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DISCREPANCY RESULTS FOR THE VAN DER CORPUT SEQUENCE

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ABSTRACT. Let $d_N = ND_N(\omega)$ be the discrepancy of the van der Corput sequence in base 2. We improve on the known bounds for the number of indices N such that $d_N \le \log N/100$. Moreover, we show that the summatory function of d_N satisfies an exact formula involving a 1-periodic, continuous function. Finally, we give a new proof of the fact that d_N is invariant under digit reversal in base 2.

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Dedicated to Henri Faure on the occasion of his 80th birthday

1. Introduction

Every nonnegative integer n admits a unique expansion $n = \sum_{i=0}^{\nu} \varepsilon_i 2^i$ such that $\varepsilon_i \in \{0,1\}$ and ($\nu = 0$ or $\varepsilon_\nu \neq 0$). We let $\varepsilon_i(n)$ denote the i-th digit in base 2. The van der Corput sequence is defined via the *radical inverse* of n in base 2: define $\omega_n = \sum_{i=0}^{\nu} \varepsilon_i(n) 2^{-i-1}$.

Let $x = (x_n)_{n\ge 0}$ be a sequence in $[0,1)$. The *discrepancy* $D_N(x)$ of x is defined by

$$D_N(x) = \sup_{0\le a\le b\le 1} \bigl|A_N(x,a,b)/N - (b-a)\bigr|$$

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$$\text{for} \quad N \geq 1, \quad \text{where} \quad A_N(x,a,b) = |\{n < N : a \leq x_n < b\}|.$$

Also, we set $D_0(x) = 0$. The *star-discrepancy* (or discrepancy at the origin) of a sequence x in $[0,1)$ is defined by $D_N^*(x) = \sup_{0 \leq b \leq 1} |A_N(x,0,b)/N - b|$, for $N \geq 1$, and we set $D_0^*(x) = 0$.

In this paper, we are concerned with the discrepancy of the van der Corput sequence. We define

$$d_N = N D_N(\omega),$$

we will use this notation throughout this paper. It is well known [3, Théorème 1] that the star discrepancy of the van der Corput sequence equals its discrepancy: we have $D_N^*(\omega) = D_N(\omega)$. The van der Corput sequence is a *low discrepancy sequence*, that is, we have $d_N \leq C \log N$ for some constant C. More precise results are known: Béjian and Faure [3] proved the following theorem.

Theorem A.

$$d_N \leq \frac{1}{3} \log_2 N + 1 \quad \textit{for all} \quad N \geq 1;$$

moreover,

$$\limsup_{N\to\infty} \left(d_N - \frac{1}{3} \log_2 N \right) = \frac{4}{9} + \frac{1}{3} \log_2 3,$$

where $\log_2$ *denotes the logarithm in base* 2.

In the proof of these statements, they implicitly show that d_N is bounded above by the polygonal path connecting the first maxima on the intervals $I_k = [2^{k-1}, 2^k]$, given by the points $\left(\frac{1}{3}\left(2^{k+1} + (-1)^k\right), \frac{k}{3} + \frac{7}{9} + (-1)^k/(9 \cdot 2^{k-1})\right)$. This should be compared to the argument given by Coons and Tyler [6] concerning Stern's diatomic sequence (also called Stern-Brocot sequence), see also the paper by Coons and the author [5] and the recent paper by Coons [4].

Concerning the "usual" order of magnitude of the discrepancy of the van der Corput sequence, Drmota, Larcher and Pillichshammer [8, Theorem 2] proved a central limit theorem for d_N.

Theorem B. *For every real* y*, we have*

$$\lim_{M\to\infty} \frac{1}{M} \left| \left\{ N < M : d_N \leq \frac{1}{4} \log_2 N + y \frac{1}{4\sqrt{3}} \sqrt{\log_2 N} \right\} \right| = \Phi(y), \tag{1}$$

where $\quad \Phi(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-t^2/2} \, \mathrm{d}t.$

We note that this implies in particular that d_N is usually of order $\log N$. More precisely, letting $A_{M,y}$ denote the expression on the left hand side of (1), we trivially have $A_{M,y'} \leq A_{M,y}$ if $y' \leq y$. This implies, for any sequence $(y_M)_{M\geq 1}$ of reals such that $y_M \to -\infty$ for $M \to \infty$, that

$$\lim_{M\to\infty} A_{M,y_M} \leq \lim_{M\to\infty} A_{M,y} = \Phi(y) \quad \text{for all real } y,$$

therefore this limit equals 0.

In particular, if $\delta < 1/4$, the number of integers $N < M$ such that $d_N \le \delta \log_2 N$ is $o(M)$, where $\log_2$ is the base-2 logarithm.

Bounds of this type, with an explicit error term, had been proved earlier: Sós [22] proved such a statement for $\{n\alpha\}$-sequences, more generally Tijdeman and Wagner [25] showed that any sequence in $[0,1)$ has almost nowhere small discrepancy. More specifically, they proved the following theorem.

Theorem C. *Let ξ be a sequence in $[0,1)$. Let M and N be integers with $M \ge 0$ and $N > 1$. Then $nD_n(\xi) < \log N/100$ for at most $2N^{5/6}$ integers n with $M < n \le M + N$.*

In fact, it follows from Lemma 2 in their paper [25] that the exponent 5/6 can be replaced by an arbitrarily small positive value if we demand an arbitrarily small constant in place of 1/100.

Corollary. *Let ξ be a sequence in $[0,1)$. For each $\varepsilon > 0$ there exists a constant $\delta > 0$ such that for all integers $M \ge 0$ and $N > 1$ we have $nD_n(\xi) < \delta \log N$ for at most $2N^{\varepsilon}$ integers n with $M < n \le M + N$.*

Moreover, we want to note the article [12], which gives a survey on constructions of uniformly distributed sequences. Many of these constructions are related to the van der Corput sequence. We proceed to the statement of our results.

2. Results

We wish to show that the constant 5/6 in Theorem C can be improved at least for the van der Corput sequence.

Theorem 2.1. *For all large N, the number of $n < N$ satisfying $d_n \le \log n/100$ is bounded above by $N^{0.183}$.*

Moreover, Tijdeman and Wagner [25, Theorem 3] showed that for infinitely many N we have $d_n \le \log N/100$ for more than $N^{1/21}$ integers $n \in [1, N]$. We wish to improve on the exponent 1/21.

Theorem 2.2. *For all large N, the number of $n < N$ satisfying $d_n \le \log n/100$ is bounded below by $N^{0.056}$.*

It would be interesting to determine, for each given $\varepsilon > 0$, the exact "exponent of strong irregularity" of the van der Corput sequence. That is, determine the infimum of η such that the number of $n < N$ satisfying $d_n \le \varepsilon \log n$ is bounded by N^η, for all large N. By the above results this infimum, for $\varepsilon = 1/100$, lies in $[0.056, 0.183]$. We leave this as an open question.

Moreover, we wish to note that our method is applicable to constants ε other than $1/100$, as long as this constant is smaller than $1/8$.

Next, we consider partial sums

$$S(N) = d_1 + \cdots + d_N.$$

It was shown by Béjian and Faure [3] that

$$\frac{1}{N}\sum_{k=1}^{N} d_k = \frac{\log_2 N}{4} + O(1),$$

where $\log_2 N$ denotes the base-2 logarithm of N. We are interested in the error term appearing in this expression. It turns out that there exists an exact formula involving a 1-periodic, continuous function (see, for example, the papers by Delange [7] and Flajolet et al. [13]).

Theorem 2.3. *There exists a continuous, 1-periodic function $\psi : \mathbb{R} \to \mathbb{R}$ such that*

$$\frac{1}{N}S(N) = \frac{\log_2 N}{4} + \frac{d_N}{2N} + \psi(\log_2 N). \tag{2}$$

The function ψ is uniquely determined.

In particular, we obtain the boundedness result of the error term given by Béjian and Faure. Our third result is concerned with *digit reversal*: If $\varepsilon_\nu \cdots \varepsilon_0$ is the proper binary expansion of n, we define $n^R = \sum_{0\le i\le\nu} \varepsilon_{\nu-i} 2^i$. Then the following theorem holds.

Theorem 2.4. *Assume that α, β, γ are complex numbers and that the sequence x satisfies $x_{2n} = x_n$ and $x_{2n+1} = \alpha x_n + \beta x_{n+1} + \gamma$ for $n \ge 1$. Then for $n \ge 1$ we have*

$$x_n = x_{n^R}.$$

This theorem generalizes Theorem 2.1 in the paper [24] by the author, see also Morgenbesser and the author [18] and the recent paper by the author [23]. We obtain the following, somewhat curious, corollary.

Corollary 2.5.

$$N D_N(\omega) = N^R D_{N^R}(\omega).$$

Remark. While Corollary 2.5 does not seem to be stated explicitly in the literature, it follows easily from known facts. The discrepancy d_N can also be written in the following form: we have

$$d_N = 2 \sum_{0\le k<N} (1/2 - \omega_k)$$

(see Proinov and Atanassov [20]). Moreover, Beck [2] gives an explicit formula for $\sum_{0\le k<N}(\omega_k - 1/2)$, which implies

$$d_N = \sum_{0\le i\le\nu}\varepsilon_i \; - \sum_{0\le i<j\le\nu}\varepsilon_i\varepsilon_j 2^{i-j},$$

where $N = \sum_{0\le i\le\nu}\varepsilon_i 2^i$ is the base-2 expansion of N. This representation immediately gives a direct proof of Corollary 2.5.

We note, however, that this digit reversal property seems to be restricted to base 2. That is, the van der Corput sequence in base q, where $q \ge 3$, does not seem to satisfy an analogous property with respect to digit reversal in base q. We refer the reader to [10, 11, 15, 19] concerning results on the discrepancy and diaphony of digital sequences. Among these one can find explicit formulas for the star discrepancy analogous to (3).

For illustration, we list the first values of $d_N = ND_N(\omega)$:

N	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
d_N	0	1	1	$\frac{3}{2}$	1	$\frac{7}{4}$	$\frac{3}{2}$	$\frac{7}{4}$	1	$\frac{15}{8}$	$\frac{7}{4}$	$\frac{17}{8}$	$\frac{3}{2}$	$\frac{17}{8}$	$\frac{7}{4}$	$\frac{15}{8}$

N	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31
d_N	1	$\frac{31}{16}$	$\frac{15}{8}$	$\frac{37}{16}$	$\frac{7}{4}$	$\frac{39}{16}$	$\frac{17}{8}$	$\frac{37}{16}$	$\frac{3}{2}$	$\frac{37}{16}$	$\frac{17}{8}$	$\frac{39}{16}$	$\frac{7}{4}$	$\frac{37}{16}$	$\frac{15}{8}$	$\frac{31}{16}$

Apart from the identity $d_N = d_{2^k-N}$, which is valid for $2^{k-1} \le N \le 2^k$ and which can be shown easily by induction, we see the notable identity $d_{19} = d_{25}$. Note that $19^R = 25$.

The remainder of this paper is dedicated to the proofs of our results.

3. Proofs

We will use the following explicit formula due to Béjian and Faure [3].

$$d_N = \sum_{j=1}^{\infty}\left\|N/2^j\right\|. \tag{3}$$

Here $\|x\| = \min_{n\in\mathbb{Z}}|x-n|$ is the distance of x to the nearest integer. Based on this result Béjian and Faure proved that d_N satisfies the following recurrence:

$$d_0 = 0, \quad d_1 = 1, \quad d_{2N} = d_N, \quad d_{2N+1} = \frac{d_N + d_{N+1} + 1}{2}, \tag{4}$$

which is valid for all $N \ge 0$.

We note that $(d_n)_{n\geq 0}$ is a 2-regular sequence in the sense of A l l o u c h e and S h a l l i t [1]. Moreover, the recurrence is of the discrete divide-and-conquer type [9, 14].

3.1. Proof of Theorems 2.1 and 2.2

In order to prove these theorems, we state a couple of lemmas. We let $|N|_{\mathtt{01}}$ denote the number of occurrences of `01` in the binary expansion of N, extended by zeros to the left; this equals the number of blocks of consecutive `1`s in the binary expansion of N. We state the following lemma, which is essentially contained in Lemma 5 of the paper [8] by D r m o t a, L a r c h e r, and P i l l i c h s h a m m e r.

Lemma 3.1. *We have*

$$\frac{1}{2}|N|_{\mathtt{01}} \leq d_N \leq 2|N|_{\mathtt{01}}. \tag{5}$$

P r o o f. We use the formula $d_N = \sum_{j=1}^{\infty} \left\| \frac{N}{2^j} \right\|$. Assume that $m = |N|_{\mathtt{01}}$. For $0 \leq k < m$ let a_k be the index corresponding to the beginning of the k-th block of `1`s, and b_k be the index corresponding to the end. We prove the first inequality first. We have

$$\sum_{j\geq a_0+2} \left\|N/2^j\right\| = \sum_{j\geq 0} \left|N/2^{a_0+2+j}\right| \geq \sum_{j\geq 0} \left|1/2^{2+j}\right| = 1/2,$$

moreover for $0 \leq k < m-1$,

$$\left\|N/2^{b_k+1}\right\| \geq \|1/2+1/8+1/16+\cdots\| = 1/4$$

and for $1 \leq k < m$,

$$\left\|N/2^{a_k+2}\right\| \geq \|1/4\| = 1/4.$$

To conclude the proof of the first inequality, we note that the indices b_k+1 and a_k+2 are pairwise different.

As for the second inequality, we bound the contribution of each block of `1`s by 2 as follows. For simplicity of the argument, we set $b_{-1} = \infty$. We have

$$d_N = \sum_{j=1}^{\infty} \left\|\frac{N}{2^j}\right\| = \sum_{-1\leq k<m-1} \left(\sum_{j=a_{k+1}+2}^{b_k} \left\|\frac{N}{2^j}\right\| + \sum_{j=b_{k+1}+1}^{a_{k+1}+1} \left\|\frac{N}{2^j}\right\| \right).$$

The summands are bounded above by geometric series with quotient $q = 1/2$, which yields the second inequality. $\square$

We note that the constant 2 is optimal, which can be seen by considering integers having the binary expansion $(\mathtt{0}^s\mathtt{1}^s)^k$ and letting $s \to \infty$. The constant $1/2$ probably can be improved, but not beyond $2/3$, which follows by considering integers of the form $(\mathtt{01})^k$ and letting $k \to \infty$. The next lemma is concerned with counting occurrences of `01` in the binary expansion.

Lemma 3.2. *For $k \geq 0$, $\ell \geq 1$ set*

$$a_{k,\ell} = \left|\left\{n \in [2^k, 2^{k+1}) : |n|_{\mathtt{01}} = \ell\right\}\right|.$$

Then

$$a_{k,\ell} = \binom{k+1}{2\ell-1}.$$

P r o o f. We are interested in the set $\mathcal{A}$ of integers $n \in [2^k, 2^{k+1})$ having exactly ℓ blocks of consecutive $\mathtt{1}$s. We define a bijection φ from $\mathcal{A}$ onto the set of $(2\ell-1)$-element subsets of $\{0, \ldots, k\}$ as follows. The binary expansion $\varepsilon_k \cdots \varepsilon_0$ of n consists of ℓ blocks of consecutive $\mathtt{1}$s and $\ell - 1$ or ℓ blocks of consecutive $\mathtt{0}$s. Let $\varphi(n)$ consist of those indices $i \in \{0, \ldots, k\}$ corresponding to the rightmost element of a block of $\mathtt{1}$s or to the rightmost element of one of the first $\ell - 1$ blocks of $\mathtt{0}$s. It is clear how to construct the inverse function. $\square$

We are interested in the quantity

$$A_{k,\varepsilon} = \left|\left\{N \in [2^k, 2^{k+1}) : d_N \leq \varepsilon \log N\right\}\right|.$$

By (5) and Lemma 3.2 we have

$$\begin{aligned} A_{k,\varepsilon} &\leq \left|\left\{N \in [2^k, 2^{k+1}) : |N|_{\mathtt{01}} \leq 2\varepsilon \log 2^{k+1}\right\}\right| \\ &= \sum_{\ell=1}^{2\varepsilon(k+1)\log 2} a_{k,\ell} = \sum_{\ell=1}^{2\varepsilon(k+1)\log 2} \binom{k+1}{2\ell-1} \end{aligned} \tag{6}$$

and

$$\begin{aligned} A_{k,\varepsilon} &\geq \left|\left\{N \in [2^k, 2^{k+1}) : |N|_{\mathtt{01}} \leq (\varepsilon/2) \log 2^{k}\right\}\right| \\ &= \sum_{\ell=1}^{(\varepsilon/2)k\log 2} a_{k,\ell} = \sum_{\ell=1}^{(\varepsilon/2)k\log 2} \binom{k+1}{2\ell-1}. \end{aligned} \tag{7}$$

We are therefore interested in large deviations of the binomial distribution. To this end, we state the following two lemmas. The first one is a well-known estimate.

Lemma 3.3. *For each integer $n \geq 0$ and all $\alpha \in [0, 1/2]$ we have*

$$\sum_{0 \leq k \leq \alpha n} \binom{n}{k} \leq 2^{H(\alpha)n},$$

where $H(\alpha)$ is the binary entropy function, defined by

$$H(x) = x \log_2 \frac{1}{x} + (1-x) \log_2 \frac{1}{1-x}$$

for $0 < x < 1$, and $H(0) = H(1) = 0$.

Note that
$$\frac{n^n}{k^k(n-k)^{n-k}} = 2^{H(k/n)n}.$$

Lemma 3.4. *Assume that $k, \ell \ge 1$ are integers, $\alpha, \beta \in (0,1)$ real numbers, and $\alpha k \le \ell \le \beta k$. Then*
$$\binom{k}{\ell} \ge \frac{1}{3\sqrt{\ell}} \left(\frac{\beta^{-\alpha}}{(1-\alpha)^{1-\beta}}\right)^k.$$

Proof. For all $n \ge 1$, we have the estimate (see Robbins [21])
$$\sqrt{2\pi} n^{n+1/2} e^{-n} \le n! \le e n^{n+1/2} e^{-n}.$$

Therefore
$$\begin{aligned}\binom{k}{\ell} &\ge \frac{\sqrt{2\pi}}{e^2} \frac{\sqrt{k}}{\sqrt{\ell}\sqrt{k-\ell}} \left(\frac{k}{\ell}\right)^{\ell} \left(\frac{k}{k-\ell}\right)^{k-\ell} \\ &\ge \frac{1}{3\sqrt{\ell}} \left(\frac{1}{\beta}\right)^{\alpha k} \left(\frac{k}{k(1-\alpha)}\right)^{k(1-\beta)}.\end{aligned}$$

This implies the statement. □

In order to prove Theorems 2.1 and 2.2, we combine Lemmas 3.3 and 3.4 and the estimates (6), (7). From (6) and Lemma 3.4,
$$A_{k,1/100} \le 2^{H(4\log 2/100)(k+1)} \ll 2^{0.1829k},$$
which implies the first theorem.

To prove Theorem 2.2, we note that we obtain from (7) and Lemma 3.4, setting $\beta = \log 2/100$ and $\alpha = \beta - \delta$,
$$A_{k,1/100} \ge \binom{k+1}{2\lfloor (1/200)k\log 2\rfloor - 1} \ge \frac{1}{3\sqrt{k+1}} \left(\frac{\beta^{-\alpha}}{(1-\alpha)^{1-\beta}}\right)^k$$
for large k. This implies the statement of Theorem 2.2, noting that
$$\log\bigl(\beta^{-\beta}/(1-\beta)^{1-\beta}\bigr)/\log(2) > 0.056.$$

3.2. Proof of Theorem 2.3

We define $S'(N) = S(N) - d_N/2 = d_1 + \cdots + d_{N-1} + d_N/2$. By splitting the sum into even and odd indices and using the recurrence (4), we obtain
$$\begin{aligned}S'(2N) &= \sum_{k=1}^{N-1} d_{2k} + \sum_{k=0}^{N-1} d_{2k+1} + \frac{d_{2N}}{2} = S'(N) + \frac{1}{2}\sum_{k=0}^{N-1}(d_k + d_{k+1} + 1) \\ &= S'(N) + \frac{1}{2}\sum_{k=1}^{N-1} d_k + \frac{1}{2}\sum_{k=1}^{N} d_k + \frac{N}{2} = 2S'(N) + \frac{N}{2}.\end{aligned} \tag{8}$$

Define
$$R(N)=\frac{1}{N}S'(N)-\frac{1}{4}\log_2 N.$$
By a simple calculation using (8) we obtain
$$R(2N)=R(N). \tag{9}$$
We may therefore define a 1-periodic function ψ defined on the set $\{\log_2 N : N\in\mathbb{N}\}+\mathbb{Z}$ as follows: if $x=\log_2 N+\ell$, where N is odd and $\ell\in\mathbb{Z}$, we set $\psi(x)=R(N)$. Using the identity $R(2N)=R(N)$, it is easy to see that this is well-defined, moreover (2) holds.

We need to show that ψ has a continuous extension to $\mathbb{R}$. Since the points $\{\log_2 N\}$ are dense in $[0,1)$ such a extension is necessarily unique.

We define auxiliary functions $F_k:[2^{k-1},2^k]\to\mathbb{R}$ by $F_k(x)=S'(\lfloor x\rfloor)$. Note that by Theorem A the maximal height of a jump of F_k is $k/3+O(1)$. We define $\psi_k:[0,1]\to\mathbb{R}$ in such a way that
$$\psi_k\big(\{\log_2(x)\}\big)=\frac{1}{x}F_k(x)-\frac{1}{4}\log_2 x \quad\text{for}\quad 2^{k-1}\le x<2^k$$
for $0\le x<1$, and $\psi_k(1)=F_k(2^k)/2^k-k/4$. We have $\psi_k(0)=\psi_k(1)=1/2$ by (9). Note that each $z\in[0,1)$ is hit exactly once by the function $\{\log_2(x)\}$, therefore ψ_k is uniquely determined. Moreover the height of the jumps of $\psi_k:[0,1]\to\mathbb{R}$ is bounded by $O(k/2^k)$. We first show pointwise convergence of the sequence $(\psi_k)_k$. Assume first that $z=\{\log_2 N\}$ or $z=1$. At such points we have identically $\psi_k(\{\log_2 N\})=\psi(\{\log_2(N)\})$ for all k, therefore the statement is clear.

Assume that $z\in[0,1)$ is not of this form. Choose, for each $k\ge 1$,
$$N_k=\max\big\{N\in[2^{k-1},2^k):\{\log_2 N\}\le z\big\}.$$
We consider the sequence of values $\psi_k(\{\log_2 N_k\})$. Note that
$$N_{k+1}\in\{2N_k,2N_k+1\}.$$
Trivially, we have $|\psi_{k+1}\big(\{\log_2(2N_k)\}\big)-\psi_k(\{\log_2 N_k\})|=0$. By (8) we have
$$\begin{aligned}
&\psi_{k+1}\big(\{\log_2(2N_k+1)\}\big)-\psi_k(\{\log_2 N_k\})\\
&=\frac{1}{2N_k+1}S'(2N_k+1)-\frac{1}{4}\log_2(2N_k+1)-\left(\frac{1}{N_k}S'(N_k)-\frac{1}{4}\log_2 N_k\right)\\
&=\frac{1}{2N_k}\left(2S'(N_k)+\frac{N_k}{2}+\frac{d_{2N_k}+d_{2N_k+1}}{2}\right)+\left(\frac{1}{2N_k+1}-\frac{1}{2N_k}\right)S'(2N_k+1)\\
&\quad+\frac{1}{4}\big(\log_2(2N_k)-\log_2(2N_k+1)\big)-\frac{1}{4}\big(\log_2(2N_k)-\log_2 N_k\big)-\frac{S'(N_k)}{N_k}
\end{aligned}$$

$$= \frac{d_{2N_k} + d_{2N_k+1}}{4N_k} - \frac{S'(2N_k+1)}{2N_k(2N_k+1)} + \frac{1}{4}\bigl(\log_2(2N_k) - \log_2(2N_k+1)\bigr).$$

Using the estimate $S'(N_k) = O(N_k \log(N_k))$, which follows from Theorem A, we obtain

$$\bigl|\psi_{k+1}(\{\log_2 N_{k+1}\}) - \psi_k(\{\log_2 N_k\})\bigr| \le C' \frac{\log N_k}{N_k} \le C\frac{k}{2^k}, \tag{10}$$

where the constant C is independent of z.

Moreover, let $x \in [2^{k-1}, 2^k)$ be such that $z = \{\log_2 x\}$. Note that $N_k < x < N_k + 1$. We have

$$\begin{aligned} \bigl|\psi_k(\{\log_2 x\}) - \psi_k(\{\log_2 N_k\})\bigr| &\le \left|\frac{1}{x}S'(\lfloor x\rfloor) - \frac{1}{N_k}S'(N_k)\right| + \frac{1}{4}\bigl|\log_2 x - \log_2 N_k\bigr| \\ &\le S'(N_k)\left(\frac{1}{N_k} - \frac{1}{x}\right) + \frac{1}{4N_k} \le C'' \frac{\log N_k}{N_k} \le C\frac{k}{2^k}, \end{aligned} \tag{11}$$

where the constant C, without loss of generality, is the same as in (10).

We define $K_\ell = C\sum_{i\ge\ell} \frac{i}{2^i}$. We note that $K_\ell \to 0$ as $\ell \to \infty$. Let I_k be the symmetric interval of length $2K_k$ around $\psi_k(\{\log_2 N_k\})$. By (10) and the triangle inequality we have $I_{k+1} \subseteq I_k$, moreover (11) implies $\psi_k(z) \in I_k$. By the nested intervals theorem the sequence $(\psi_k)_{k\ge1}$ converges pointwise, and at $x = \{\log_2 N\}$ this limit equals $\psi(x)$. This limiting function is therefore an extension of ψ, and we call it ψ as well, by abuse of notation. Since both of $\psi_k(z)$ and $\psi(z)$ lie in the interval $\psi_k(\{\log_2 N_k\}) \pm K_k = I_k$, the number $\psi_k(z)$ lies in the interval $\psi(z) \pm 2K_k$ for all $z \in [0,1)$ and $k \ge 1$, therefore the sequence $(\psi_k)_{k\ge1}$ of functions converges uniformly to ψ. We need to show continuity of ψ. Let $z \in [0,1]$ and assume that $\varepsilon > 0$. Choose k so large that the height of the jumps of ψ_k is bounded by $\varepsilon/6$ and also such that $\sup_{0\le y\le1}|\psi(y) - \psi_k(y)| < \varepsilon/3$. There exists a δ such that ψ_k has at most one jump in the interval $[z-\delta, z+\delta]\cap[0,1]$, and by the choice of k we can choose δ so small that for any $y \in [z-\delta, z+\delta] \cap [0,1]$ we have $|\psi_k(y) - \psi_k(z)| < \varepsilon/3$. Application of the triangle inequality finishes the proof of continuity. Moreover, we have $\psi_k(0) = \psi_k(1) = 1/2$, therefore the extension to $\mathbb{R}$ is continuous. □

Remark. We note that similar reasoning can be applied to Stern's diatomic sequence defined by $s_1 = 1$, $s_{2n} = s_n$ and $s_{2n+1} = s_n + s_{n+1}$ for $n \ge 1$. The partial sums $S'(N) = s_1 + \cdots + s_{N-1} + s_N/2$ satisfy $S'(2N) = 3S'(N)$, moreover the maximum of s_n on dyadic intervals $[2^k, 2^{k+1})$ is F_{k+2}, where F_k is the k-th Fibonacci number (see Lehmer [16] and Lind [17]). We obtain a representation of the partial sums $S(N) = s_1 + \cdots + s_N$:

$$S_N = N^{\log_2 3}\psi(\log_2 N) + \frac{s_N}{2},$$

where ψ is continuous and 1-periodic.

3.3. Proof of Theorem 2.4

The proof is an adaption of the proof of [24, Theorem 2.1]. The central property that we will need in our proof is given by the following lemma.

Lemma 3.5. *Let*

$$A = \begin{pmatrix} 1 & 0 & 0 \\ \alpha & \beta & \gamma \\ 0 & 0 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} \alpha & \beta & \gamma \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$v = \begin{pmatrix} \alpha & \beta & \gamma \end{pmatrix} \quad \text{and} \quad w = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}^T.$$

Then the following identities for 1×3*-matrices hold.*

$$\begin{aligned}
v\;AAA &= 0\cdot v\;AA + (\beta^2+\beta+1)\,v\;A + (-\beta^2-\beta)\,v,\\
w^TA^TA^TA^T &= 0\cdot w^T A^TA^T + (\beta^2+\beta+1)\,w^T A^T + (-\beta^2-\beta)\,w^T,\\[1ex]
v\;AAB &= (\beta+1)\,v\;AB + (-\beta)\,v\;B + 0\cdot v,\\
w^TA^TA^TB^T &= (\beta+1)\,w^T A^TB^T + (-\beta)\,w^T B^T + 0\cdot w^T,\\[1ex]
v\;ABA &= (\beta+1)\,v\;BA + 0\cdot v\;A + (-\beta)\,v,\\
w^TA^TB^TA^T &= (\beta+1)\,w^T B^TA^T + 0\cdot w^T A^T + (-\beta)\,w^T,\\[1ex]
v\;ABB &= (\alpha+1)\,v\;AB + (-\alpha)\,v\;A + 0\cdot v,\\
w^TA^TB^TB^T &= (\alpha+1)\,w^T A^TB^T + (-\alpha)\,w^T A^T + 0\cdot w^T,\\[1ex]
v\;BAA &= (\beta+1)\,v\;BA + (-\beta)\,v\;B + 0\cdot v,\\
w^TB^TA^TA^T &= (\beta+1)\,w^T B^TA^T + (-\beta)\,w^T B^T + 0\cdot w^T,\\[1ex]
v\;BAB &= (\alpha+1)\,v\;AB + 0\cdot v\;B + -\alpha\,v,\\
w^TB^TA^TB^T &= (\alpha+1)\,w^T A^TB^T + 0\cdot w^T B^T + (-\alpha)\,w^T,\\[1ex]
v\;BBA &= (\alpha+1)\,v\;BA + (-\alpha)\,v\;A + 0\cdot v,\\
w^TB^TB^TA^T &= (\alpha+1)\,w^T B^TA^T + (-\alpha)\,w^T A^T + 0\cdot w^T,\\[1ex]
v\;BBB &= 0\cdot v\;BB + (\alpha^2+\alpha+1)\,v\;B + (-\alpha^2-\alpha)\,v,\\
w^TB^TB^TB^T &= 0\cdot w^T B^TB^T + (\alpha^2+\alpha+1)\,w^T B^T + (-\alpha^2-\alpha)\,w^T.
\end{aligned}$$

The proof is too trivial and tiresome to reproduce here. □

Finally, we prove Theorem 2.4 by induction.

First we note that it is sufficient to assume $x_1 = 1$: if $x_1 \neq 0$, we study the sequence x' defined by $x'_n = x_n/x_1$ instead, which satisfies the recurrence with $\alpha, \beta, \gamma/x_1$ instead of α, β, γ. If $x_1 = 0$, we note that, for each nonnegative integer n, x_n depends in a continuous way on x_1, and so this case follows from taking the limit. Set $A(\mathtt{0}) = \left(\begin{smallmatrix} 1 & 0 & 0 \\ \alpha & \beta & \gamma \\ 0 & 0 & 1 \end{smallmatrix}\right)$ and $A(\mathtt{1}) = \left(\begin{smallmatrix} \alpha & \beta & \gamma \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)$. As in [24], we have for odd $n \geq 3$ such that $n = (\varepsilon_\nu \cdots \varepsilon_0)_2$,

$$x_n = \begin{pmatrix} \alpha & \beta & \gamma \end{pmatrix} A(\varepsilon_1) \cdots A(\varepsilon_{\nu-1}) \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}^T \tag{12}$$

and the statement of the theorem is equivalent to the assertion that

$$\begin{aligned} &\begin{pmatrix} \alpha & \beta & \gamma \end{pmatrix} A(\varepsilon_1) \cdots A(\varepsilon_{\nu-1}) \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}^T \\ &\qquad = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} A(\varepsilon_1)^T \cdots A(\varepsilon_{\nu-1})^T \begin{pmatrix} \alpha & \beta & \gamma \end{pmatrix}^T \end{aligned} \tag{13}$$

for all $\nu \geq 1$ and all finite sequences $(\varepsilon_1, \ldots, \varepsilon_{\nu-1})$ in $\{\mathtt{0}, \mathtt{1}\}$. This can be checked for $\nu \leq 3$ by simple calculation. Let therefore $\nu \geq 4$. Assume that $\varepsilon_1\varepsilon_2\varepsilon_3 = \mathtt{000}$. We consider the first pair of identities in Lemma 3.5. We multiply the first of these equations by $A(\varepsilon_4) \cdots A(\varepsilon_{\nu-1})w$ from the right and the second one by $A(\varepsilon_4)^T \cdots A(\varepsilon_{\nu-1})^T v^T$, also from the right. Then the left hand sides give the two constituents of (13), and the right hand sides are equal by the induction hypothesis. The other 7 cases are analogous, and the proof Theorem 2.4 is complete. $\square$

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