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ON M. B. LEVIN'S PROOFS FOR THE EXACT LOWER DISCREPANCY BOUNDS OF SPECIAL SEQUENCES AND POINT SETS (A SURVEY)

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ABSTRACT. The goal of this overview article is to give a tangible presentation of the breakthrough works in discrepancy theory [3, 5] by M. B. Levin. These works provide proofs for the exact lower discrepancy bounds of Halton's sequence and a certain class of (t, s)-sequences. Our survey aims at highlighting the major ideas of the proofs and we discuss further implications of the employed methods. Moreover, we derive extensions of Levin's results.

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1. Introduction and statement of main results

In [3] and [5] M. B. Levin proved optimal lower discrepancy bounds for certain shifted (t, m, s)-nets and for the s-dimensional Halton sequence. The main ideas of these proofs are also basis for later, even deeper works of Levin on this topic, see [4, 6]. However, these papers will not be discussed in our survey. In [3] and [5] Levin showed the subsequent Theorems 1 and 2, which we will state below in a simplified version. We start with fixing the notation for basic quantities and concepts, which will be needed for the formulation of Levin's results and of our extensions.

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Let $(\boldsymbol{x}_n)_{n\in\mathbb{N}}$ be an infinite sequence in the s-dimensional unit cube $[0,1)^s$,

$$\boldsymbol{y} = \left(y^{(1)}, \dots, y^{(s)}\right),$$

and

$$[\boldsymbol{0}, \boldsymbol{y}) = \left[0, y^{(1)}\right) \times \cdots \times \left[0, y^{(s)}\right) \subseteq [0,1)^s.$$

We call $\Delta\big(\cdot, (\boldsymbol{x}_n)_{n=1}^N\big) : [0,1]^s \to \mathbb{R}$,

$$\Delta\big(\boldsymbol{y}, (\boldsymbol{x}_n)_{n=1}^N\big) = \sum_{n=1}^{N} \big(\chi_{[\boldsymbol{0},\boldsymbol{y})}(\boldsymbol{x}_n) - y^{(1)} \cdots y^{(s)}\big),$$

the discrepancy function of the sequence $(\boldsymbol{x}_n)_{n\in\mathbb{N}}$. We define the star-discrepancy of an N-point set $(\boldsymbol{x}_n)_{n=1}^N$ as

$$D^*\big((\boldsymbol{x}_n)_{n=1}^N\big) = \sup_{\boldsymbol{y}\in[0,1)^s} \left| \frac{1}{N} \Delta\big(\boldsymbol{y}, (\boldsymbol{x}_n)_{n=1}^N\big) \right|.$$

Further, we need the definition of a (t,m,s)-net in base b introduced by H. Niederreiter [2] and the so-called d-admissibility property of nets.

Definition 1. For integers $b \geq 2$, $s \geq 1$, m and t, with $0 \leq t \leq m$, a (t,m,s)-*net in base* b is defined as a set of points $\mathcal{P} = \{\boldsymbol{x}_0, \dots, \boldsymbol{x}_{b^m-1}\}$ in $[0,1)^s$, which satisfies the condition that every interval with volume b^{-m+t} of the form

$$\mathcal{J} = \prod_{i=1}^{s} \left[\frac{a_i}{b^{d_i}}, \frac{a_i+1}{b^{d_i}}\right) \quad \text{with} \quad d_i \in \mathbb{N}_0,\ a_i \in \{0,1,\dots,b^{d_i}-1\}, \quad \text{for } i = 1,\dots,s,$$

contains exactly b^t points of $\mathcal{P}$. We will call these intervals $\mathcal{J}$ elementary intervals.

Definition 2. For $x = \sum_{i\geq 1} \frac{x_i}{b^i}$, where $x_i \in \{0,1,\dots,b-1\}$ and $m \in \mathbb{N}$, the truncation is defined as

$$[x]_m = \sum_{i=1}^{m} \frac{x_i}{b^i}.$$

For $\boldsymbol{x} = \big(x^{(1)}, \dots, x^{(s)}\big)$ the truncation is defined as $[\boldsymbol{x}]_m = \big([x^{(1)}]_m, \dots, [x^{(s)}]_m\big)$. Moreover, we define $[x]_0 := 0$.

Keep in mind that for an arbitrary number $x \in \mathbb{R}$, $[x]$ denotes the integer part of x. For the next definition recall the concept of the digital shift.

For a point $x = \sum_{i\geq 1} \frac{x_i}{b^i}$ and a shift $\sigma = \sum_{i\geq 1} \frac{\sigma_i}{b^i}$ we have that

$$x \oplus \sigma := \sum_{i\geq 1} \frac{y_i}{b^i}, \qquad \text{where} \qquad y_i \equiv x_i + \sigma_i \bmod b$$

and analogously,

$$x \ominus \sigma := \sum_{i\geq 1} \frac{y_i}{b^i}, \qquad \text{where} \qquad y_i \equiv x_i - \sigma_i \bmod b.$$

For $\boldsymbol{x} = \big(x^{(1)}, \dots, x^{(s)}\big)$ and $\boldsymbol{\sigma} = \big(\sigma^{(1)}, \dots, \sigma^{(s)}\big)$ the b-adic digitally shifted point is defined by $\boldsymbol{x} \oplus \boldsymbol{\sigma} = \big(x^{(1)} \oplus \sigma^{(1)}, \dots, x^{(s)} \oplus \sigma^{(s)}\big)$. Analogously, we define $\boldsymbol{x} \ominus \boldsymbol{\sigma}$.

Definition 3. For $x = \sum_{i\geq 1} \frac{x_i}{b^i}$, where $x_i = 0$ for $i = 1, \dots, k$ and $x_{k+1} \neq 0$, the absolute valuation of x is defined as

$$\|x\|_b = \frac{1}{b^{k+1}}.$$

For $\boldsymbol{x} = (x^{(1)}, \dots, x^{(s)})$ the absolute valuation is defined as $\|\boldsymbol{x}\|_b := \prod_{j=1}^{s} \|x^{(j)}\|_b$.

With this definition we can introduce point sets with a special property which is essential for the further considerations of this chapter.

Definition 4. For an integer d, we say that a point set $\mathcal{P} = \{\boldsymbol{x}_0, \dots, \boldsymbol{x}_{b^m-1}\}$ in $[0,1)^s$ is d-admissible in base b if

$$\min_{0\leq k<n<b^m} \|\boldsymbol{x}_n \ominus \boldsymbol{x}_k\|_b > \frac{1}{b^{m+d}}.$$

We remind the definition of the Halton sequence in bases $b_1, \dots, b_s$, where $s \geq 1$. Throughout this survey all occurring bases $b_1, \dots, b_s$, are assumed to be pairwise coprime integers.

Definition 5. Let $b_1, \dots, b_s,\ b_i \geq 2$ $(i = 1, \dots, s)$, for some dimension $s \geq 1$, be integers. Then the *s-dimensional Halton sequence in bases* $b_1, \dots, b_s$, denoted by $\big(H_s(n)\big)_{n\in\mathbb{N}_0}$, is defined as

$$H_s(n) := \big(\phi_{b_1}(n), \dots, \phi_{b_s}(n)\big), \quad n = 0, 1, \dots,$$

where ϕ_{b_i} denotes the radical inverse function in base b_i, i.e, the function $\phi_{b_i} : \mathbb{N}_0 \to [0,1)$, defined as

$$\phi_{b_i}(n) := \sum_{j=0}^{\infty} n_j b_i^{-j-1},$$

where $n = n_0 + n_1 b_i + n_2 b_i^2 + \cdots,$ with $n_0, n_1, n_2, \dots \in \{0, 1, \dots, b_i - 1\}$.

It is well known in discrepancy theory that the Halton sequence (requiring that the underlying bases are pairwise coprime) is a low discrepancy sequence, i.e., the star-discrepancy is of order $\mathcal{O}\big(\frac{(\log N)^s}{N}\big)$ (see, e.g., [1]). Succeeding in showing that the discrepancy of the Halton sequence satisfies $D^*\big((H_s(n))_{n=1}^N\big) \geq c_s \frac{(\log N)^s}{N}$, for infinitely many N, with a constant $c_s > 0$, would prove that this order is exact.

For (t, m, s)-nets in base b, denoted by $\mathcal{P}$, we know that their discrepancy always satisfies $D^*(\mathcal{P}) \leq c_{s,b} b^t \frac{(\log N)^{s-1}}{N}$. We will show that the order $\mathcal{O}\big(\frac{(\log N)^{s-1}}{N}\big)$ is exact for certain (t, m, s)-nets.

Now, we can state Levin's main results from [3] and [5] (in a simplified form).

Theorem 1. *Let $s \geq 2, d \geq 1, m \geq 9(d+t)(s-1)^2$ and let $(\boldsymbol{x}_n)_{0 \leq n < b^m}$ be a d-admissible (t, m, s)-net in base b. Then, we can provide an explicitly given $\boldsymbol{w}$ such that*

$$b^m D^*\big((\boldsymbol{x}_n \oplus \boldsymbol{w})_{0 \leq n < b^m}\big) \geq \frac{(4(d+t)(s-1)^2)^{-s+1}}{b^d} m^{s-1}.$$

In particular, we have

$$D^*\big((\boldsymbol{x}_n \oplus \boldsymbol{w})_{0 \leq n < N}\big) \geq c_{s,d} \frac{(\log N)^{s-1}}{N},$$

with a constant $c_{s,d} > 0$ and $N = b^m$.

Theorem 2. *Put $B = b_1 \cdots b_s$, $s \geq 2$ and $m_0 = \lfloor 2B \log_2 B \rfloor + 2$, then the estimate for the star-discrepancy of the Halton sequence*

$$\sup_{1 \leq N \leq 2^{m m_0}} N D^*\big((H_s(n))_{n=1}^N\big) \geq m^s (8B)^{-1},$$

is valid for $m \geq B$. In particular, there exists some constant $c_s > 0$, such that

$$D^*\big((H_s(n))_{n=1}^N\big) \geq c_s \frac{(\log N)^s}{N}, \text{ for infinitely many } N \in \mathbb{N}.$$

The implied constant c_s also depends on the bases but not on N.

The aim of this paper is two-fold. **First**, we will give an easier and simpler access to the ideas of Levin. To this end, we are eager to give a clear and illustrative re-proof of Theorems 1 and 2. We use absolutely the same ideas as Levin, but focus on a clearer presentation. To achieve this goal, we restrict the re-proof of Theorem 1 to the two-dimensional case and carry out the steps in detail. For this case of course, the exact lower discrepancy bound follows (for an arbitrary $\boldsymbol{w}$) by the general lower bound for the discrepancy of two-dimensional point sets by W. M. Schmidt [7]. For simplicity we will also restrict ourselves to base $b = 2$. Moreover, we focus on the optimal quality parameter $t = 0$ and for ease of presentation we formulate and prove the result for $m \equiv 0 \bmod 4$. We also state the result without the shift and require a certain condition on $\boldsymbol{x}_0$ instead. (The ideas for the proof in the general case are the same as in this special version.) This gives Theorem 3:

Theorem 3. *Let $(\boldsymbol{x}_n)_{0 \le n < 2^m}$ be a $(0, m, 2)$-net in the base 2 with $m \ge 4$, $m \equiv 0 \bmod 4$ and $\boldsymbol{x}_0 = \boldsymbol{\gamma} = (\gamma^{(1)}, \gamma^{(2)})$,*

$$\gamma^{(1)} = \frac{1}{2^2} + \frac{1}{2^4} + \cdots + \frac{1}{2^{m/2}},$$
$$\gamma^{(2)} = \frac{1}{2^{m/2+2}} + \frac{1}{2^{m/2+4}} + \cdots + \frac{1}{2^m}.$$

Then it holds for the interval $J_{\boldsymbol{\gamma}} = [0, \gamma^{(1)}) \times [0, \gamma^{(2)})$ that

$$\frac{1}{N} \Delta\big(\boldsymbol{\gamma}, (\boldsymbol{x}_n)_{0 \le n < 2^m}\big) \le -\frac{1}{4} \frac{1}{2^{m+2}} m,$$

and consequently,

$$D^*\big((\boldsymbol{x}_n)_{0 \le n < N}\big) \ge \frac{1}{16 \log 2} \frac{\log N}{N}, \quad \textit{with} \quad N = 2^m.$$

The **second aim** is to give, in a certain sense, a quantitative extension of Theorems 1 and 2. We will show:

Theorem 4. *Let $m \ge 2s^s(s-1)^s$. Then, there is a set $\Gamma \subseteq [0,1)^s$, $s \ge 2$, with the following properties:*

- *For all $\boldsymbol{x} \in [0,1)^s$ there exists a $\boldsymbol{\gamma} \in \Gamma$ with*

$$\|\boldsymbol{x} - \boldsymbol{\gamma}\| < b\sqrt{s} \frac{1}{b^{\frac{m}{2(s-1)s}}}.$$

  Here, $\|\cdot\|$ denotes the euclidean norm.

- *If $\mathcal{P} = \{\boldsymbol{x}_0, \dots, \boldsymbol{x}_{b^m-1}\}$ is a $(0, m, s)$-net in base b, and if $\boldsymbol{x}_i \in \Gamma$ for some $i \in \{0, \dots, b^m - 1\}$, then, with $N = b^m$,*

$$D^*(\mathcal{P}) \ge \frac{(b-1)^s (2s-3)^{s-1}}{b^s (4s^2(s-1)^2 \log b)^{s-1}} \frac{(\log N)^{s-1}}{N}.$$

Theorem 5. *There are constants c_1 and $c_2 > 0$, such that for infinitely many N there exists a set $\Lambda_N \subseteq [0,1)^2$ with the following properties:*

- *We have $\lambda_2(\Lambda_N) \ge c_1$, where λ_2 denotes the 2-dimensional Lebesgue measure.*
- *For all $\boldsymbol{x} \in \Lambda_N$ there exists a $\boldsymbol{y} \in [0,1)^2$ with $\|\boldsymbol{x} - \boldsymbol{y}\| < \sqrt{8} \frac{1}{N^{\frac{1}{14}}}$ and*

$$\big|\Delta\big(\boldsymbol{y}, (H_2(n))_{n=1}^N\big)\big| \ge c_2 (\log N)^2.$$

Remark 1. An analogous result can be obtained for arbitrary dimensions. For sake of simplicity our considerations will be restricted to the two-dimensional case. The basic ideas become better visible in this case and can be adopted to higher dimensions in a straightforward manner.

The remainder of this paper is organised as follows: In Chapter 2, we will discuss the d-admissibility property in more detail. Of course, the proof of Theorem 3 will be the major part of this chapter. We relax some of the conditions of Theorem 3 in Chapter 3 and derive a more general result (Theorem 4). In Chapter 4, we will prove Theorem 2 in detail. Chapter 5 will be solely dedicated to the proof of Theorem 5.

2. Remarks on admissibility of nets and Re-proof of Theorem 3

Before stating the proof of Theorem 3, we discuss the d-admissibility property for $(0, m, s)$-nets, since in this theorem we restrict ourselves to the quality parameter $t = 0$.

Lemma 2.1. *A point set $\mathcal{P} = \{\boldsymbol{x}_0, \dots, \boldsymbol{x}_{b^m-1}\}$ in $[0,1)^s$ is s-admissible if and only if $\mathcal{P}$ is a $(0, m, s)$-net in base b. Moreover, $\mathcal{P}$ cannot be d-admissible for $d < s$.*

Proof. Let $\mathcal{P}$ be a $(0, m, s)$-net in base b. First, we show that

$$\frac{1}{b^{m+s-1}} \geq \min_{0 \leq k < n < b^m} \|\boldsymbol{x}_n \ominus \boldsymbol{x}_k\|_b,$$

by taking special elementary intervals into account. Since $\mathcal{P}$ is a $(0, m, s)$-net, we know by definition that every elementary interval of order m in base b, i.e., every elementary interval with volume $\frac{1}{b^m}$, contains exactly one point of $\mathcal{P}$. Therefore, this is also true for intervals of the form

$$\left[\frac{k}{b^m}, \frac{k+1}{b^m}\right) \times [0,1)^{s-1}, \qquad k \in \{0, \dots, b^m - 1\}.$$

Now let $\boldsymbol{x} = (x^{(1)}, \dots, x^{(s)})$ be the unique point of $\mathcal{P}$ for which it holds that $x^{(1)} \in \left[0, \frac{1}{b^m}\right)$. Moreover, let $\boldsymbol{y} = (y^{(1)}, \dots, y^{(s)})$ be the point of $\mathcal{P}$ such that $y^{(1)} \in \left[\frac{b-1}{b^m}, \frac{b}{b^m}\right)$. This is equivalent to

$$0 \leq x^{(1)} < \frac{1}{b^m}, \qquad \frac{b-1}{b^m} \leq y^{(1)} < \frac{1}{b^{m-1}}.$$

Therefore, we know that $x^{(1)}$ and $y^{(1)}$ can be written as

$$\begin{aligned} x^{(1)} &= \frac{\alpha_1}{b^{m+1}} + \frac{\alpha_2}{b^{m+2}} + \cdots , \\ y^{(1)} &= \frac{b-1}{b^m} + \frac{\beta_1}{b^{m+1}} + \frac{\beta_2}{b^{m+2}} + \cdots , \end{aligned}$$

where $\alpha_i, \beta_i \in \{0, 1, \ldots, b-1\}$ for $i \geq 1$. Thus, $\|y^{(1)} \ominus x^{(1)}\|_b = \frac{1}{b^m}$. Moreover, for $x^{(i)}$ and $y^{(i)}$, $i = 2, \ldots, s$, it holds that $\|y^{(i)} \ominus x^{(i)}\|_b \leq \frac{1}{b}$. Therefore, it follows, that

$$\|\boldsymbol{y} \ominus \boldsymbol{x}\|_b \leq \frac{1}{b^{m+s-1}}.$$

If we can prove that $\min_{0 \leq k < n < b^m} \|\boldsymbol{x}_n \ominus \boldsymbol{x}_k\|_b > \frac{1}{b^{m+s}}$, then the first implication of the assertion immediately follows. Suppose that there exist points

$$\boldsymbol{x} = \left(x^{(1)}, \ldots, x^{(s)}\right), \boldsymbol{x} \in \mathcal{P} \quad \text{and} \quad \boldsymbol{y} = \left(y^{(1)}, \ldots, y^{(s)}\right), \boldsymbol{y} \in \mathcal{P}$$

such that $\|\boldsymbol{y} \ominus \boldsymbol{x}\|_b \leq \frac{1}{b^{m+s}}$. Then, there exist integers $l^{(1)}, \ldots, l^{(s-1)}$ such that

$$\|y^{(i)} \ominus x^{(i)}\|_b \leq \frac{1}{b^{l^{(i)}}}, \quad \text{for } i = 1, \ldots, s-1,$$

and

$$\|y^{(s)} \ominus x^{(s)}\|_b \leq \frac{1}{b^{m+s-l^{(1)}-\ldots-l^{(s-1)}}}.$$

This implies that the first $l^{(i)} - 1$ digits of the b-adic expansion of $x^{(i)}$ and $y^{(i)}$, $i = 1, \ldots, s-1$ are identical. Also, the first $m + s - l^{(1)} - \cdots - l^{(s-1)} - 1$ digits of the b-adic expansion of $x^{(s)}$ and $y^{(s)}$ are identical. Consequently, $\boldsymbol{x}$ and $\boldsymbol{y}$ are contained in an elementary interval of volume $\frac{1}{b^m}$. This contradicts our assumption that $\mathcal{P}$ is a $(0, m, s)$-net.

Let now $\mathcal{P}$ be an arbitrary b^m-point set in $[0, 1)^s$ which is not a $(0, m, s)$-net. Then there exists an elementary interval $\mathcal{J}_1 \subseteq [0, 1)^s$ of volume $1/b^m$ which contains no point of $\mathcal{P}$ or at least two points of $\mathcal{P}$. In the second case it immediately follows (by the same considerations as above) that $\mathcal{P}$ is not s-admissible. Consider now the first case: We can partition $[0, 1)^s$ into b^m elementary intervals $\mathcal{J}_i$ of the same shape as $\mathcal{J}_1$. Since $\mathcal{J}_1$ contains no point of $\mathcal{P}$ there exists at least one i such that $\mathcal{J}_i$ contains at least two points, and this again contradicts the s-admissibility. □

Remark 2. Note, that it might happen that a $(1, m, s)$-net in base b is non-admissible for any integer d. To see this, just take b copies of a $(0, m-1, s)$-net in base b. This gives an example of a $(1, m, s)$-net in base b which is not d-admissible for any $d \in \mathbb{N}$.

These preliminary considerations put us in the position to prove Theorem 3. In Chapter 3 we give the proof for a more general result in the general case. Note, that for (t, m, s)-nets with nonzero quality parameter the d-admissibility condition has to be required additionally. The idea underlying the proof of the theorem in the general case is exactly the same.

Proof of Theorem 3. Note that by Lemma 2.1 $(\boldsymbol{x}_n)_{0\le n<2^m}$ is 2-admissible. To begin with, we want to find a suitable partition of the interval $J_{\boldsymbol{\gamma}}$. Let therefore $\boldsymbol{r} = (r_1, r_2) \in \mathbb{N}^2$. For

$$r_1 = 2j_1 \quad \text{and} \quad r_2 = m/2 + 2j_2 \quad \text{with} \quad j_1, j_2 \in \{1, \dots, m/4\}$$

it holds that

$$\gamma^{(1)} = \sum_{r_1} \frac{1}{2^{r_1}} \quad \text{and} \quad \gamma^{(2)} = \sum_{r_2} \frac{1}{2^{r_2}}.$$

Now define the set A which contains all combinations of the indices r_1 and r_2, i.e.,

$$A = \big\{(r_1, r_2) |\; r_1 = 2j_1,\; r_2 = m/2 + 2j_2,\; j_1, j_2 \in \{1, \dots, m/4\}\big\}.$$

The partition of $J_{\boldsymbol{\gamma}}$ is then given by

$$J_{\boldsymbol{r},\boldsymbol{\gamma}} = \left[\big[\gamma^{(1)}\big]_{r_1-1}, \big[\gamma^{(1)}\big]_{r_1-1} + \frac{1}{2^{r_1}}\right) \times \left[\big[\gamma^{(2)}\big]_{r_2-1}, \big[\gamma^{(2)}\big]_{r_2-1} + \frac{1}{2^{r_2}}\right),$$

for $(r_1, r_2) \in A$. Furthermore, let

$$\begin{aligned} A_1 &= \{\boldsymbol{r} \in A |\; r_1 + r_2 \le m\}, \\ A_2 &= \{\boldsymbol{r} \in A |\; r_1 + r_2 = m + 1\}, \\ A_3 &= \{\boldsymbol{r} \in A |\; r_1 + r_2 \ge m + 2\}, \end{aligned}$$

such that $A = A_1 \cup A_2 \cup A_3$. The intervals $J_{\boldsymbol{r},\boldsymbol{\gamma}}$ are elementary intervals in base 2 with volume $\frac{1}{2^{r_1+r_2}}$, i.e., of order $r_1 + r_2$. Moreover, all $J_{\boldsymbol{r},\boldsymbol{\gamma}}$ are disjoint and therefore, we obtain with

$$\mathcal{A}(\boldsymbol{r}) := \sum_{n=0}^{2^m-1} \chi_{J_{\boldsymbol{r},\boldsymbol{\gamma}}}(\boldsymbol{x}_n),$$

$$\begin{aligned} \frac{1}{N}\Delta\big(\boldsymbol{\gamma}, (\boldsymbol{x}_n)_{0\le n<2^m}\big) &= \sum_{\boldsymbol{r}\in A} \left(\frac{\mathcal{A}(\boldsymbol{r})}{2^m} - \lambda_2(J_{\boldsymbol{r},\boldsymbol{\gamma}})\right) = \sum_{\boldsymbol{r}\in A_1} \left(\frac{\mathcal{A}(\boldsymbol{r})}{2^m} - \lambda_2(J_{\boldsymbol{r},\boldsymbol{\gamma}})\right) \\ &\quad + \sum_{\boldsymbol{r}\in A_2} \left(\frac{\mathcal{A}(\boldsymbol{r})}{2^m} - \lambda_2(J_{\boldsymbol{r},\boldsymbol{\gamma}})\right) + \sum_{\boldsymbol{r}\in A_3} \left(\frac{\mathcal{A}(\boldsymbol{r})}{2^m} - \lambda_2(J_{\boldsymbol{r},\boldsymbol{\gamma}})\right) \\ &=: \Delta_1(\boldsymbol{\gamma}) + \Delta_2(\boldsymbol{\gamma}) + \Delta_3(\boldsymbol{\gamma}). \end{aligned}$$

Consider Δ_1. Since $(\boldsymbol{x}_n)_{0\le n<2^m}$ is a $(0,m,2)$-net, it is fair with respect to all elementary intervals of order $\le m$. For $\boldsymbol{r}\in A_1$ it holds that $r_1+r_2\le m$ and therefore

$$\Delta_1(\boldsymbol{\gamma})=\sum_{\boldsymbol{r}\in A_1}\frac{\mathcal{A}(\boldsymbol{r})}{2^m}-\lambda_2(J_{\boldsymbol{r},\boldsymbol{\gamma}})=0.$$

Consider Δ_2. From the condition that $\boldsymbol{r}\in A_2\subseteq A$ we get that

$$r_1=2j_1\qquad\text{and}\qquad r_2=m/2+2j_2,$$

where $j_1,j_2\in\{1,\dots,m/4\}$. It follows that

$$r_1+r_2=m+2(j_1+j_2-m/4).$$

Since $j_1+j_2-m/4\in\mathbb{Z}$ we know that $2(j_1+j_2-m/4)\neq 1$ which is a contradiction to the assumption that $r_1+r_2=m+1$ for all $\boldsymbol{r}\in A_2$. Therefore, $A_2=\emptyset$ and $\Delta_2=0$.

Consider Δ_3. As a first step we want to show that $J_{\boldsymbol{r},\boldsymbol{\gamma}}$ with $r_1+r_2\ge m+2$ cannot contain any point of $(\boldsymbol{x}_n)_{0\le n<2^m}$ and we will do that by deriving a contradiction.

Suppose there exists $\boldsymbol{x}_k\in J_{\boldsymbol{r},\boldsymbol{\gamma}}$ for some $k<2^m$ and some $\boldsymbol{r}\in A_3$. Then we know for the first coordinate

$$\left[\gamma^{(1)}\right]_{r_1-1}\le x_k^{(1)}<\left[\gamma^{(1)}\right]_{r_1-1}+\frac{1}{2^{r_1}}$$

which is equivalent to

$$\begin{aligned}\frac{1}{2^2}+\frac{1}{2^4}+\dots+\frac{1}{2^{r_1-2}}\le\frac{x_{k,1}^{(1)}}{2}+\dots+\frac{x_{k,r_1-1}^{(1)}}{2^{r_1-1}}+\frac{x_{k,r_1}^{(1)}}{2^{r_1}}+\cdots\\<\frac{1}{2^2}+\frac{1}{2^4}+\dots+\frac{1}{2^{r_1-2}}+\frac{1}{2^{r_1}}.\end{aligned}$$

Therefore, it has to hold that

$$x_{k,2}^{(1)}=x_{k,4}^{(1)}=\dots=x_{k,r_1-2}^{(1)}=1\quad\text{and}\quad x_{k,1}^{(1)}=x_{k,3}^{(1)}=\dots=x_{k,r_1-1}^{(1)}=0.$$

An analogous procedure can be done for the second coordinate. Hence,

$$\left[\gamma^{(1)}\right]_{r_1-1}=\left[x_k^{(1)}\right]_{r_1-1}\qquad\text{and}\qquad\left[\gamma^{(2)}\right]_{r_2-1}=\left[x_k^{(2)}\right]_{r_2-1}.\tag{2.1}$$

Combining (2.1) and the assumption that $\boldsymbol{x}_0=\boldsymbol{\gamma}$ leads to

$$\left[(\boldsymbol{x}_k\ominus\boldsymbol{x}_0)^{(1)}\right]_{r_1-1}=0\qquad\text{and}\qquad\left[(\boldsymbol{x}_k\ominus\boldsymbol{x}_0)^{(2)}\right]_{r_2-1}=0.$$

Thus, we get $\|x_k^{(i)}\ominus x_0^{(i)}\|_2\le\frac{1}{2^{r_i}}$. Since $\boldsymbol{r}\in A_3$, i.e., $r_1+r_2\ge m+2$, it follows that

$$\|\boldsymbol{x}_k\ominus\boldsymbol{x}_0\|_2\le\frac{1}{2^{r_1+r_2}}\le\frac{1}{2^{m+2}}.$$

This is a contradiction to the assumption that $(\boldsymbol{x}_n)_{0\le n<2^m}$ is a 2-admissible $(0,m,2)$-net in base 2.

Hence, $\mathcal{A}(\boldsymbol{r}) = 0$ for all $\boldsymbol{r} \in A_3$ and

$$\begin{aligned}\Delta_3(\boldsymbol{\gamma}) &= \sum_{\boldsymbol{r}\in A_3} \left(\frac{\mathcal{A}(\boldsymbol{r})}{2^m} - \lambda_2(J_{\boldsymbol{r},\boldsymbol{\gamma}}) \right) \\ &= -\sum_{\boldsymbol{r}\in A_3} \frac{1}{2^{r_1+r_2}} \le -\sum_{\substack{\boldsymbol{r}\in A_3 \\ r_1+r_2=m+2}} \frac{1}{2^{m+2}} = -|A_4| \frac{1}{2^{m+2}}\end{aligned}$$

with

$$A_4 = \{\boldsymbol{r} \in A_3 |\ r_1 + r_2 = m+2\}.$$

It is easy to see that

$$|A_4| = \frac{m}{4} \quad \text{for } \ m \ge 4 \quad \text{and} \quad m \equiv 0 \bmod 4,$$

and so we finally get

$$\begin{aligned}\frac{1}{N}\Delta\big(\boldsymbol{\gamma}, (\boldsymbol{x}_n)_{0\le n<2^m}\big) &= \Delta_3(\boldsymbol{\gamma}) \le -\frac{1}{2^{m+2}}|A_4| \\ &= -\frac{1}{4}\frac{1}{2^{m+2}}m.\end{aligned}$$

□

3. Proof of Theorem 4

The first aim of this section is to focus on the assumption of Theorem 3 that there exists a point $\boldsymbol{x}_0 \in \mathcal{P}$ such that $\boldsymbol{x}_0 = \boldsymbol{\gamma}$ (of course the condition $\boldsymbol{x}_0 = \boldsymbol{\gamma}$ can be replaced by $\boldsymbol{x}_n = \boldsymbol{\gamma}$ for any $n \in \{0, \dots, 2^m - 1\}$). This restriction on the point set is weakened by showing that there are many possible choices for $\boldsymbol{\gamma}$ such that the proof of Theorem 3 can still be performed in an analogous way. In fact, it turns out that $\boldsymbol{\gamma}$ only has to fulfill some simple properties as the following lemma shows:

Lemma 3.1. *Let $(\boldsymbol{x}_n)_{0\le n<b^m}$ be a $(0,m,s)$-net in base b. Let*

$$\boldsymbol{x}_0 \in \prod_{j=1}^{s} [\gamma^{(j)}, \gamma^{(j)} + \frac{1}{b^{\max(R_j)}}),$$

where

$$\gamma^{(j)} = \sum_{r\in R_j} \frac{a_r^{(j)}}{b^r},$$

$a_r^{(j)} \in \{1, 2, \dots, b-1\}$ and $R_j \subseteq \{1, 2, \dots, m\}$ for $j = 1, \dots, s$. Here the R_j are arbitrary, but for $\boldsymbol{r} = (r_1, r_2, ..., r_s) \in R_1 \times R_2 \times ... \times R_s$, the following constraints need to be satisfied:

- $|\{\boldsymbol{r}|\ m+1 \leq \sum_{j=1}^{s} r_j < m+s\}| \leq \frac{m^{s-1}}{\delta}$,
- $|\{\boldsymbol{r}|\ \sum_{j=1}^{s} r_j = m+\alpha\}| \geq \frac{m^{s-1}}{\beta}$,

for some constant $\beta > 0$, some integer $\alpha \geq s$ and for $\delta > \frac{b^{\alpha}(b^{s-1}-1)\beta}{b^{s-1}}$. Then, it holds for the interval $J_{\boldsymbol{\gamma}} = \prod_{j=1}^{s}[0, \gamma^{(j)})$ that

$$\frac{1}{N}\Delta\big(\boldsymbol{\gamma}, (\boldsymbol{x}_n)_{0\leq n<b^m}\big) \leq -\frac{m^{s-1}}{b^m}\left(-\frac{(b-1)^s}{\delta}\frac{b^{s-1}-1}{b^{s-1}} + \frac{(b-1)^s}{\beta}\frac{1}{b^{\alpha}}\right),$$

where

$$\left(-\frac{(b-1)^s}{\delta}\frac{b^{s-1}-1}{b^{s-1}} + \frac{(b-1)^s}{\beta}\frac{1}{b^{\alpha}}\right) > 0.$$

P r o o f. Let $A = \{\boldsymbol{r}|\ r_j \in R_j,\ j = 1, \ldots, s\}$ be the set of indices which can be split into three disjoint subsets

$$A_1 = \{\boldsymbol{r} \in A|\ \sum_{j=1}^{s} r_j \leq m\},$$
$$A_2 = \{\boldsymbol{r} \in A|\ m+1 \leq \sum_{j=1}^{s} r_j < m+s\},$$
$$A_3 = \{\boldsymbol{r} \in A|\ \sum_{j=1}^{s} r_j \geq m+s\}.$$

Further let

$$A_4 = \{\boldsymbol{r}|\ \sum_{j=1}^{s} r_j = m+\alpha\}.$$

A partition of the interval $J_{\boldsymbol{\gamma}}$ is given by the subintervals

$$J_{\boldsymbol{r},\boldsymbol{\gamma},\boldsymbol{g}} = \prod_{j=1}^{s}\left[[\gamma^{(j)}]_{r_j-1} + \frac{g_j}{b^{r_j}}, [\gamma^{(j)}]_{r_j-1} + \frac{g_j+1}{b^{r_j}}\right),$$

where $\boldsymbol{g} = (g_1, \ldots, g_s)$ with $g_j \in \{0, 1, \ldots, a_{r_j} - 1\}$.
The intervals $J_{\boldsymbol{r},\boldsymbol{\gamma},\boldsymbol{g}}$ are disjoint elementary intervals of order $\sum_{j=1}^{s} r_j$ in base b. We define

$$\mathcal{A}(\boldsymbol{r}, \boldsymbol{g}) := \sum_{n=0}^{b^m-1} \chi_{J_{\boldsymbol{r},\boldsymbol{\gamma},\boldsymbol{g}}}(\boldsymbol{x}_n).$$

Then, it is possible to split the estimation of the discrepancy function into three parts corresponding to the sets A_1, A_2 and A_3,

$$\begin{aligned}\frac{1}{N}\Delta(\boldsymbol{\gamma},(\boldsymbol{x}_n)_{0\le n<b^m}) &= \sum_{\substack{\boldsymbol{r}\in A_1\\ \boldsymbol{g}}}\left(\frac{\mathcal{A}(\boldsymbol{r},\boldsymbol{g})}{b^m}-\lambda_s(J_{\boldsymbol{r},\boldsymbol{\gamma},\boldsymbol{g}})\right)\\ &\quad+\sum_{\substack{\boldsymbol{r}\in A_2\\ \boldsymbol{g}}}\left(\frac{\mathcal{A}(\boldsymbol{r},\boldsymbol{g})}{b^m}-\lambda_s(J_{\boldsymbol{r},\boldsymbol{\gamma},\boldsymbol{g}})\right)\\ &\quad+\sum_{\substack{\boldsymbol{r}\in A_3\\ \boldsymbol{g}}}\left(\frac{\mathcal{A}(\boldsymbol{r},\boldsymbol{g})}{b^m}-\lambda_s(J_{\boldsymbol{r},\boldsymbol{\gamma},\boldsymbol{g}})\right)\\ &=\Delta_1+\Delta_2+\Delta_3.\end{aligned}$$

It follows by the net property and the fact that $J_{\boldsymbol{r},\boldsymbol{\gamma},\boldsymbol{g}}$ are elementary intervals that

$$\Delta_1=\sum_{\substack{\boldsymbol{r}\in A_1\\ \boldsymbol{g}}}\left(\frac{\mathcal{A}(\boldsymbol{r},\boldsymbol{g})}{b^m}-\lambda_s(J_{\boldsymbol{r},\boldsymbol{\gamma},\boldsymbol{g}})\right)=0.$$

Since $J_{\boldsymbol{r},\boldsymbol{\gamma},\boldsymbol{g}}$, $\boldsymbol{r}\in A_2$, are elementary intervals of order greater or equal to $m+1$, they either contain one point of the $(0,m,s)$-net or they are empty. Let us consider these two cases:

(1) $\exists\,\boldsymbol{x}_k\in J_{\boldsymbol{r},\boldsymbol{\gamma},\boldsymbol{g}}$. Then it holds that

$$\frac{1}{b^m}-\frac{1}{b^{m+1}}\le\frac{\mathcal{A}(\boldsymbol{r},\boldsymbol{g})}{b^m}-\lambda_s(J_{\boldsymbol{r},\boldsymbol{\gamma},\boldsymbol{g}})=\frac{1}{b^m}-\frac{1}{b^{\sum_{j=1}^s r_j}}\le\frac{1}{b^m}-\frac{1}{b^{m+s-1}}.$$

(2) $\nexists\,\boldsymbol{x}_k\in J_{\boldsymbol{r},\boldsymbol{\gamma},\boldsymbol{g}}$. In this case it holds that

$$-\frac{1}{b^{m+1}}\le\frac{\mathcal{A}(\boldsymbol{r},\boldsymbol{g})}{b^m}-\lambda_s(J_{\boldsymbol{r},\boldsymbol{\gamma},\boldsymbol{g}})=-\frac{1}{b^{\sum_{j=1}^s r_j}}\le-\frac{1}{b^{m+s-1}}.$$

Then, by the assumptions on A_2 we obtain the estimate

$$-\frac{1}{b^{m+1}}\frac{m^{s-1}}{\delta}(b-1)^s\le\Delta_2\le\left(\frac{1}{b^m}-\frac{1}{b^{m+s-1}}\right)\frac{m^{s-1}}{\delta}(b-1)^s.$$

Now, consider Δ_3. The first step is again to show that $J_{\boldsymbol{r},\boldsymbol{\gamma},\boldsymbol{g}}$ with $\boldsymbol{r}\in A_3$ and for all associated $\boldsymbol{g}$, cannot contain any point of a $(0,m,s)$-net which has an element $\boldsymbol{x}_0\in\prod_{j=1}^s[\gamma^{(j)},\gamma^{(j)}+\frac{1}{b^{\max(R_j)}})$. The condition that $\boldsymbol{x}_0$ is contained in this set, is equivalent to

$$[\gamma^{(j)}]_{r_j}=[x_0^{(j)}]_{r_j},\qquad\text{for } j=1,\dots,s. \tag{3.1}$$

Suppose that there exists $\boldsymbol{x}_k \in J_{\boldsymbol{r},\boldsymbol{\gamma},\boldsymbol{g}}$ for some $k < b^m$, some $\boldsymbol{r} \in A_3$ and some $\boldsymbol{g}$. It then follows that

$$[\gamma^{(j)}]_{r_j-1} = [x_k^{(j)}]_{r_j-1}, \qquad \text{for } j = 1, \ldots, s.$$

Therefore,

$$\|\boldsymbol{x}_k \ominus \boldsymbol{x}_0\|_b \leq \frac{1}{b^{\sum_{j=1}^s r_j}} \leq \frac{1}{b^{m+s}}.$$

This is a contradiction to the assumption that $\boldsymbol{x}_k$ and $\boldsymbol{x}_0$ are elements of a $(0, m, s)$-net in base b because from Lemma 2.1 we know that

$$\min_{\boldsymbol{x},\boldsymbol{y}\in\mathcal{P}} \|\boldsymbol{x} \ominus \boldsymbol{y}\|_b = \frac{1}{b^{m+s-1}}.$$

Hence, all $J_{\boldsymbol{r},\boldsymbol{\gamma},\boldsymbol{g}}$, where $\boldsymbol{r} \in A_3$ are empty. Using the fact that $|A_4| \geq \frac{m^{s-1}}{\beta}$, we then get

$$\begin{aligned}
\Delta_3 &= \sum_{\substack{\boldsymbol{r}\in A_3\\ \boldsymbol{g}}} \left(\frac{\mathcal{A}(\boldsymbol{r},\boldsymbol{g})}{b^m} - \lambda_s(J_{\boldsymbol{r},\boldsymbol{\gamma},\boldsymbol{g}}) \right) \\
&= - \sum_{\substack{\boldsymbol{r}\in A_3\\ \boldsymbol{g}}} \frac{1}{b^{\sum_{j=1}^s r_j}} \\
&\leq - \sum_{\substack{\boldsymbol{r}\in A_4\\ \boldsymbol{g}}} \frac{1}{b^{m+\alpha}} \\
&\leq - \frac{m^{s-1}}{\beta}(b-1)^s \frac{1}{b^{m+\alpha}}.
\end{aligned}$$

Finally, we get the estimate

$$\begin{aligned}
\frac{1}{N}\Delta(\boldsymbol{\gamma}, (\boldsymbol{x}_n)_{0\leq n<b^m}) &= \Delta_1 + \Delta_2 + \Delta_3 \\
&\leq \left(\frac{1}{b^m} - \frac{1}{b^{m+s-1}} \right) \frac{m^{s-1}}{\delta}(b-1)^s \\
&\quad - \frac{m^{s-1}}{\beta}(b-1)^s \frac{1}{b^{m+\alpha}} \\
&= -\frac{m^{s-1}}{b^m} \left(-\frac{(b-1)^s}{\delta} \frac{b^{s-1}-1}{b^{s-1}} + \frac{(b-1)^s}{\beta} \frac{1}{b^\alpha} \right) < 0
\end{aligned}$$

for $\delta > \frac{b^\alpha(b^{s-1}-1)\beta}{b^{s-1}}$. $\square$

Subsequently, we now derive Theorem 4, which in some sense describes how dense possible choices of $\boldsymbol{\gamma}$ are in $[0,1)^s$.

P r o o f o f T h e o r e m 4. Let Γ be defined as the set, which contains all points of the form

$$\boldsymbol{\gamma} = \left(\sum_{r_1} \frac{1}{b^{r_1}}, \dots, \sum_{r_s} \frac{1}{b^{r_s}}\right),$$

where $r_i \in R_i \subseteq \{1, 2, \dots, m\}$ for $i = 1, \dots, s$ and the sets R_i fulfill the following conditions:

- $|\{(r_1, \dots, r_s)|\ m + 1 \le \sum_{i=1}^{s} r_i < m + s\}| = 0,$
- $|\{(r_1, \dots, r_s)|\ \sum_{i=1}^{s} r_i = m + s\}| \ge \frac{m^{s-1}(2s-3)^{s-1}}{(4s^2(s-1)^2)^{s-1}}.$

Consider now the b-adic digit expansion of some $\boldsymbol{x} = \left(x^{(1)}, \dots, x^{(s)}\right) \in [0,1)^s$,

$$x^{(i)} = \sum_{s_i \in S_i} \frac{a_{s_i}}{b^{s_i}},$$

where $S_i \subseteq \mathbb{N}$ is the set of indices for which we have $a_{s_i} \in \{1, 2, \dots, b-1\}$ for $i = 1, \dots, s$. Now we have to construct a point $\boldsymbol{\gamma}$ with the following properties:

$$\|\boldsymbol{x} - \boldsymbol{\gamma}\| < b\sqrt{s}\frac{1}{b^{\frac{m}{2(s-1)s}}}, \tag{3.2}$$

$$\boldsymbol{\gamma} \in \Gamma, \text{ where } \boldsymbol{\Gamma} \text{ is defined as above.} \tag{3.3}$$

Let $\boldsymbol{\gamma} = \left(\gamma^{(1)}, \dots, \gamma^{(s)}\right)$,

$$\gamma^{(i)} = \sum_{r_i \in R_i} \frac{a_{r_i}}{b^{r_i}},$$

where

$$R_i = \{s_i \in S_i|\ s_i \le k\} \cup T_i, \quad \text{where } k := \left[\frac{m}{2(s-1)s}\right],$$

and where $t_i \in T_i$ has the form

$$t_i = \left[\frac{m}{2s(s-1)}\right] + sj_i$$

for $i = 1, \dots, s-1$ and $t_s \in T_s$ has the form

$$t_s = m - (s-1)\left(\left[\frac{m}{2s(s-1)}\right] + s\bar{m}\right) + sj_s.$$

Here, $j_1, \dots, j_s \in \{1, \dots, \bar{m}\}$ with

$$\bar{m} = \left[\frac{m(2s-3)}{2s^2(s-1)}\right].$$

Moreover, we choose $a_{r_i} = a_{s_i}$ for all $r_i \in \{s_i \in S_i|\ s_i \le k\}$ and otherwise, $a_{r_i} = 1$.

By the choice of S_i it then holds that $[x^{(i)}]_k = [\gamma^{(i)}]_k$ for all $i = 1, \ldots, s$. This implies that $\boldsymbol{x}$ and $\boldsymbol{\gamma}$ are contained in the same square elementary interval of order sk, i.e.,

$$\boldsymbol{x}, \boldsymbol{\gamma} \in \prod_{i=1}^{s} \left[\frac{A_i}{b^k}, \frac{A_i + 1}{b^k}\right)$$

for some $A_i \in \{0, 1, \ldots, b^k - 1\}$. Therefore, it holds that

$$\|\boldsymbol{x} - \boldsymbol{\gamma}\| < \sqrt{s}\frac{1}{b^k} \leq b\sqrt{s}\frac{1}{b^{\frac{m}{2(s-1)s}}}.$$

Hence, (3.2) is shown. It remains to check, whether the condition on $\boldsymbol{\gamma}$, mentioned at the beginning of the proof, is satisfied, i.e., if $\boldsymbol{\gamma} \in \Gamma$. Obviously, $R_i \subseteq \{1, 2, \ldots, m\}$ for all $i = 1, \ldots, s$.

To begin with, observe that for any $r_i \in R_i$, where $i = 1, \ldots, s-1$, and for any $s_s \in S_s, s_s \leq k$ we have that

$$\begin{aligned}\sum_{i=1}^{s-1} r_i + s_s &\leq (s-1)\left[\frac{m}{2s(s-1)}\right] + \bar{m}s + k \\ &\leq (s-1)\left(\frac{m}{2s(s-1)}\right) + \frac{m(2s-3)}{2s^2(s-1)}s + \frac{m}{2s(s-1)} \leq m.\end{aligned}$$

Additionally, for any $s_1 \in S_1, s_1 \leq k$ and $r_i \in R_i$, where $i = 2, \ldots, s$ it holds that

$$\begin{aligned}s_1 + \sum_{i=2}^{s} r_i &\leq k + (s-1)\left(\left[\frac{m}{2s(s-1)}\right] + s\bar{m}\right) + s\bar{m} \\ &\leq s\frac{m}{2s(s-1)} + (s-1)s\frac{m(2s-3)}{2s^2(s-1)} + s\frac{m(2s-3)}{2s^2(s-1)} = m.\end{aligned}$$

Hence, we can conclude that

$$\left|\left\{(r_1, \ldots, r_s) \middle|\ \sum_{i=1}^{s} r_i > m, r_i \in R_i\right\}\right| = \left|\left\{(t_1, \ldots, t_s) \middle|\ \sum_{i=1}^{s} t_i > m, t_i \in T_i\right\}\right|.$$

Therefore, let us consider $t_i \in T_i$ for $i = 1, \ldots, s$. We have that

$$\sum_{i=1}^{s} t_i = m + s\big(j_1 + \cdots + j_s - (s-1)\bar{m}\big) \neq m + s,$$

because of the fact that $\bar{m} \in \mathbb{Z}$. It follows that

$$\left|\left\{(r_1, \ldots, r_s) \middle|\ m + 1 \leq \sum_{i=1}^{s} r_i < m + s\right\}\right| = 0.$$

For the case $t_1 + \cdots + t_s = m + s$ it holds that

$$j_s = 1 + (s-1)\bar{m} - j_1 - \cdots - j_{s-1}.$$

This implies that the following inequality must be fulfilled:

$$1 \le 1 + (s-1)\bar{m} - j_1 - \cdots - j_{s-1} \le \bar{m}.$$

Obviously, the left inequality holds for any choice of $j_1, \ldots, j_{s-1}$. For the right inequality consider the case that $j_1 = \ldots = j_{s-1}$. Then we can conclude that it has to hold

$$j_1 \ge \left[\frac{(s-2)\bar{m}}{s-1}\right] + 1.$$

Hence, we obtain

$$\begin{aligned}
\left|\left\{(r_1, \ldots, r_s) \middle| \sum_{i=1}^{s} r_i = m + s\right\}\right| &= \left|\left\{(t_1, \ldots, t_s) \middle| \sum_{i=1}^{s} t_i = m + s\right\}\right| \\
&= \left(\bar{m} - \left[\frac{(s-2)\bar{m}}{s-1}\right]\right)^{s-1} \\
&\ge \left[\frac{\bar{m}}{s-1}\right]^{s-1} \\
&\ge \frac{m^{s-1}(2s-3)^{s-1}}{(4s^2(s-1)^2)^{s-1}}
\end{aligned}$$

by using the estimate

$$\left[\frac{\bar{m}}{s-1}\right] = \left[\frac{\left[\frac{m(2s-3)}{2s^2(s-1)}\right]}{s-1}\right] \ge \frac{m(2s-3)}{4s^2(s-1)^2} \qquad \text{for } m \ge \frac{2s^2(s-1)^2}{2s-3}.$$

Thus, also (3.3) is shown. Now we finish the proof of Theorem 4. It remains to show the second item. Let $\mathcal{P} = \{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{b^m-1}\}$ be a $(0, m, s)$-net in base b for which some element $\boldsymbol{x}_i$ belongs to the set Γ. Therefore, the conditions of Lemma 3.1 are satisfied with

$$\alpha = s, \beta = \frac{(4s^2(s-1)^2)^{s-1}}{(2s-3)^{s-1}} \quad \text{and for any} \quad \delta > \frac{b(b^{s-1}-1)(4s^2(s-1)^2)^{s-1}}{(2s-3)^{s-1}}.$$

By considering the limit $\delta \to \infty$ we obtain

$$\frac{1}{N}\Delta(\boldsymbol{\gamma}, (\boldsymbol{x}_n)_{0 \le n < b^m}) \le -\frac{m^{s-1}}{b^m} \frac{(b-1)^s(2s-3)^{s-1}}{b^s(4s^2(s-1)^2)^{s-1}},$$

and the assertion follows with $N = b^m$. □

4. Re-proof of Theorem 2

In the interest of clear presentation, the proof of Theorem 2 will be split into several auxiliary lemmas. The necessity of the following two results should be motivated. In a later step, we will define a special axes-parallel box $[\mathbf{0}, \boldsymbol{y})$ and partition this multi-dimensional interval into several disjoint axes-parallel boxes (see, equation (4.1)). Lemma 4.1 and Lemma 4.2 show under which condition on n a sequence element $H_s(n)$ of the Halton sequence is contained in one of these disjoint intervals.

Lemma 4.1. *Define* $x_i := \sum_{j=1}^{\infty} x_{i,j} b_i^{-j}$, $x_{i,j} \in \{0, 1, \ldots, b_i - 1\}$, *and its truncation* $[x_i]_r := \sum_{j=1}^{r} x_{i,j} b_i^{-j}$, *for* $i = 1, \ldots, s$, $r = 1, 2, \ldots$ *Then, we have*

$$\phi_{b_i}(n) \in \big[[x_i]_r, [x_i]_r + b_i^{-r}\big) \Longleftrightarrow n \equiv \dot{x}_{i,r} \bmod b_i^r, \quad \textit{where} \quad \dot{x}_{i,r} = \sum_{j=1}^{r} x_{i,j} b_i^{j-1}.$$

Proof. The result follows immediately from the definition of the Halton sequence. □

Lemma 4.2. *For a vector* $\boldsymbol{r} = (r_1, \ldots, r_s)$ *of positive integers, let* $B_{\boldsymbol{r}} := \prod_{i=1}^{s} b_i^{r_i}$, *and the integer* $M_{i,\boldsymbol{r}}$, *be defined such that* $M_{i,\boldsymbol{r}}(B_{\boldsymbol{r}} b_i^{-r_i}) \equiv 1 \bmod b_i^{r_i}$, *then we have*

$$\phi_{b_i}(n) \in \big[[x_i]_{r_i}, [x_i]_{r_i} + b_i^{-r_i}\big) \quad \textit{for} \;\; i = 1, \ldots, s \Longleftrightarrow n \equiv \ddot{x}_{\boldsymbol{r}} \bmod B_{\boldsymbol{r}},$$

$$\textit{with} \quad \ddot{x}_{\boldsymbol{r}} = \sum_{i=1}^{s} M_{i,\boldsymbol{r}} B_{\boldsymbol{r}} b_i^{-r_i} \dot{x}_{i,r_i}.$$

Proof. This follows immediately from Lemma 4.1 and the Chinese remainder theorem. □

In order to obtain further information about the discrepancy function of the Halton sequence, i.e., about $\Delta(\cdot, (H_s(n))_{n=1}^N)$, we will investigate this function for a special setting of the interval $[\mathbf{0}, \boldsymbol{y})$ and thereby exploit the information gained by the previous lemmas. Accordingly, let y_i, $i = 1, \ldots, s$, be defined as

$$y_i := \sum_{j=1}^{m} b_i^{-j\tau_i}, \quad \text{with} \quad \tau_i = \min\big\{1 \le k < B^{(i)} | b_i^k \equiv 1 \bmod B^{(i)}\big\},$$

where $m \in \mathbb{N}, m \ge B$ and $B^{(i)} = \frac{B}{b_i}$. If we consider, for instance, the two-dimensional Halton sequence in bases $b_1 = 2$ and $b_2 = 3$, we obtain $\tau_1 = 2$ and $\tau_2 = 1$.

Having gathered these tools, we put $[\mathbf{0},\boldsymbol{y}) = [0,y^{(1)})\times\cdots\times[0,y^{(s)})\subset[0,1)^s$. The pertinence of introducing the integers τ_i will be revealed at a later step in Lemma 4.5. For a further analysis concerning $[\mathbf{0},\boldsymbol{y})$, it turns out to be beneficial to consider a disjoint partitioning of this interval. To achieve the goal of a disjoint decomposition, a truncation of the one-dimensional interval borders y_i, of the form $[y_i]_{\tau_i k_i} = \sum_{j=1}^{k_i} b_i^{-j\tau_i}$, $k_i \geq 1$, $i = 1,\dots,s$, is taken into account. Collecting the integers k_i in a vector $\boldsymbol{k} = (k_1,\dots,k_s)$ we arrive at

$$[\mathbf{0},\boldsymbol{y}) \;=\; \bigcup_{1\leq k_1,\dots,k_s\leq m} P_{\boldsymbol{k}}, \text{ with } P_{\boldsymbol{k}} := \prod_{i=1}^{s}\big[[y_i]_{\tau_i k_i} - b_i^{-k_i\tau_i}, [y_i]_{\tau_i k_i}\big). \tag{4.1}$$

We apply Lemma 4.2 to the interval $P_{\boldsymbol{k}}$ and obtain:

Lemma 4.3. *An element $H_s(n)$ of the Halton sequence is contained in $P_{\boldsymbol{k}}$ if and only if $\phi_{b_i}(n) \in \big[[y_i]_{\tau_i k_i} - b_i^{-\tau_i k_i}, [y_i]_{\tau_i k_i}\big)$, for $i = 1,\dots,s$, or equivalently,*

$$n \equiv \sum_{i=1}^{s} M_{i,\boldsymbol{\tau}\cdot\boldsymbol{k}} B_{\boldsymbol{\tau}\cdot\boldsymbol{k}} b_i^{-\tau_i k_i}\dot{y}_{i,\tau_i(k_i-1)} \bmod B_{\boldsymbol{\tau}\cdot\boldsymbol{k}}, \tag{4.2}$$

where $\dot{y}_{i,\tau_i k_i} := \sum_{j=1}^{k_i} b_i^{j\tau_i - 1}$. Here, $\boldsymbol{\tau} = (\tau_1,\dots,\tau_s)$ and the product $\boldsymbol{\tau}\cdot\boldsymbol{k}$ denotes the vector $(\tau_1 k_1,\dots,\tau_s k_s)$.

A slight reformulation of relation (4.2) is required. Although, by the previous lemma, we have found a criterion for a sequence element to be contained in $P_{\boldsymbol{k}}$, key steps of the proof of Theorem 2 will be based on a congruence of the form $n \equiv \tilde{y}_m + A_{\boldsymbol{k}} \bmod B_{\boldsymbol{\tau}\cdot\boldsymbol{k}}$, with $\tilde{y}_m$ **independent** of $\boldsymbol{k}$ and $A_{\boldsymbol{k}}$ the least positive remainder modulo $B_{\boldsymbol{\tau}\cdot\boldsymbol{k}}$, i.e.,

$$A_{\boldsymbol{k}} :\equiv \sum_{i=1}^{s} -M_{i,\boldsymbol{\tau}\cdot\boldsymbol{k}} B_{\boldsymbol{\tau}\cdot\boldsymbol{k}} b_i^{-1} \bmod B_{\boldsymbol{\tau}\cdot\boldsymbol{k}},\ A_{\boldsymbol{k}} \in [0, B_{\boldsymbol{\tau}\cdot\boldsymbol{k}}).$$

This form is obtained as follows: We have

$$\begin{aligned}
&\sum_{i=1}^{s} M_{i,\boldsymbol{\tau}\cdot\boldsymbol{k}} B_{\boldsymbol{\tau}\cdot\boldsymbol{k}} b_i^{-\tau_i k_i}\dot{y}_{i,\tau_i(k_i-1)}\\
&= \sum_{i=1}^{s} M_{i,\boldsymbol{\tau}\cdot\boldsymbol{k}} B_{\boldsymbol{\tau}\cdot\boldsymbol{k}} b_i^{-\tau_i k_i}\dot{y}_{i,\tau_i k_i} - \sum_{i=1}^{s} M_{i,\boldsymbol{\tau}\cdot\boldsymbol{k}} B_{\boldsymbol{\tau}\cdot\boldsymbol{k}} b_i^{-1}\\
&\equiv \sum_{i=1}^{s} M_{i,\boldsymbol{\tau}(m+1)} B_{\boldsymbol{\tau}(m+1)} b_i^{-\tau_i(m+1)}\dot{y}_{i,\boldsymbol{\tau}(m+1)} - \sum_{i=1}^{s} M_{i,\boldsymbol{\tau}\cdot\boldsymbol{k}} B_{\boldsymbol{\tau}\cdot\boldsymbol{k}} b_i^{-1}\\
&\equiv: \tilde{y}_m + A_{\boldsymbol{k}} \bmod B_{\boldsymbol{\tau}\cdot\boldsymbol{k}}.
\end{aligned}$$

Here $\tilde{y}_m$ is chosen such that $\tilde{y}_m \in [0, B_{\boldsymbol{\tau}(m+1)})$. The first of the congruences above follows by elementary computations. We summarize:

$$H_s(n) \in P_{\boldsymbol{k}} \Longleftrightarrow n \equiv \tilde{y}_m + A_{\boldsymbol{k}} \bmod B_{\boldsymbol{\tau}\cdot\boldsymbol{k}}.$$

Note that the multiplication $\boldsymbol{\tau}(m+1)$ has to be understood componentwise, i.e., we have $\boldsymbol{\tau}(m+1) = \big(\tau_1(m+1), \ldots, \tau_s(m+1)\big)$.

Employing the information received from Lemma 4.3, the equality

$$\sum_{n=N_1 B_{\boldsymbol{\tau}\cdot\boldsymbol{k}}}^{(N_1+1)B_{\boldsymbol{\tau}\cdot\boldsymbol{k}}-1} \big(\chi_{P_{\boldsymbol{k}}}(H_s(n)) - B_{\boldsymbol{\tau}\cdot\boldsymbol{k}}^{-1}\big) = 0,$$

holds for any integer $N_1 \geq 0$, since amongst $B_{\boldsymbol{\tau}\cdot\boldsymbol{k}}$ consecutive integers the congruence of relation (4.2) has exactly one solution. Moreover, for an integer $N_2 \in [0, B_{\boldsymbol{\tau}\cdot\boldsymbol{k}})$, we have

$$\sum_{n=\tilde{y}_m+N_1 B_{\boldsymbol{\tau}\cdot\boldsymbol{k}}}^{\tilde{y}_m+N_1 B_{\boldsymbol{\tau}\cdot\boldsymbol{k}}+N_2-1} \big(\chi_{P_{\boldsymbol{k}}}(H_s(n)) - B_{\boldsymbol{\tau}\cdot\boldsymbol{k}}^{-1}\big) = \sum_{n\in[\tilde{y}_m, \tilde{y}_m+N_2)} \big(\chi_{P_{\boldsymbol{k}}}(H_s(n)) - B_{\boldsymbol{\tau}\cdot\boldsymbol{k}}^{-1}\big). \quad (4.3)$$

Recalling that

$$\begin{aligned} H_s(n) \in P_{\boldsymbol{k}} \Longleftrightarrow n \equiv \tilde{y}_m + A_{\boldsymbol{k}} \bmod B_{\boldsymbol{\tau}\cdot\boldsymbol{k}} \Longleftrightarrow \\ \exists\, l \in \mathbb{Z}, \quad \text{such that} \quad n = lB_{\boldsymbol{\tau}\cdot\boldsymbol{k}} + \tilde{y}_m + \underbrace{A_{\boldsymbol{k}}}_{\in[0,B_{\boldsymbol{\tau}\cdot\boldsymbol{k}})}, \end{aligned}$$

the characteristic function in the sum (4.3) only has a nonzero contribution for $n = \tilde{y}_m + A_{\boldsymbol{k}}$, i.e., $l = 0$, since for all other values of l, n does not belong to the interval $[\tilde{y}_m, \tilde{y}_m + N_2)$. Hence, these arguments enable to restate (4.3) by the expression

$$\begin{aligned} \sum_{\substack{n\in[\tilde{y}_m,\tilde{y}_m+N_2)\\ n=\tilde{y}_m+A_{\boldsymbol{k}}}} 1 - N_2 B_{\boldsymbol{\tau}\cdot\boldsymbol{k}}^{-1} &= \begin{cases} 1 - N_2 B_{\boldsymbol{\tau}\cdot\boldsymbol{k}}^{-1}, & 0 \leq A_{\boldsymbol{k}} < N_2, \\ -N_2 B_{\boldsymbol{\tau}\cdot\boldsymbol{k}}^{-1}, & \text{else.} \end{cases} \\ &= \chi_{[0,N_2)}(A_{\boldsymbol{k}}) - N_2 B_{\boldsymbol{\tau}\cdot\boldsymbol{k}}^{-1}. \end{aligned}$$

So far, we have constructed a special interval $[\mathbf{0}, \boldsymbol{y})$, partitioned this box into subintervals and derived criteria to verify if some sequence element $H_s(n)$ is contained in a fixed box $P_{\boldsymbol{k}}$. To make the star-discrepancy of the Halton sequence sufficiently large, we additionally have to construct infinitely many values for N, which are bad in the sense that they yield (in combination with the special interval $[\mathbf{0}, \boldsymbol{y})$) a large discrepancy. The decisive idea is to show the existence of such N, rather to give an explicit construction. This consideration is realised by taking a quantity α_m into account, which represents the average of the discrepancy function, evaluated for the sequence elements $(H_s(n))_{n=\tilde{y}_m}^{\tilde{y}_m+N-1}$ for several different values of N. Succeeding in showing that $|\alpha_m| \geq c_s m^s$, with $c_s > 0$, would allow to conclude Theorem 2.

Lemma 4.4. *Let*

$$\alpha_m := \frac{1}{B_{\boldsymbol{\tau} m}} \sum_{N=1}^{B_{\boldsymbol{\tau} m}} \Delta\left(\boldsymbol{y}, (H_s(n))_{n=\tilde{y}_m}^{\tilde{y}_m+N-1}\right),$$

then

$$\alpha_m = \sum_{1 \leq k_1,\dots,k_s \leq m} \left(\frac{1}{2} - \frac{A_{\boldsymbol{k}}}{B_{\boldsymbol{\tau}\cdot\boldsymbol{k}}} - \frac{1}{2B_{\boldsymbol{\tau}\cdot\boldsymbol{k}}}\right). \tag{4.4}$$

P r o o f. We have

$$\begin{aligned}\alpha_m &= \frac{1}{B_{\boldsymbol{\tau} m}} \sum_{N=1}^{B_{\boldsymbol{\tau} m}} \Delta\left(\boldsymbol{y}, (H_s(n))_{n=\tilde{y}_m}^{\tilde{y}_m+N-1}\right) \\ &= \sum_{1 \leq k_1,\dots,k_s \leq m} \underbrace{\frac{1}{B_{\boldsymbol{\tau} m}} \sum_{N=1}^{B_{\boldsymbol{\tau} m}} \sum_{n=\tilde{y}_m}^{\tilde{y}_m+N-1} \left(\chi_{P_{\boldsymbol{k}}}(H_s(n)) - B_{\boldsymbol{\tau}\cdot\boldsymbol{k}}^{-1}\right)}_{=:\alpha_{m,\boldsymbol{k}}}.\end{aligned}$$

The summands $\alpha_{m,\boldsymbol{k}}$ can be reformulated in the following way:

$$\begin{aligned}\alpha_{m,\boldsymbol{k}} &= \frac{1}{B_{\boldsymbol{\tau} m}} \sum_{N=1}^{B_{\boldsymbol{\tau} m}} \sum_{n=\tilde{y}_m}^{\tilde{y}_m+N-1} \left(\chi_{P_{\boldsymbol{k}}}(H_s(n)) - B_{\boldsymbol{\tau}\cdot\boldsymbol{k}}^{-1}\right) \\ &= \frac{1}{B_{\boldsymbol{\tau} m}} \sum_{N_1=0}^{B_{\boldsymbol{\tau} m}/B_{\boldsymbol{\tau}\cdot\boldsymbol{k}}-1} \sum_{N_2=1}^{B_{\boldsymbol{\tau}\cdot\boldsymbol{k}}} \Bigg(\underbrace{\sum_{n=\tilde{y}_m}^{\tilde{y}_m+N_1B_{\boldsymbol{\tau}\cdot\boldsymbol{k}}-1} \left(\chi_{P_{\boldsymbol{k}}}(H_s(n)) - B_{\boldsymbol{\tau}\cdot\boldsymbol{k}}^{-1}\right)}_{=0} \\ &\qquad\qquad + \underbrace{\sum_{n=\tilde{y}_m+N_1B_{\boldsymbol{\tau}\cdot\boldsymbol{k}}}^{\tilde{y}_m+N_1B_{\boldsymbol{\tau}\cdot\boldsymbol{k}}+N_2-1} \left(\chi_{P_{\boldsymbol{k}}}(H_s(n)) - B_{\boldsymbol{\tau}\cdot\boldsymbol{k}}^{-1}\right)}_{=\chi_{[0,N_2)}(A_{\boldsymbol{k}}) - N_2B_{\boldsymbol{\tau}\cdot\boldsymbol{k}}^{-1}} \Bigg) \\ &= \frac{1}{B_{\boldsymbol{\tau} m}} \sum_{N_1=0}^{B_{\boldsymbol{\tau} m}/B_{\boldsymbol{\tau}\cdot\boldsymbol{k}}-1} \sum_{N_2=1}^{B_{\boldsymbol{\tau}\cdot\boldsymbol{k}}} \left(\chi_{[0,N_2)}(A_{\boldsymbol{k}}) - N_2B_{\boldsymbol{\tau}\cdot\boldsymbol{k}}^{-1}\right) \\ &= \frac{1}{B_{\boldsymbol{\tau}\cdot\boldsymbol{k}}} \left(\sum_{N_2=1}^{B_{\boldsymbol{\tau}\cdot\boldsymbol{k}}} \chi_{[0,N_2)}(A_{\boldsymbol{k}}) - \sum_{N_2=1}^{B_{\boldsymbol{\tau}\cdot\boldsymbol{k}}} N_2B_{\boldsymbol{\tau}\cdot\boldsymbol{k}}^{-1} \right).\end{aligned} \tag{4.5}$$

By virtue of the fact that $A_{\boldsymbol{k}} \in [0, B_{\boldsymbol{\tau}\cdot\boldsymbol{k}})$ the first sum of (4.5) is not vanishing and simplifies to $B_{\boldsymbol{\tau}\cdot\boldsymbol{k}} - A_{\boldsymbol{k}}$. We therefore arrive at

$$\alpha_{m,\boldsymbol{k}} = \frac{1}{2} - \frac{A_{\boldsymbol{k}}}{B_{\boldsymbol{\tau}\cdot\boldsymbol{k}}} - \frac{1}{2B_{\boldsymbol{\tau}\cdot\boldsymbol{k}}},$$

and consequently

$$\alpha_m = \sum_{1\le k_1,\dots,k_s\le m} \Big(\frac{1}{2} - \frac{A_{\boldsymbol{k}}}{B_{\boldsymbol{\tau}\cdot\boldsymbol{k}}} - \frac{1}{2B_{\boldsymbol{\tau}\cdot\boldsymbol{k}}}\Big).$$

$\square$

Lemma 4.5. *Let α_m be defined as in the previous lemma. Then we have*

$$|\alpha_m| \ge c_s m^s, \quad \textit{with} \quad c_s > 0.$$

Proof. For simplicity reasons, we will prove this lemma only for the two--dimensional Halton sequence in bases $b_1 = 2$ and $b_2 = 3$. The general case works analogously with a bit more technical effort. To estimate the absolute value of α_m from below, we investigate the three occurring sums in (4.4) separately. We have $\sum_{1\le k_1,k_2\le m} \frac{1}{2} = \frac{m^2}{2}$. The definition of $A_{\boldsymbol{k}}$ gives

$$\frac{A_{\boldsymbol{k}}}{B_{\boldsymbol{\tau}\cdot\boldsymbol{k}}} \equiv -\sum_{i=1}^{2} \frac{M_{i,\boldsymbol{\tau}\cdot\boldsymbol{k}} B_{\boldsymbol{\tau}\cdot\boldsymbol{k}} b_i^{-1}}{B_{\boldsymbol{\tau}\cdot\boldsymbol{k}}} \bmod 1, \tag{4.6}$$

and therefore it is necessary to examine the expression $M_{i,\boldsymbol{\tau}\cdot\boldsymbol{k}} b_i^{-1} \bmod 1$ in detail. According to the choice of the integer $M_{i,\boldsymbol{\tau}\cdot\boldsymbol{k}}$ and τ_i, we obtain in our special case:

$$M_{1,\boldsymbol{\tau}\cdot\boldsymbol{k}} 3^{k_2} \equiv 1 \bmod 2^{2k_1},$$

hence

$$M_{1,\boldsymbol{\tau}\cdot\boldsymbol{k}} 3^{k_2} \equiv 1 \bmod 2$$

and consequently,

$$M_{1,\boldsymbol{\tau}\cdot\boldsymbol{k}} \equiv 1 \bmod 2.$$

Further

$$M_{2,\boldsymbol{\tau}\cdot\boldsymbol{k}} 2^{2k_1} \equiv 1 \bmod 3^{k_2},$$

hence

$$M_{2,\boldsymbol{\tau}\cdot\boldsymbol{k}} 2^{2k_1} \equiv 1 \bmod 3$$

and consequently,

$$M_{2,\boldsymbol{\tau}\cdot\boldsymbol{k}} \equiv 1 \bmod 3.$$

Combining this result with (4.6) yields

$$\frac{A_{\boldsymbol{k}}}{B_{\boldsymbol{\tau}\cdot\boldsymbol{k}}} \equiv -\frac{1}{b_1} - \frac{1}{b_2} = -\frac{1}{2} - \frac{1}{3} \bmod 1 = 1 - \frac{1}{2} - \frac{1}{3} = \frac{1}{6}.$$

Summing up the reformulated addends of equation (4.4), gives

$$|\alpha_m| = \left| m^2\Big(\frac{1}{2} - \frac{1}{6}\Big) - \sum_{1 \le k_1, k_2 \le m} \frac{1}{2B_{\boldsymbol{\tau}\cdot\boldsymbol{k}}} \right| \ge c_2 m^2, \quad \text{with} \quad c_2 > 0,$$

and m sufficiently large. □

This estimate gives us the necessary tools to conclude Theorem 2.

Proof of Theorem 2. From the definition of α_m (see formulation of Lemma 4.4) and from Lemma 4.5 we conclude that for every m there is an N with $1 \le N \le B_{\boldsymbol{\tau}m}$ such that

$$\left|\Delta\left(\boldsymbol{y}, (H_s(n))_{n=\tilde{y}_m}^{\tilde{y}_m+N-1}\right)\right| \ge c_s m^s.$$

Hence,

$$\left|\Delta\left(\boldsymbol{y}, (H_s(n))_{n=0}^{\tilde{y}_m-1}\right)\right| \ge \frac{c_s}{2} m^s \ \vee\ \left|\Delta\left(\boldsymbol{y}, (H_s(n))_{n=0}^{\tilde{y}_m+N-1}\right)\right| \ge \frac{c_s}{2} m^s.$$

Assume, the second estimate holds (the other case is treated analogously) and set $N_m := \tilde{y}_m + N$, i.e.,

$$\left|\Delta\left(\boldsymbol{y}, (H_s(n))_{n=0}^{N_m-1}\right)\right| \ge \frac{c_s}{2} m^s.$$

Now note that

$$N_m = \tilde{y}_m + N \le B_{\boldsymbol{\tau}(m+1)} + B_{\boldsymbol{\tau}m} \le B^{3m(\tau_1+\ldots+\tau_s)},$$

i.e.,

$$m \ge \frac{\log N_m}{\log B^{3(\tau_1+\ldots+\tau_s)}},$$

and therefore

$$\left|\Delta\left(\boldsymbol{y}, (H_s(n))_{n=0}^{N_m-1}\right)\right| \ge \frac{c_s}{2(\log B^{3(\tau_1+\ldots+\tau_s)})^s} (\log N_m)^s.$$

It can easily be argued that we can obtain infinitely many such N_m, with this property and the result follows. □

5. Proof of Theorem 5

The investigations of the current section are restricted to the two-dimensional Halton sequence in bases $b_1 = 2$ and $b_2 = 3$. In the following, we survey possible options to modify the intervals $[0, y^{(1)})$ and $[0, y^{(2)})$, and discuss whether these changes still allow to derive the estimate $|\alpha_m| \ge c_2 m^2$ or not. A way to obtain further possible values for $y^{(1)}$ or $y^{(2)}$ would be to remove some addends of the specification of $y^{(1)}$ or $y^{(2)}$, i.e., to consider for example

$$\tilde{y}^{(1)} = \sum_{\substack{j=1 \\ j\neq l}}^{m} 2^{-j\tau_1} \quad \text{or} \quad \tilde{y}^{(2)} = \sum_{\substack{j=1 \\ j\neq l}}^{m} 3^{-j\tau_2} \quad \text{with} \quad l \in \mathbb{N} \quad \text{and} \quad 1 \leq l \leq m.$$

Recalling equation (4.4), the choice of the modified box $[0, \tilde{y}^{(1)}) \times [0, y^{(2)})$ would have the consequence that (4.4) amounts to

$$\alpha_m = \sum_{\substack{1\leq k_1,k_2\leq m \\ k_1\neq l}} \left(\frac{1}{2} - \frac{A_{\boldsymbol{k}}}{B_{\boldsymbol{\tau}\cdot\boldsymbol{k}}} - \frac{1}{2B_{\boldsymbol{\tau}\cdot\boldsymbol{k}}}\right).$$

Note, that all previous steps of the proof of Theorem 2 can easily be adapted to this modified choice of the axes-parallel box. Since k_1 only takes on $(m-1)$ different values, we get

$$\alpha_m = \frac{1}{3}m(m-1) \ - \sum_{\substack{1\leq k_1,k_2\leq m \\ k_1\neq l}} \frac{1}{2B_{\boldsymbol{\tau}\cdot\boldsymbol{k}}}$$

and therefore we are still in the position to derive a lower bound for $|\alpha_m|$ of the form c_2m^2. The next corollary focuses on the questions of how many addends can be removed from the representation of $y^{(1)}$ (or $y^{(2)}$).

Corollary 5.1. *Let $\epsilon > 0$ and fix an $m > \hat{c}_2(\epsilon)$, with a sufficiently large constant $\hat{c}_2(\epsilon)$. If we remove at most $m(1-\epsilon)$ addends from the representation of $y^{(1)}$ $(y^{(2)})$, while $y^{(2)}$ $(y^{(1)})$ remains unchanged, then we still have*

$$|\alpha_m| \geq c_2(\epsilon)m^2 \quad \textit{with} \quad c_2(\epsilon) > 0.$$

Up to now we have only modified $y^{(1)}$ $(y^{(2)})$ and kept $y^{(2)}$ $(y^{(1)})$ unchanged. If we remove addends from the representation of $y^{(1)}$ and from the one of $y^{(2)}$, we obtain the following corollary.

Corollary 5.2. *Let $\epsilon > 0$ and fix an $m > \hat{c}_3(\epsilon)$, with a sufficiently large constant $\hat{c}_3(\epsilon)$. If we remove at most $m(1-\epsilon)$ addends from the representation of $y^{(1)}$ and $y^{(2)}$ then we still have*

$$|\alpha_m| \geq c_3(\epsilon)m^2 \quad \textit{with} \quad c_3(\epsilon) > 0.$$

Based on these preliminary considerations, we will derive the following lemma, which states, that there are, in some sense, many feasible choices for the interval borders $y^{(1)}$ and $y^{(2)}$.

Lemma 5.1. *Let m be sufficiently large (as in Corollary 5.2). Then, there is a set $\Upsilon \subseteq [0,1)^2$ with the following property: For all $\boldsymbol{x} \in [0,1)^2$ there exists an $\boldsymbol{y} \in \Upsilon$ with*

$$\|\boldsymbol{x} - \boldsymbol{y}\| < \sqrt{8}\frac{1}{2^{m/2}}.$$

Furthermore, for such a $\boldsymbol{y}$, we have $|\alpha_m| \geq c_2 m^2$, with some constant $c_2 > 0$.

Proof. Let $y^{(1)} = 0.\underbrace{010101\ldots01}_{2m}$ in base 2, and $y^{(2)} = 0.\underbrace{11\ldots1}_{m}$ in base 3, the original choice of the interval borders of the two-dimensional box $[0, y^{(1)}) \times [0, y^{(2)})$. We now consider modified interval borders of the form

$$\tilde{y}^{(1)} = 0.\underbrace{a_1 \ldots a_{l_1} 0101 \ldots 01}_{2m} \quad \text{with} \quad a_1, \ldots, a_{l_1} \in \{0,1\}$$

and

$$\tilde{y}^{(2)} = 0.\underbrace{b_1 \ldots b_{l_2} 11 \ldots 11}_{m} \qquad \text{with} \quad b_1, \ldots, b_{l_2} \in \{0,1,2\}.$$

The question is of course, how large $l_1 = l_1(m)$ and $l_2 = l_2(m)$ can be chosen for a given m, such that we still have $|\alpha_m| \geq c_2 m^2$ for this modified choice of the interval. The set Υ is then defined as the set of all feasible choices of $(\tilde{y}^{(1)}, \tilde{y}^{(2)})$. Let $\tilde{k}_1^{(i)}$ and $\tilde{k}_1^{(i-1)} \leq l_1/2$ be integers, for which $a_{2\tilde{k}_1^{(i)}} = a_{2\tilde{k}_1^{(i-1)}} = 1$. If one of the digits $a_{2\tilde{k}_1^{(i-1)}+1}, \ldots, a_{2\tilde{k}_1^{(i)}-1}$ is one, we split an interval of the form

$$\left[[\tilde{y}^{(1)}]_{2\tilde{k}_1^{(i-1)}}, [\tilde{y}^{(1)}]_{2\tilde{k}_1^{(i)}}\right)$$

into the two disjoint intervals

$$\left[[\tilde{y}^{(1)}]_{2\tilde{k}_1^{(i-1)}}, [\tilde{y}^{(1)}]_{2\tilde{k}_1^{(i)}} - 2^{-2\tilde{k}_1^{(i)}}\right) \ \wedge \ \left[[\tilde{y}^{(1)}]_{2\tilde{k}_1^{(i)}} - 2^{-2\tilde{k}_1^{(i)}}, [\tilde{y}^{(1)}]_{2\tilde{k}_1^{(i)}}\right).$$

Now, let $\tilde{k}_2^{(i)} \leq l_2$, be an integer, for which $b_{\tilde{k}_2^{(i)}} = 2$. Then, we split an interval of the form

$$\left[[\tilde{y}^{(2)}]_{\tilde{k}_2^{(i)}} - 2 \cdot 3^{-\tilde{k}_2^{(i)}}, [\tilde{y}^{(2)}]_{\tilde{k}_2^{(i)}}\right)$$

into the two disjoint intervals

$$\left[[\tilde{y}^{(2)}]_{\tilde{k}_2^{(i)}} - 2 \cdot 3^{-\tilde{k}_2^{(i)}}, [\tilde{y}^{(2)}]_{\tilde{k}_2^{(i)}} - 3^{-\tilde{k}_2^{(i)}}\right) \ \wedge \ \left[[\tilde{y}^{(2)}]_{\tilde{k}_2^{(i)}} - 3^{-\tilde{k}_2^{(i)}}, [\tilde{y}^{(2)}]_{\tilde{k}_2^{(i)}}\right).$$

We investigate the influence of this additional interval on the quantity α_m. Therefore, we consider the average of the discrepancy function for the interval

$$J_1 = \left[[\tilde{y}^{(1)}]_{2\tilde{k}_1^{(i-1)}}, [\tilde{y}^{(1)}]_{2\tilde{k}_1^{(i)}} - 2^{-2\tilde{k}_1^{(i)}}\right) \times [0, \tilde{y}^{(2)}),$$

i.e., we study:

$$\begin{aligned}
\tilde{\alpha}_m^{(1)} &= \frac{1}{B_{\boldsymbol{\tau}m}} \sum_{N=1}^{B_{\boldsymbol{\tau}m}} \left(\sum_{n=\tilde{y}_m}^{\tilde{y}_m+N-1} \chi_{J_1}(H_s(n)) - N\lambda_2(J_1) \right) \\
&= \frac{1}{B_{\boldsymbol{\tau}m}} \sum_{N=1}^{B_{\boldsymbol{\tau}m}} \left(\sum_{n=\tilde{y}_m}^{\tilde{y}_m+N-1} \chi_{J_1}(H_s(n)) \right) \\
&\quad - \frac{B_{\boldsymbol{\tau}m}+1}{2} \left(\sum_{j=2\tilde{k}_1^{(i-1)}+1}^{2\tilde{k}_1^{(i)}-1} \frac{a_j}{2^j} \left(\sum_{i=1}^{l_2} \frac{b_i}{3^i} + \sum_{i=l_2+1}^{m} \frac{1}{3^i} \right) \right) \\
&\geq \frac{1}{B_{\boldsymbol{\tau}m}} \sum_{N=1}^{B_{\boldsymbol{\tau}m}} \left(\sum_{j=2\tilde{k}_1^{(i-1)}+1}^{2\tilde{k}_1^{(i)}-1} \sum_{i=1}^{l_2} a_j b_i \left\lfloor \frac{N}{2^j 3^i} \right\rfloor + \sum_{j=2\tilde{k}_1^{(i-1)}+1}^{2\tilde{k}_1^{(i)}-1} \sum_{i=l_2+1}^{m} a_j \left\lfloor \frac{N}{2^j 3^i} \right\rfloor \right) \\
&\quad - \frac{B_{\boldsymbol{\tau}m}+1}{2} \left(\sum_{j=2\tilde{k}_1^{(i-1)}+1}^{2\tilde{k}_1^{(i)}-1} \frac{a_j}{2^j} \left(\sum_{i=1}^{l_2} \frac{b_i}{3^i} + \sum_{i=l_2+1}^{m} \frac{1}{3^i} \right) \right).
\end{aligned}$$

Estimating the floor function yields:

$$\begin{aligned}
\tilde{\alpha}_m^{(1)} &\geq \frac{1}{B_{\boldsymbol{\tau}m}} \sum_{N=1}^{B_{\boldsymbol{\tau}m}} \left(\sum_{j=2\tilde{k}_1^{(i-1)}+1}^{2\tilde{k}_1^{(i)}-1} \sum_{i=1}^{l_2} a_j b_i \left(\frac{N}{2^j 3^i} - 1 \right) \right. \\
&\quad \left. + \sum_{j=2\tilde{k}_1^{(i-1)}+1}^{2\tilde{k}_1^{(i)}-1} \sum_{i=l_2+1}^{m} a_j \left(\frac{N}{2^j 3^i} - 1 \right) \right) \\
&\quad - \frac{B_{\boldsymbol{\tau}m}+1}{2} \left(\sum_{j=2\tilde{k}_1^{(i-1)}+1}^{2\tilde{k}_1^{(i)}-1} \frac{a_j}{2^j} \left(\sum_{i=1}^{l_2} \frac{b_i}{3^i} + \sum_{i=l_2+1}^{m} \frac{1}{3^i} \right) \right) \\
&= \frac{B_{\boldsymbol{\tau}m}+1}{2} \sum_{j=2\tilde{k}_1^{(i-1)}+1}^{2\tilde{k}_1^{(i)}-1} \sum_{i=1}^{l_2} \frac{a_j}{2^j} \frac{b_i}{3^i} + \frac{B_{\boldsymbol{\tau}m}+1}{2} \sum_{j=2\tilde{k}_1^{(i-1)}+1}^{2\tilde{k}_1^{(i)}-1} \sum_{i=l_2+1}^{m} \frac{a_j}{2^j} \frac{1}{3^i} \\
&\quad - \frac{B_{\boldsymbol{\tau}m}+1}{2} \left(\sum_{j=2\tilde{k}_1^{(i-1)}+1}^{2\tilde{k}_1^{(i)}-1} \frac{a_j}{2^j} \left(\sum_{i=1}^{l_2} \frac{b_i}{3^i} + \sum_{i=l_2+1}^{m} \frac{1}{3^i} \right) \right) \\
&\quad - \left(\sum_{i=1}^{l_2} b_i + (m - l_2) \right) \sum_{j=2\tilde{k}_1^{(i-1)}+1}^{2\tilde{k}_1^{(i)}-1} a_j \\
&\geq (-m - l_2) \sum_{j=2\tilde{k}_1^{(i-1)}+1}^{2\tilde{k}_1^{(i)}-1} a_j \;\geq\; -2m \sum_{j=2\tilde{k}_1^{(i-1)}+1}^{2\tilde{k}_1^{(i)}-1} a_j .
\end{aligned}$$

We get an analogue upper bound for $\tilde{\alpha}_m^{(1)}$, by estimating $\sum_{n=\tilde{y}_m}^{\tilde{y}_m+N-1} \chi_{J_1}(H_s(n))$ with the expression

$$\sum_{j=2\tilde{k}_1^{(i-1)}+1}^{2\tilde{k}_1^{(i)}-1} \sum_{i=1}^{l_2} a_j b_i \left(\left\lfloor \frac{N}{2^j 3^i} \right\rfloor + 1 \right) + \sum_{j=2\tilde{k}_1^{(i-1)}+1}^{2\tilde{k}_1^{(i)}-1} \sum_{i=l_2+1}^{m} a_j \left(\left\lfloor \frac{N}{2^j 3^i} \right\rfloor + 1 \right).$$

To sum up, we get

$$\left| \tilde{\alpha}_m^{(1)} \right| \leq 2m \sum_{j=2\tilde{k}_1^{(i-1)}+1}^{2\tilde{k}_1^{(i)}-1} a_j.$$

In total, all intervals of this form yield therefore a contribution of at most $l_1 m$.

Studying the average of the discrepancy function for an interval of the form

$$J_2 = \left[0, \tilde{y}^{(1)}\right) \times \left[\left[\tilde{y}^{(2)}\right]_{\tilde{k}_2^{(i)}} - 3^{-\tilde{k}_2^{(i)}}, \left[\tilde{y}^{(2)}\right]_{\tilde{k}_2^{(i)}} \right),$$

we get, analogously to above, an additional contribution to α_m of at most $l_2 m$. In total, we thus have, an contribution of the magnitude

$$m(l_1 + l_2).$$

Therefore, if $l_1 + l_2 < m$, we still can derive an estimate of the form $|\alpha_m| \geq c_2 m^2$ for the modified box $[0, \tilde{y}^{(1)}) \times [0, \tilde{y}^{(2)})$. Let now m be given and $\boldsymbol{x} = (x_1, x_2) \in [0,1)^2$, arbitrary but fixed, where

$$x_1 = \sum_{i \geq 1} \frac{a_i}{2^i}, \; a_i \in \{0,1\} \quad \text{and} \quad x_2 = \sum_{i \geq 1} \frac{b_i}{3^i}, \; b_i \in \{0,1,2\}.$$

Due to above considerations, we can find $\boldsymbol{y} \in \Upsilon$, which satisfies

$$\|\boldsymbol{x} - \boldsymbol{y}\| < \sqrt{\left(\frac{1}{2^{\lfloor \frac{m}{2} \rfloor - 1}} \right)^2 + \left(\frac{2}{3^{\lfloor \frac{m}{2} \rfloor - 1}} \right)^2} < \sqrt{8} \frac{1}{2^{m/2}},$$

and also allows to derive $|\alpha_m| \geq c_2 m^2$. $\square$

Based on the previous lemma, we are in the position to prove Theorem 5, which gives a lower bound for the discrepancy for a specific N and not just for the average.

P r o o f o f T h e o r e m 5. Fix an m, which satisfies the condition of Lemma 5.1 and recall $N_m = N + \tilde{y}_m$, as in the proof of Theorem 2. Consider now squares $Q_i \subseteq [0,1)^2$ of side length $\frac{2\sqrt{8}}{2^{m/2}}$. Due to Lemma 5.1, we know that each such square contains elements of the set Υ (defined as in Lemma 5.1). We partition $[0,1)^2$ into $\frac{2^m}{32}$ such squares Q_i. Choose, for each Q_i, $\boldsymbol{y}_i \in Q_i \cap \Upsilon$. For some fixed $\boldsymbol{y}_i$, we have

$$|\alpha_m(\boldsymbol{y}_i)| \geq c_2 m^2. \tag{5.1}$$

Let $c_2 > 0$ be small enough, such that this estimate holds for all other choices $\boldsymbol{y}_j \in Q_j \neq Q_i$ as well.
Note, that we always have $|\alpha_m| \leq cm^2$ for a fixed constant $c > 0$, since

$$D^*\Big((H_2(n))_{n=1}^N\Big) \leq c\frac{(\log N)^2}{N}, \text{ for all } N.$$

Now, we claim that the number of Ns with $1 \leq N \leq B_{\boldsymbol{\tau}m}$ and

$$\Big|\Delta\Big(\boldsymbol{y}_i, (H_2(n))_{n=1}^{N_m}\Big)\Big| < \frac{c_2}{2}m^2$$

is at most $\kappa B_{\boldsymbol{\tau}m}$, with $\kappa := \frac{c-c_2}{c-c_2/2}$.
Suppose the number of Ns with $1 \leq N \leq B_{\boldsymbol{\tau}m}$ and

$$\Big|\Delta\Big(\boldsymbol{y}_i, (H_2(n))_{n=1}^{N_m}\Big)\Big| < \frac{c_2}{2}m^2$$

would be larger than $\kappa B_{\boldsymbol{\tau}m}$. Then, we would have

$$|\alpha_m(\boldsymbol{y}_i)B_{\boldsymbol{\tau}m}| < \kappa B_{\boldsymbol{\tau}m}\frac{c_2}{2}m^2 + (1-\kappa)B_{\boldsymbol{\tau}m}cm^2 = c_2 B_{\boldsymbol{\tau}m}m^2,$$

which is a contradiction to inequality (5.1).
Therefore, the number of Ns with $1 \leq N \leq B_{\boldsymbol{\tau}m}$ and

$$\Big|\Delta\Big(\boldsymbol{y}_i, (H_2(n))_{n=1}^{N_m}\Big)\Big| \geq \frac{c_2}{2}m^2$$

is at least $(1-\kappa)B_{\boldsymbol{\tau}m} = \frac{c_2}{2c-c_2}B_{\boldsymbol{\tau}m}$.
To sum up, we have $\frac{2^m}{32}$ squares Q_i, and for each of them, we have identified $(1-\kappa)B_{\boldsymbol{\tau}m}$ distinct values for N, $1 \leq N \leq B_{\boldsymbol{\tau}m}$, which give a sufficiently large discrepancy. Thus, in total we have identified $\frac{2^m}{32}(1-\kappa)B_{\boldsymbol{\tau}m}$ many N and this implies that at least one of those N is identified at least $\frac{2^m}{32}(1-\kappa)$-times. Let N_0 be an N with this certain multiplicity. Further, this means that there exist at least $\frac{2^m}{32}(1-\kappa)$ distinct $\boldsymbol{y}_i \in \cup_i\, Q_i \cap \Upsilon$, such that

$$\Big|\Delta\big(\boldsymbol{y}_i, (H_2(n))_{n=1}^{N_m^{(0)}}\big)\Big| \geq \frac{c_2}{2}m^2,$$

where $N_m^{(0)} := N_0 + \tilde{y}_m$. Note, that the union of all squares Q_i containing the $\boldsymbol{y}_i$ with this property, forms the set Λ_{N_0} and therefore $\lambda_2(\Lambda_N) \geq 1-\kappa$. It remains to verify, that for all $\boldsymbol{x} \in \Lambda_{N_0}$ there exists a $\boldsymbol{y} \in [0,1)^2$ having a distance less than $\sqrt{8}\frac{1}{N^{\frac{1}{14}}}$. Since $1 \leq N_0 \leq B_{\boldsymbol{\tau}m}$, the claim immediately follows by Lemma 5.1 and the estimate $\tilde{y}_m + B_{\boldsymbol{\tau}m} < 2^{7m}$. □

Remark 3. We note, that the considerations of this section can also be adopted to an arbitrary dimension $s > 2$. For ease of notation, we have only presented them in the two-dimensional case for the bases $b_1 = 2$ and $b_2 = 3$.

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