{a}{2}\right)\,,\,\left(\frac{1}{2}\,,\,\frac{1+\alpha a}{2}\right)\,,\,\left(\frac{1}{2}+\frac{a}{2}\,,\,\frac{1}{2}\right)\,,\,\left(\frac{1}{2}\,,\,\frac{1-\alpha a}{2}\right).$$

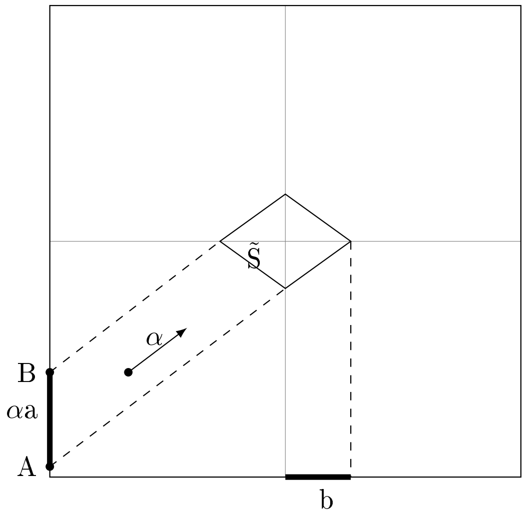

Figure 5.

Note that two sides of $\tilde{S}$ have slope α, the other two have slope $-\alpha$. Its vertical diagonal $[A,B]$ has a length of $a\alpha \neq \{k\alpha\}$ for all $k \in \mathbb{N}$. We show that for no starting point $\mathbf{x}$ the set $\tilde{S}$ is of bounded remainder for the irrational continuous rotation with slope α. Hence the set S is not of bounded remainder for the billiard with starting slope α (and any starting point $\mathbf{x}$).
Indeed, it is easy to see that for this set $\tilde{S}$ we have

$$\left|\int\limits_0^T \chi_{\tilde{S}}(\{t\},\{\alpha t\})\,\mathrm{d}t - b\cdot\sum_{n=1}^{[T]}\chi_{[A,B]}(\{n\alpha\})\right| \leq 1.$$

It was shown by Kesten in [6] that

$$\left|\sum_{n=1}^{[T]}\chi_{[A,B]}(\{n\alpha\}) - a\alpha\cdot[T]\right|$$

is unbounded since $B-A=a\alpha \neq \{k\alpha\}$ for all $k\in\mathbb{N}$.

Hence (note that $\lambda(\tilde{S}) = a\alpha \cdot b$)

$$\left| \int_0^T \chi_{\tilde{S}}(\{t\}, \{\alpha t\})\, \mathrm{d}t - T \cdot \lambda(\tilde{S}) \right|$$

is unbounded. $\square$

3. Proof of Theorem 2 b

The proof of Theorem 2b will need the most work. We start with some auxiliary results. Especially we will have to deal with functions of the form

$$g_m(x) := \frac{1}{2m} \sum_{k=0}^{2m-1} \sqrt{1 - (1 - \frac{k}{m} - x)^2} \tag{4}$$

for $x \in [0, \frac{1}{m}]$, where m is a given positive integer. The function g_m is illustrated in Figure 6.

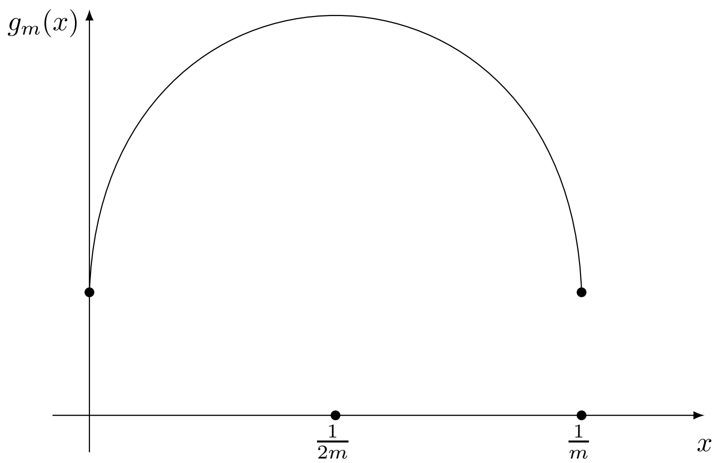

FIGURE 6.

Let now m be fixed, and $h_m(x) := g_m(x) - g_m(0)$. It is easy to see that

$$h_m(0) = h_m(\frac{1}{m}) = 0,\ h_m$$

is arbitrarily often differentiable on $(0, \frac{1}{m})$, continuous on $[0, \frac{1}{m}]$, symmetric around $\frac{1}{2m}$ and strictly convex on $[0, \frac{1}{m}]$.

Further we have

Lemma 1.

a) *There exist c', c'' with $0 < c' < c''$ such that for all m large enough we have*

$$c' \cdot \frac{1}{m^{3/2}} \quad < \quad h_m(\frac{1}{2m}) \quad < \quad c'' \cdot \frac{1}{m^{3/2}}. \tag{5}$$

b) *There exists $c''' > 0$ such that*

$$h_m(x) \quad \geq \quad c''' \cdot \frac{1}{m}\sqrt{x} \tag{6}$$

for all $x \in [0, \frac{1}{2m}]$.

P r o o f. This is shown by some tedious but elementary analysis. We do not give all details but just give two hints:

• To prepare part a) note that

$$\begin{aligned}
h_m(x) \quad &= \quad g_m(x) - g_m(0) \\
&= \quad \tfrac{1}{2m} \textstyle\sum_{k=0}^{2m-1} \left(\sqrt{1 - \left(1 - \frac{k}{m} - x\right)^2} \quad - \quad \sqrt{1 - \left(1 - \frac{k}{m}\right)^2} \right) \\
&= \quad \tfrac{1}{2m} \textstyle\sum_{k=0}^{m} \left(\sqrt{1 - \left(1 - \frac{k}{m} - x\right)^2} \quad - \quad \sqrt{1 - \left(1 - \frac{k}{m}\right)^2} \right. \\
&\qquad \left. + \quad \sqrt{1 - \left(1 - \frac{2m-1-k}{m} - x\right)^2} \quad - \quad \sqrt{1 - \left(1 - \frac{2m-1-k}{m}\right)^2} \right) \\
&=: \quad \tfrac{1}{2m} \textstyle\sum_{k=0}^{m} w_m(k, x)
\end{aligned}$$

and note that simple calculation shows that for $k \geq 0$ there are absolute constants $c_1', c_2' > 0$ such that

$$c_1' \cdot \frac{1}{m^{1/2} \max(k,1)^{3/2}} \quad \leq \quad w_m(k, \frac{1}{2m}) \quad \leq \quad c_2' \cdot \frac{1}{m^{1/2} \max(k,1)^{3/2}},$$

always.

• To show part b, note that for x small, only the first summand of h_m, i.e., the summand for k=0,

$$\frac{1}{m}\sqrt{1 - (1-x)^2} \quad \sim \quad \frac{1}{m}\sqrt{x}$$

is of relevance. □

Lemma 2.

a) *There exists $c_1 > 0$ such that*

$$\left|h'_m(x)\right| \quad \leq \quad c_1 \cdot \frac{1}{\sqrt{m}} \tag{7}$$

for all $x \in [\frac{1}{10m}, \frac{9}{10m}]$ and all m large enough.

Especially, it holds that

$$h'_m(x) \quad \geq \quad -c_1 \cdot \frac{1}{\sqrt{m}} \tag{8}$$

for all $x \in [0, \frac{9}{10m}]$.

b) *There exists $c_2 > 0$ such that for all m large enough we have*

$$h'_m(x) \quad \geq \quad 5c_1 \frac{1}{\sqrt{m}} \tag{9}$$

for all $x \in \left(0, c_2 \frac{1}{m}\right)$.

(Here c_1 is the constant from part a).)

P r o o f. This follows immediately from Lemma 1 and the convexity of h_m. □

Let m large enough be fixed. Moreover, in the following let a, b, c be given reals with

$$0 \leq a \leq b \leq c < \frac{1}{m},$$

and

$$\begin{aligned} G_m(x) \quad := \quad & g_m\Big(x\Big) + g_m\left((x+a) \bmod \frac{1}{m}\right) \\ & + g_m\left((x+b) \bmod \frac{1}{m}\right) + g_m\left((x+c) \bmod \frac{1}{m}\right). \end{aligned}$$

Then we have:

Lemma 3. *There are $c_3, c_4 > 0$ such that for all m large enough and all a, b, c as above there is an $x_0 \in \{0, a, b, c\}$ such that G_m is strictly increasing on $[x_0, x_0 + c_4 \cdot \frac{1}{m}]$ and*

$$G_m\left(x_0 + c_4 \cdot \frac{1}{3m}\right) - G_m(x_0) \quad > \quad c_3 \cdot \frac{1}{m^{3/2}}, \tag{10}$$

$$G_m\left(x_0 + c_4 \cdot \frac{2}{3m}\right) - G_m\left(x_0 + c_2 \cdot \frac{1}{3m}\right) \quad > \quad c_3 \cdot \frac{1}{m^{3/2}}, \tag{11}$$

$$G_m\left(x_0 + c_4 \cdot \frac{1}{m}\right) - G_m\left(x_0 + c_2 \cdot \frac{2}{3m}\right) \quad > \quad c_3 \cdot \frac{1}{m^{3/2}}\,. \tag{12}$$

P r o o f. At least one of the following relations holds:

$$c < \frac{4}{5m} \quad \textit{or} \quad b < c - \frac{1}{5m}, \quad \textit{or} \quad a < b - \frac{1}{5m}, \quad \textit{or} \quad a > \frac{1}{5m}\,.$$

Assume for example that $c < \frac{4}{5m}$ holds (the other cases are treated quite analogously). Then set $x_0 = 0$. Let $c_4 := \min(c_2, \frac{1}{10})$ where c_2 is like in Lemma 2 b). Then for any $x \in \left[0, c_4\frac{1}{m}\right)$ it holds that

$$(a+x) \bmod \frac{1}{m}, \text{ and } (b+x) \bmod \frac{1}{m}, \text{ and } (c+x) \bmod \frac{1}{m} \text{ are all in } \left[0, \frac{9}{10m}\right).$$

Hence by Lemma 2 a) we have that g'_m at these places is at least $-c_1 \cdot \frac{1}{\sqrt{m}}$. By Lemma 2 b) for $x \in [0, c_4\frac{1}{m}]$ we have $g'_m(x) \geq 5c_1 \cdot \frac{1}{\sqrt{m}}$ and hence $G'_m(x) \geq 2c_1\frac{1}{\sqrt{m}}$ for all those x.

From this the assertions of Lemma 3 immediately follow. □

Lemma 4. *For all m large enough there is a sub-interval Λ_m of $[0, \frac{1}{m}]$ of length at least $\frac{c_4}{3} \cdot \frac{1}{m}$ such that either*

$$G_m(x) > 2\int_0^2 \sqrt{1-(1-y)^2}\,\mathrm{d}y + \frac{c_3}{2m^{3/2}} = \pi + \frac{c_3}{2m^{3/2}} \tag{13}$$

or

$$G_m(x) < 2\int_0^2 \sqrt{1-(1-y)^2}\,\mathrm{d}y - \frac{c_3}{2m^{3/2}} = \pi - \frac{c_3}{2m^{3/2}} \tag{14}$$

holds for all $x \in \Lambda_m$.

P r o o f. This follow immediately from Lemma 3. □

P r o o f o f T h e o r e m 2 b. For the proof we proceed in analogy to the proof of Theorem 1.7 b in [5], where the corresponding result was shown for the continuous irrational rotation, and in the following we sometimes refer to this proof.

Fix an irrational $\alpha \in (\frac{1}{8}, \frac{1}{4})$ with continued fraction expansion $\alpha = [0; a_1, a_2, \dots]$ and convergents $\frac{p_n}{q_n}$ satisfying $a_{l+1} > q_l^{100}$ and p_l even, for infinitely many l. There exist uncountably many such αs. Let S be a disk with diameter $d := 2\alpha/\sqrt{1+\alpha^2}$. (Note that the set of α with the above properties is dense in $(\frac{1}{8}, \frac{1}{4})$, hence the set of diameters d is dense in $(\frac{2}{\sqrt{65}}, \frac{2}{\sqrt{17}})$.)

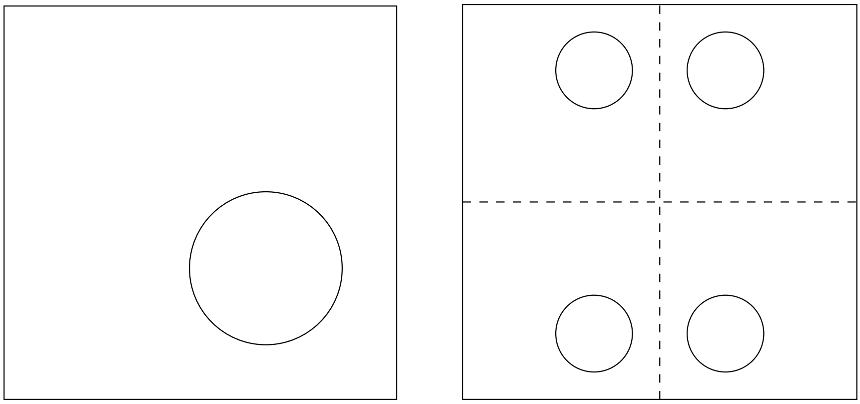

Figure 7.

Studying the billiard path with respect to S means to study the continuous rotation with respect to four copies of S with diameter $\alpha/\sqrt{1+\alpha^2}$ each (see Figure 7)
For such α and one copy of such disks it was shown in Theorem 1.7 b in [5] that the continuous rotation is not of bounded remainder for all starting points $\mathbf{x}$. In the proof of this Theorem the result was shown by studying the function g_m as defined in this current paper at the beginning of Section 3 (in [5] our g_m is denoted by G_m), and it was shown that the validity of the result of Theorem 1.7 b in [5] is due to the fact that for every m there exists a subinterval $\Lambda_m \subseteq [0, \frac{1}{2m}]$ of length at least $\frac{1}{6m}$ such that either

$$g_m(x) > \frac{1}{2}\int_0^2 \sqrt{1-(1-y)^2}\ \mathrm{d}y + \frac{\tilde{c}}{m^{3/2}} = \frac{\pi}{4} + \frac{\tilde{c}}{m^{3/2}} \tag{15}$$

or

$$g_m(x) < \frac{1}{2}\int_0^2 \sqrt{1-(1-y)^2}\ \mathrm{d}y - \frac{\tilde{c}}{m^{3/2}} = \frac{\pi}{4} - \frac{\tilde{c}}{m^{3/2}} \tag{16}$$

holds for an absolute constant $\tilde{c} > 0$ and all $x \in \Lambda_m$.

By following the proof of Theorem 1.7 b in [5] it becomes obvious that studying now four copies of disks instead of one copy means to study

$$g_m\Big(x\Big) + g_m\Big((x+a)\mathrm{mod}\frac{1}{m}\Big) + g_m\Big((x+b)\mathrm{mod}\frac{1}{m}\Big) + g_m\Big((x+c)\mathrm{mod}\frac{1}{m}\Big)$$

for some a, b, c.

This means in particular to analyse the corresponding function G_m as it has been done in Lemma 3 and Lemma 4 of the current paper. In Lemma 4 it was shown that for G_m an analogous property (independent of the choices for a, b, and c) holds as stated above for g_m.

Again by following the proof of Theorem 1.7 b in [5] it is obvious that from this property for G_m (Lemma 4) the result of our Theorem 2 b follows.

□

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Ida Aichinger
CERN, European Organization
for Nuclear Research
385 Route de Meyrin
CH-1217 Meyrin
SWITZERLAND
E-mail: ida.aichinger@cern.ch

Gerhard Larcher
Institute of Financial Mathematics
and Applied Number Theory
University Linz
Altenbergerstr. 69
A-4040 Linz
AUSTRIA
E-mail: gerhard.larcher@jku.at