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MOTZKIN'S MAXIMAL DENSITY AND RELATED CHROMATIC NUMBERS

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ABSTRACT. This paper concerns the problem of determining or estimating the maximal upper density of the sets of nonnegative integers S whose elements do not differ by an element of a given set M of positive integers. We find some exact values and some bounds for the maximal density when the elements of M are generalized Fibonacci numbers of odd order. The generalized Fibonacci sequence of order r is a generalization of the well known Fibonacci sequence, where instead of starting with two predetermined terms, we start with r predetermined terms and each term afterwards is the sum of r preceding terms. We also derive some new properties of the generalized Fibonacci sequence of order r. Furthermore, we discuss some related coloring parameters of distance graphs generated by the set M.

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1. Introduction

For a given set M of positive integers, a problem of Motzkin's asks to find the maximal upper density of sets S of nonnegative integers in which no two elements of S are allowed to differ by an element of M. Following Motzkin, if M be a given set of positive integers, a set S of nonnegative integers is said to be an M-set if $a \in S$, $b \in S$ implies $a - b \notin M$. Let S be any set of nonnegative integers and $S(x)$ be the number of elements $n \in S$ such that $n \leq x$, $x \in \mathbb{R}$. We define the upper and lower densities of S, denoted respectively by $\overline{\delta}(S)$ and $\underline{\delta}(S)$, by

$$\overline{\delta}(S) = \limsup_{x\to\infty} \frac{S(x)}{x}, \qquad \underline{\delta}(S) = \liminf_{x\to\infty} \frac{S(x)}{x}.$$

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We say that S has density $\delta(S)$, when $\overline{\delta}(S) = \underline{\delta}(S) = \delta(S)$. The parameter of interest is the maximal density of an M-set, defined by

$$\mu(M) := \sup \overline{\delta}(S),$$

where the supremum is taken over all M-sets S. Motzkin posed the problem of determining the quantity $\mu(M)$. In 1973, Cantor and Gordon [1] proved that there exists a set S such that $\delta(S) = \mu(M)$, when M is finite. The following two lemmas proved in [1] and [7], respectively, are useful results for bounding $\mu(M)$.

Lemma 1.1. *Let $M = \{m_1, m_2, m_3, \ldots\}$, and c and m be positive integers such that $\gcd(c,m) = 1$. Then*

$$\mu(M) \geq \kappa(M) := \sup_{(c,m)=1} (1/m) \min_{k \geq 1} |cm_k|_m,$$

where $|x|_m$ denotes the absolute value of the absolutely least remainder of x $\pmod{m}$.

Lemma 1.2. *Let α be a real number, $\alpha \in [0,1]$. If for any M-set S with $0 \in S$ there exists a positive integer k such that $S(k) \leq (k+1)\alpha$, then $\mu(M) \leq \alpha$.*

For a finite set M, by a remark of Haralambis [7], we can write $\kappa(M)$ as,

$$\kappa(M) = \max_{\substack{m = m_i + m_j \\ 1 \leq k \leq \frac{m}{2}}} (1/m) \min_i |km_i|_m, \tag{1.1}$$

where m_i, m_j are distinct elements of M.

Motzkin's density problem has wide connections to some coloring problems. The study of Motzkin's density problem is equivalent to the study of the fractional chromatic number of distance graphs. A fractional coloring of a graph G is a mapping c which assigns to each independent set I of G a non-negative weight $c(I)$ such that for each vertex x, $\sum_{x \in I} c(I) \geq 1$. The *fractional chromatic number* of G, denoted by $\chi_f(G)$, is the least total weight of a fractional coloring of G. Let D be a set of positive integers. The distance graph generated by D, denoted by $G(Z,D)$, has set Z as the vertex set, and two vertices x and y are adjacent whenever $|x - y| \in D$. It is proved by Chang et al. [2] that for any finite set D, finding the fractional chromatic number of distance graphs or the maximal density is the same problem. Precisely, they proved the next theorem.

Theorem 1.1. *For any finite set D of positive integers, $\mu(D) = 1/\chi_f\big(G(Z,D)\big)$.*

Further, the fractional chromatic number is related with another useful chromatic number called the *circular chromatic number* defined as follows: Let $k \geq 2d$ be positive integers. A (k,d)-coloring of a graph G is a mapping,

$c: V(G) \to \{0, 1, \dots k-1\}$, such that $d \le |c(u)-c(v)| \le k-d$ for any $uv \in E(G)$. The circular chromatic number of G, denoted by $\chi_c(G)$, is the minimum ratio k/d such that G admits a (k,d)-coloring. It is proved [15] that for any graph G,

$$\chi_f(G) \le \chi_c(G) \le \chi(G) = \lceil \chi_c(G) \rceil,$$

Moreover, for distance graphs $G(Z,D)$ with distance set D, the following theorem [15] relates circular chromatic number with $\kappa(D)$:

Theorem 1.2. *For any finite set D of positive integers, $\chi_c\big(G(Z,D)\big) \le \frac{1}{\kappa(D)}$.*

The values and bounds of $\mu(M)$ have been studied for several special families of sets M ([1], [2], [5], [6], [7], [4], [8], [9], [13], [12], [11]) but, in general, only for $|M| \le 2$, complete solution was given by Cantor and Gordon [1]. In this paper, we intend to study the problem of estimating the maximal density $\mu(U)$ when the set U is finite and consists of the first consecutive generalized Fibonacci numbers of odd order. The Fibonacci sequence has been generalized in many ways. One of them is the Fibonacci sequence $\{U_n\}$ of order r. Let $r \ge 1$ be an integer. The Fibonacci sequence $\{U_n\}$ of order r is given by the recurrence relation

$$U_n = U_{n-1} + U_{n-2} + \cdots + U_{n-r}, \quad \text{where} \quad n \ge r,$$

with the r initial terms

$$U_n = 0 \quad \text{for } 0 \le n \le r-2, \quad \text{and} \quad U_{r-1} = 1.$$

These generalized Fibonacci numbers are also known as the Fibonacci r-step numbers. The usual Fibonacci numbers can be obtained by fixing $r = 2$. For small values of r, these sequences are sometimes called by individual names. For $r = 3$, *tribonacci sequence*; for $r = 4$, *tetranacci sequence*, and so on. When $r = 2$, we know that the sequence $(\frac{U_n}{U_{n-1}})$ (the ratio of two consecutive Fibonacci numbers) converges to the golden ratio. A fact about this generalization is that, like the usual Fibonacci sequence, $\lim\limits_{n\to\infty} \frac{U_n}{U_{n-1}}$ exists and is the real positive root of the equation $x^r - x^{r-1} - \cdots - x - 1 = 0$. All other roots of this equation lie inside the unit circle. The polynomial $x^r - x^{r-1} - \cdots - x - 1$ has been extensively studied. For detailed work on this polynomial, one may refer to ([3], [10], [14]).

In Section 2, we give some new properties related to r-step Fibonacci numbers. These properties are then applied to derive the main results about the maximal density of odd order r-step Fibonacci numbers in Section 3. Further, in Section 4, we relate our results with the two chromatic numbers, defined earlier, of distance graphs.

2. Some properties of the Fibonacci r-step numbers

In this section, we prove some lemmas concerning the Fibonacci r-step numbers. These lemmas are then applied to prove the results of the next section.

Lemma 2.1. *For $n \geq r+1$, the elements of the set $\{U_{n-r-1}, U_{n-1}, U_n\}$ are in arithmetic progression.*

Proof. We have,

$$\begin{aligned} U_n &= U_{n-1} + U_{n-2} + \cdots + U_{n-r} \\ &= U_{n-1} + U_{n-2} + \cdots + U_{n-r} + U_{n-r-1} - U_{n-r-1} \\ &= 2U_{n-1} - U_{n-r-1}. \end{aligned}$$

Hence, the lemma. □

Lemma 2.2. *Let $r > 2$ be an odd integer. Then,*

(i) *for $r \leq i \leq 2r-1$,*

$$U_i = 2^{i-r} = 3\left(\sum_{k=1}^{\frac{r-1}{2}} U_{i-2k}\right) + (-1)^{i+1};$$

(ii) *for $i > 2r-1$,*

$$U_i = 3\left(\sum_{k=1}^{\frac{r-1}{2}} U_{i-2k}\right) + \left(\sum_{k=\frac{r+1}{2}}^{r} U_{i-2k}\right).$$

Proof.

(i) Clearly, if $i = r$, then $U_r = 1$, satisfies the formula. So, let $r+1 \leq i \leq 2r-1$. Using the recurrence $U_i = 2U_{i-1} - U_{i-r-1}$ and $U_r = U_{r-1} = 1$, we have $U_i = 2^{i-r}$. So, if i is even,

$$\begin{aligned} \sum_{k=1}^{\frac{r-1}{2}} U_{i-2k} &= (2^{i-2-r} + 2^{i-4-r} + \cdots + 2) + 1 \\ &= 2\left(\frac{2^{i-1-r} - 1}{3}\right) + 1 = \frac{2^{i-r} + 1}{3}, \end{aligned}$$

and if i is odd,

$$\sum_{k=1}^{\frac{r-1}{2}} U_{i-2k} = (2^{i-2-r} + 2^{i-4-r} + \cdots + 1) = \frac{2^{i-r} - 1}{3}.$$

Therefore, for each i,

$$U_i = 3\left(\sum_{k=1}^{\frac{r-1}{2}} U_{i-2k}\right) + (-1)^{i+1}.$$

(ii) Since $U_i = U_{i-1} + U_{i-2} + \cdots + U_{i-r}$, we have, for $i > 2r-1$,

$$\begin{aligned} U_i &= (U_{i-1} + U_{i-3} + \cdots + U_{i-r+2}) \\ &\quad + (U_{i-2} + U_{i-4} + v + U_{i-r+1}) + U_{i-r} \\ &= (2U_{i-2} - U_{i-r-2}) + \cdots + (2U_{i-r+1} - U_{i-2r+1}) \\ &\quad + (U_{i-2} + U_{i-4} + \cdots + U_{i-r+1}) + U_{i-r} \\ &= 3\left(\sum_{k=1}^{\frac{r-1}{2}} U_{i-2k}\right) + (U_{i-r} - U_{i-r-2} - U_{i-r-4} - \cdots - U_{i-2r+1}) \\ &= 3\left(\sum_{k=1}^{\frac{r-1}{2}} U_{i-2k}\right) + \left(\sum_{k=\frac{r+1}{2}}^{r} U_{i-2k}\right). \end{aligned}$$

This completes the proof of the lemma. □

Lemma 2.3. *Let $r > 2$ be an odd integer. Then, for $i \geq r$,*

$$U_i \quad \textit{and} \quad (U_{i-2} + U_{i-4} + \cdots + U_{i-r+1}) \quad \textit{are of opposite parity.}$$

Proof. We take the following two cases:

Case 1: $(r \leq i \leq 2r-1)$. Since,

$$U_i = 2^{i-r} = 3\left(\sum_{k=1}^{\frac{r-1}{2}} U_{i-2k}\right) + (-1)^{i+1},$$

we have U_i and $\sum_{k=1}^{\frac{r-1}{2}} U_{i-2k}$ are of opposite parity.

Case 2: $(2r \geq i)$. We prove this by induction on i. The basis step, $i = 2r$ is clearly true as

$$U_{2r} = 3\left(\sum_{k=1}^{\frac{r-1}{2}} U_{2r-2k}\right) + \left(\sum_{k=\frac{r+1}{2}}^{r} U_{2r-2k}\right),$$

where the second sum on the right hand side is equal to 1.

Now, let the result be true for all l such that $2r \leq l < i$. Then, we need to prove this for i as well. We have,

$$\begin{aligned} \sum_{k=\frac{r+1}{2}}^{r} U_{i-2k} &= U_{i-r-1} + (U_{i-r-3} + U_{i-r-5} + \cdots + U_{i-2r}) \\ &= U_{i-r-1} + \sum_{k=1}^{\frac{r-1}{2}} U_{(i-r-1)-2k}. \end{aligned}$$

By the induction hypothesis, U_{i-r-1} and $\sum_{k=1}^{\frac{r-1}{2}} U_{(i-r-1)-2k}$ are of opposite parity. This implies that $\sum_{k=\frac{r+1}{2}}^{r} U_{i-2k}$ is odd. Thus, U_i and $\sum_{k=1}^{\frac{r-1}{2}} U_{i-2k}$ are of opposite parity by Lemma 2.2. Hence, the lemma. $\square$

We use the following elementary property of positive integers in the proof of the next lemma.

Remark 2.1. Let $a, b, c,$ and d be positive integers with $b > d$, and $\frac{a}{b} < \frac{c}{d}$. Then

$$\frac{a-c}{b-d} < \frac{a}{b} < \frac{c}{d} .$$

Lemma 2.4. *Let $r > 2$ be an odd integer and $i \geq 2r-1$. Then,*

$$\frac{2^{r-3}-1}{3(2^{r-3})} < \frac{U_{i-2}+U_{i-4}+\cdots+U_{i-r+1}}{U_i} < \frac{2^{r-2}+1}{3(2^{r-2})}.$$

Further, for $n \geq 2r+1$, the sequence $\big(f(n)\big)$, where

$$f(n) = \frac{U_{n-2}+U_{n-4}+\cdots+U_{n-r+1}+\frac{2^{r-2}+1}{3}}{U_n+2^{r-2}},$$

is strictly decreasing.

Proof. Since $U_n = 2U_{n-1} - U_{n-r-1}$ for $n \geq r+1$, we have for $n \geq 2r$, $\frac{U_n}{U_{n-1}} < 2$. Hence, $\frac{U_n}{U_{n-i}} < 2^i$, if $n \geq r+i$. This implies that

$$\frac{U_{i-2}+U_{i-4}+\cdots+U_{i-r+1}}{U_i} > \frac{1}{2^2}+\frac{1}{2^4}+\cdots+\frac{1}{2^{r-1}},$$

if $i \geq 2r-1$. Thus, if $i \geq 2r-1$, then

$$\frac{U_{i-2}+U_{i-4}+\cdots+U_{i-r+1}}{U_i} > \frac{1}{2^2}+\frac{1}{2^4}+\cdots+\frac{1}{2^{r-1}} > \frac{2^{r-3}-1}{3(2^{r-3})}.$$

On the other hand, for $i \geq 2r-1$,

$$U_i > 3\left(\sum_{k=1}^{\frac{r-1}{2}} U_{i-2k}\right),$$

which gives

$$\frac{U_{i-2}+U_{i-4}+\cdots+U_{i-r+1}}{U_i} < \frac{2^{r-2}+1}{3(2^{r-2})}.$$

This proves the first part of the lemma.

For the second part, we apply induction on n to show that $f(n) < f(n-1)$ for all $n \geq 2r+2$. Notice that $f(2r+2) = \frac{1}{3} - \frac{2}{3(U_{2r+2}+2^{r-2})}$, and $f(2r+1) = \frac{1}{3}$. Hence, the basis step is satisfied. Since $U_n = 2U_{n-1} - U_{n-r-1}$, we have

$$\begin{aligned}
f(n) &= \frac{U_{n-2} + U_{n-4} + \cdots + U_{n-r+1} + \frac{2^{r-2}+1}{3}}{U_n + 2^{r-2}} \\
&= \frac{2\left(\sum_{k=1}^{\frac{r-1}{2}} U_{n-1-2k}\right) - \left(\sum_{k=1}^{\frac{r-1}{2}} U_{n-r-1-2k}\right) + \frac{2^{r-2}+1}{3}}{2U_{n-1} - U_{n-r-1} + 2^{r-2}} \\
&= \frac{2\left(\sum_{k=1}^{\frac{r-1}{2}} U_{n-1-2k} + \frac{2^{r-2}+1}{3}\right) - \left(\sum_{k=1}^{\frac{r-1}{2}} U_{n-r-1-2k} + \frac{2^{r-2}+1}{3}\right)}{2(U_{n-1} + 2^{r-2}) - (U_{n-r-1} + 2^{r-2})}.
\end{aligned}$$

Now, by induction hypothesis assume that

$$\begin{aligned}
f(n-1) &= \frac{\sum_{k=1}^{\frac{r-1}{2}} U_{n-1-2k} + \frac{2^{r-2}+1}{3}}{(U_{n-1} + 2^{r-2})} < f(n-2) < \cdots f(n-r-1) \\
&= \frac{\sum_{k=1}^{\frac{r-1}{2}} U_{n-r-1-2k} + \frac{2^{r-2}+1}{3}}{(U_{n-r-1} + 2^{r-2})} < \cdots f(2r+1).
\end{aligned}$$

Letting

$$a = 2\left(\sum_{k=1}^{\frac{r-1}{2}} U_{n-1-2k} + \frac{2^{r-2}+1}{3}\right), \qquad b = 2(U_{n-1} + 2^{r-2}),$$

$$c = \sum_{k=1}^{\frac{r-1}{2}} U_{n-r-1-2k} + \frac{2^{r-2}+1}{3}, \qquad \text{and} \quad d = U_{n-r-1} + 2^{r-2}$$

in Remark (2.1), we have

$$\begin{aligned}
&\frac{2\left(\sum_{k=1}^{\frac{r-1}{2}} U_{n-1-2k} + \frac{2^{r-2}+1}{3}\right) - \left(\sum_{k=1}^{\frac{r-1}{2}} U_{n-r-1-2k} + \frac{2^{r-2}+1}{3}\right)}{2(U_{n-1} + 2^{r-2}) - (U_{n-r-1} + 2^{r-2})} \\
&< \frac{\sum_{k=1}^{\frac{r-1}{2}} U_{n-1-2k} + \frac{2^{r-2}+1}{3}}{(U_{n-1} + 2^{r-2})} < \frac{\sum_{k=1}^{\frac{r-1}{2}} U_{n-r-1-2k} + \frac{2^{r-2}+1}{3}}{(U_{n-r-1} + 2^{r-2})}.
\end{aligned}$$

This gives, $f(n) < f(n-1)$. Thus, $f(n) < f(n-1)$ for all $n \geq 2r+2$.
Hence, the lemma. $\square$

Remark 2.2. We have for $r \leq n \leq 2r-1$,

$$U_n = 2^{n-r} = 3\left(\sum_{k=1}^{\frac{r-1}{2}} U_{n-2k}\right) + (-1)^{n+1}.$$

Therefore, if n is odd, then $f(n) = \frac{1}{3}$, and if n is even, then $f(n) = \frac{1}{3} + \frac{2}{3(U_n + 2^{r-2})}$. Further, $f(2r) = \frac{1}{3}$. Thus, the finite sequence $\big(f(n)\big)_{n=r}^{2r}$ is not monotonic.

3. Main Results

Theorem 3.1. *Let $r > 2$ be an odd integer and let $U = \{U_r, U_{r+1}, \dots, U_n\}$. Then,*

(i) *if $r+1 \leq n \leq 2r+1$, then*

$$\mu(U) = \kappa(U) = \frac{1}{3};$$

(ii) *if $n > 2r+1$, then*

$$\frac{1}{3} > \mu(U) \geq \kappa(U) \geq \frac{(U_{n-2} + U_{n-4} + \cdots + U_{n-r+1} + 1)2^{r-3} - U_n\left(\frac{2^{r-3}-1}{3}\right)}{U_n + 2^{r-2}}.$$

Proof. (i) We have, for $r \leq i \leq 2r-1$,

$$U_i = 2^{i-r} \equiv \pm 1 \pmod 3.$$

In addition, we also have

$$U_{2r} = 2U_{2r-1} - U_{r-1} = 2^r - 1 \equiv 1 \pmod 3,$$
$$U_{2r+1} = 2U_{2r} - U_r = 2^{r+1} - 3 \equiv 1 \pmod 3.$$

Therefore, taking $c = 1$ and $m = 3$ in Lemma 1.1, we have $\mu(U) \geq \kappa(U) \geq \frac{1}{3}$. On the other hand, any U-set cannot contain any consecutive integers as well as consecutive integers of same parity as $\{1, 2\} \subseteq U$. This implies that, $\mu(U) \leq \frac{1}{3}$. This completes the proof in this case.

(ii) Since U_n and $(U_{n-2} + U_{n-4} + \cdots + U_{n-r+1})$ are of opposite parity and $\frac{2^{r-2}+1}{3}$ is an odd integer, so

$$x = \frac{(U_n + 2^{r-2}) - (U_{n-2} + U_{n-4} + \cdots + U_{n-r+1} + \frac{2^{r-2}+1}{3})}{2}$$

is an integer.

We claim that for $r \le i \le n$,

$$U_i x \equiv \frac{U_n + 2^{r-2}}{2} - \frac{U_i\left(\sum_{k=1}^{\frac{r-1}{2}} U_{n-2k} + \frac{2^{r-2}+1}{3}\right) - \left(\sum_{k=1}^{\frac{r-1}{2}} U_{i-2k}\right)(U_n+2^{r-2})}{2} \pmod{U_n+2^{r-2}}.$$

If U_i is even, then

$$\begin{aligned} U_i x &= U_i \frac{(U_n + 2^{r-2}) - (U_{n-2} + U_{n-4} + \cdots + U_{n-r+1} + \frac{2^{r-2}+1}{3})}{2} \\ &\equiv -U_i \frac{(U_{n-2} + U_{n-4} + \cdots + U_{n-r+1} + \frac{2^{r-2}+1}{3})}{2} \\ &\equiv \frac{(U_{i-2} + \cdots + U_{i-r+1} + 1)(U_n + 2^{r-2})}{2} \\ &\quad - U_i \frac{(U_{n-2} + \cdots + U_{n-r+1} + \frac{2^{r-2}+1}{3})}{2} \pmod{U_n+2^{r-2}}. \end{aligned}$$

Next, if U_i is odd, then

$$\begin{aligned} U_i x &= U_i \frac{(U_n + 2^{r-2}) - (U_{n-2} + U_{n-4} + \cdots + U_{n-r+1} + \frac{2^{r-2}+1}{3})}{2} \\ &\equiv \frac{U_n + 2^{r-2}}{2} - U_i \frac{(U_{n-2} + U_{n-4} + \cdots + U_{n-r+1} + \frac{2^{r-2}+1}{3})}{2} \\ &\equiv \frac{(U_{i-2} + \cdots + U_{i-r+1} + 1)(U_n + 2^{r-2})}{2} \\ &\quad - U_i \frac{(U_{n-2} + \cdots + U_{n-r+1} + \frac{2^{r-2}+1}{3})}{2} \pmod{U_n + 2^{r-2}}. \end{aligned}$$

Thus, we obtain our claim.

Next, we have for $r \le i \le 2r-1$,

$$U_i = 3\left(\sum_{k=1}^{\frac{r-1}{2}} U_{i-2k}\right) + (-1)^{i+1}.$$

Hence, it is easy to see that

$$(U_{i-2} + \cdots + U_{i-r+1} + 1)(U_n + 2^{r-2}) - U_i(U_{n-2} + \cdots + U_{n-r+1} + \frac{2^{r-2}+1}{3}) > 0.$$

Further, for $n \geq i > 2r-1$, we have, by Lemma 2.4, that

$$\frac{2^{r-3}-1}{3(2^{r-3})} < \frac{U_{i-2}+U_{i-4}+\cdots+U_{i-r+1}}{U_i}.$$

This gives

$$\frac{2^{r-3}-1}{3(2^{r-3})} < \frac{U_{i-2}+U_{i-4}+\cdots+U_{i-r+1}+1}{U_i}.$$

Therefore,

$$\frac{2^{r-3}-1}{3(2^{r-3})} < \frac{U_{i-2}+U_{i-4}+\cdots+U_{i-r+1}+\frac{2^{r-2}+1}{3}}{U_i+2^{r-2}}$$

$$< \frac{U_{i-2}+U_{i-4}+\cdots+U_{i-r+1}+1}{U_i}.$$

Thus,

$$(U_{i-2}+\cdots+U_{i-r+1}+1)(U_n+2^{r-2})-U_i(U_{n-2}+\cdots+U_{n-r+1}+\frac{2^{r-2}+1}{3})>0.$$

Now for all i, there are two cases: either

$$U_i\left(\sum_{k=1}^{\frac{r-1}{2}} U_{n-2k}+\frac{2^{r-2}+1}{3}\right) > \left(\sum_{k=1}^{\frac{r-1}{2}} U_{i-2k}\right)(U_n+2^{r-2})$$

or

$$U_i\left(\sum_{k=1}^{\frac{r-1}{2}} U_{n-2k}+\frac{2^{r-2}+1}{3}\right) < \left(\sum_{k=1}^{\frac{r-1}{2}} U_{i-2k}\right)(U_n+2^{r-2}).$$

Equality may also hold for some i, but in that case the maximum absolute remainder modulo (U_n+2^{r-2}) is $\frac{U_n+2^{r-2}}{2}$, which is not helpful in calculating $\kappa(M)$ (see (1.1)) as we shall see below in both the cases that inequality does hold for some i.

Case 1. $U_i\left(\sum_{k=1}^{\frac{r-1}{2}} U_{n-2k}+\frac{2^{r-2}+1}{3}\right) > \left(\sum_{k=1}^{\frac{r-1}{2}} U_{i-2k}\right)(U_n+2^{r-2}).$

Clearly,

$$|U_i x|_{(U_n+2^{r-2})} = \frac{U_n+2^{r-2}}{2} - \frac{U_i\left(\sum_{k=1}^{\frac{r-1}{2}} U_{n-2k}+\frac{2^{r-2}+1}{3}\right)-\left(\sum_{k=1}^{\frac{r-1}{2}} U_{i-2k}\right)(U_n+2^{r-2})}{2}.$$

In addition, by Lemma 2.4, for all i, $n \geq i \geq 2r-1$, we have

$$\frac{\sum_{k=1}^{\frac{r-1}{2}} U_{n-2k} + \frac{2^{r-2}+1}{3}}{U_n + 2^{r-2}} \leq \frac{\sum_{k=1}^{\frac{r-1}{2}} U_{i-2k} + \frac{2^{r-2}+1}{3}}{U_i + 2^{r-2}}.$$

Therefore,

$$\begin{aligned}
0 &< U_i \left(\sum_{k=1}^{\frac{r-1}{2}} U_{n-2k} + \frac{2^{r-2}+1}{3} \right) - \left(\sum_{k=1}^{\frac{r-1}{2}} U_{i-2k} \right) (U_n + 2^{r-2}) \\
&\leq (U_n + 2^{r-2}) \left(\frac{2^{r-2}+1}{3} \right) - 2^{r-2} \left(\sum_{k=1}^{\frac{r-1}{2}} U_{n-2k} + \frac{2^{r-2}+1}{3} \right) \\
&= U_n \left(\frac{2^{r-2}+1}{3} \right) - 2^{r-2} \left(\sum_{k=1}^{\frac{r-1}{2}} U_{n-2k} \right).
\end{aligned}$$

Observe here that equality holds when $i = n$, Therefore, for all i, $n \geq i \geq 2r-1$, we have

$$\begin{aligned}
\min(|U_i x|)_{(U_n+2^{r-2})} &= \frac{U_n + 2^{r-2}}{2} - \frac{U_n \left(\frac{2^{r-2}+1}{3} \right) - 2^{r-2} \left(\sum_{k=1}^{\frac{r-1}{2}} U_{n-2k} \right)}{2} \\
&= (U_{n-2} + U_{n-4} + \cdots + U_{n-r+1} + 1)2^{r-3} - \frac{U_n}{2} \left(\frac{2^{r-2}-2}{3} \right) \\
&= (U_{n-2} + U_{n-4} + \cdots + U_{n-r+1} + 1)2^{r-3} - U_n \left(\frac{2^{r-3}-1}{3} \right).
\end{aligned}$$

Now, let $2r-2 \geq i \geq r$. We observe below that the inequality condition of Case 1 is satisfied by only those U_i for which i is odd (the rest of the U_is will satisfy the reverse inequality condition mentioned ahead in Case 2).

Let i be odd such that $r \leq i \leq 2r-2$. Clearly,

$$U_i = 3 \left(\sum_{k=1}^{\frac{r-1}{2}} U_{i-2k} \right) + 1$$

and, as $n > 2r+1 > 2r-1$, we have

$$\frac{2^{r-3}-1}{3(2^{r-3})} < \frac{\sum_{k=1}^{\frac{r-1}{2}} U_{n-2k}}{U_n}.$$

This implies

$$\frac{U_n - 3\sum_{k=1}^{\frac{r-1}{2}} U_{n-2k} - 1}{U_n + 2^{r-2}} < \frac{1}{2^{r-3}} \leq \frac{1}{U_i} = \frac{U_i - 3\sum_{k=1}^{\frac{r-1}{2}} U_{i-2k}}{U_i}.$$

Therefore,

$$U_i\left(\sum_{k=1}^{\frac{r-1}{2}} U_{n-2k} + \frac{2^{r-2}+1}{3}\right) > \left(\sum_{k=1}^{\frac{r-1}{2}} U_{i-2k}\right)(U_n + 2^{r-2}).$$

Observe that, since $n > 2r+1$, we have $U_n > 3\sum_{k=1}^{\frac{r-1}{2}} U_{n-2k} + 1$. Therefore,

$$\begin{aligned} & U_n\left(\frac{2^{r-2}+1}{3}\right) - 2^{r-2}\left(\sum_{k=1}^{\frac{r-1}{2}} U_{n-2k}\right) \\ & - \left(U_i\left(\sum_{k=1}^{\frac{r-1}{2}} U_{n-2k} + \frac{2^{r-2}+1}{3}\right) - \left(\sum_{k=1}^{\frac{r-1}{2}} U_{i-2k}\right)(U_n + 2^{r-2})\right) \\ & = \frac{U_i + 2^{r-2}}{3}\left(U_n - 3\sum_{k=1}^{\frac{r-1}{2}} U_{n-2k} - 1\right) \geq 0. \end{aligned}$$

Therefore, in this case, for all i, $r \leq i \leq n$,

$$\min(|U_i x|)_{(U_n+2^{r-2})} = (U_{n-2} + U_{n-4} + \cdots + U_{n-r+1} + 1)2^{r-3} - U_n\left(\frac{2^{r-3}-1}{3}\right).$$

Case 2. $U_i\left(\sum_{k=1}^{\frac{r-1}{2}} U_{n-2k} + \frac{2^{r-2}+1}{3}\right) < \left(\sum_{k=1}^{\frac{r-1}{2}} U_{i-2k}\right)(U_n + 2^{r-2}).$

We have,

$$\begin{aligned} & U_i\left(\sum_{k=1}^{\frac{r-1}{2}} U_{n-2k} + \frac{2^{r-2}+1}{3}\right) - \left(\sum_{k=1}^{\frac{r-1}{2}} U_{i-2k}\right)(U_n + 2^{r-2}) \\ & = U_i\left(\sum_{k=1}^{\frac{r-1}{2}} U_{n-2k}\right) - \left(\sum_{k=1}^{\frac{r-1}{2}} U_{i-2k}\right)U_n + \left(U_i\frac{2^{r-2}+1}{3} - 2^{r-2}(\sum_{k=1}^{\frac{r-1}{2}} U_{i-2k})\right) \end{aligned}$$

Therefore, for all i, $n \geq i \geq 2r-1$, we have

$$U_i\left(\sum_{k=1}^{\frac{r-1}{2}} U_{n-2k} + \frac{2^{r-2}+1}{3}\right) - \left(\sum_{k=1}^{\frac{r-1}{2}} U_{i-2k}\right)(U_n + 2^{r-2})$$
$$\geq U_i\left(\sum_{k=1}^{\frac{r-1}{2}} U_{n-2k}\right) - \left(\sum_{k=1}^{\frac{r-1}{2}} U_{i-2k}\right)U_n$$
$$\geq 2^{r-2}\frac{3}{2^{r-2}+1}\left(\sum_{k=1}^{\frac{r-1}{2}} U_{i-2k}\right)\left(\sum_{k=1}^{\frac{r-1}{2}} U_{n-2k}\right) - \left(\sum_{k=1}^{\frac{r-1}{2}} U_{i-2k}\right)U_n$$
$$= \frac{3\left(\sum_{k=1}^{\frac{r-1}{2}} U_{i-2k}\right)}{2^{r-2}+1}\left(2^{r-2}\left(\sum_{k=1}^{\frac{r-1}{2}} U_{n-2k}\right) - U_n\left(\frac{2^{r-2}+1}{3}\right)\right)$$
$$\geq 2^{r-2}\left(\sum_{k=1}^{\frac{r-1}{2}} U_{n-2k}\right) - U_n\left(\frac{2^{r-2}+1}{3}\right).$$

Since, for $n > 2r+1 > 2r-1$, we have

$$\frac{2^{r-3}-1}{3(2^{r-3})} < \frac{\sum_{k=1}^{\frac{r-1}{2}} U_{n-2k}}{U_n}.$$

This implies that

$$U_n\left(\frac{2^{r-2}+1}{3}\right) - 2^{r-2}\sum_{k=1}^{\frac{r-1}{2}} U_{n-2k} < U_n + 2^{r-2}.$$

Therefore, for all i, $n \geq i \geq 2r-1$, we have

$$|U_i x|_{U_n+2^{r-2}}$$
$$= \frac{U_n + 2^{r-2}}{2} + \frac{U_i\left(\sum_{k=1}^{\frac{r-1}{2}} U_{n-2k} + \frac{2^{r-2}+1}{3}\right) - \left(\sum_{k=1}^{\frac{r-1}{2}} U_{i-2k}\right)(U_n + 2^{r-2})}{2}.$$

Observe that for $n > 2r+1$, we have $U_n > 3\sum_{k=1}^{\frac{r-1}{2}} U_{n-2k} + 1$, which implies

$$\frac{3\sum_{k=1}^{\frac{r-1}{2}} U_{n-2k} + 1 - U_n}{U_n + 2^{r-2}} < \frac{1}{2^{r-2}}.$$

Now, let $2r-2 \geq i \geq r$ and i is even. Clearly,

$$U_i = 3\left(\sum_{k=1}^{\frac{r-1}{2}} U_{i-2k}\right) - 1.$$

Therefore,

$$\frac{3\sum_{k=1}^{\frac{r-1}{2}} U_{n-2k} + 1 - U_n}{U_n + 2^{r-2}} < \frac{3\sum_{k=1}^{\frac{r-1}{2}} U_{i-2k} - U_i}{U_i},$$

and hence

$$U_i\left(\sum_{k=1}^{\frac{r-1}{2}} U_{n-2k} + \frac{2^{r-2}+1}{3}\right) - \left(\sum_{k=1}^{\frac{r-1}{2}} U_{i-2k}\right)(U_n + 2^{r-2}) < 0.$$

Moreover, when $U_i = 3\left(\sum_{k=1}^{\frac{r-1}{2}} U_{i-2k}\right) - 1$,

$$\begin{aligned}
&U_i\left(\sum_{k=1}^{\frac{r-1}{2}} U_{n-2k} + \frac{2^{r-2}+1}{3}\right) - \left(\sum_{k=1}^{\frac{r-1}{2}} U_{i-2k}\right)(U_n + 2^{r-2}) \\
&\quad - \left(2^{r-2}\left(\sum_{k=1}^{\frac{r-1}{2}} U_{n-2k}\right) - U_n\left(\frac{2^{r-2}+1}{3}\right)\right) \\
&= \frac{U_i - 2^{r-2}}{3}\left(3\sum_{k=1}^{\frac{r-1}{2}} U_{n-2k} + 1 - U_n\right) \leq 0.
\end{aligned}$$

Notice that the above inequality is sharp for $i = 2r-2$. Therefore, in this case for all i, $r \leq i \leq n$,

$$\begin{aligned}
\min(|U_i x|)_{(U_n + 2^{r-2})} &= \frac{U_n + 2^{r-2}}{2} + \frac{2^{r-2}\left(\sum_{k=1}^{\frac{r-1}{2}} U_{n-2k}\right) - U_n\left(\frac{2^{r-2}+1}{3}\right)}{2} \\
&= (U_{n-2} + U_{n-4} + \cdots +_{n-r+1} + 1)2^{r-3} - U_n\left(\frac{2^{r-3}-1}{3}\right).
\end{aligned}$$

Therefore, by the definition of $\kappa(U)$ (see (1.1)), in both the cases,

$$\mu(U) \geq \kappa(U) \geq \frac{(U_{n-2} + U_{n-4} + \cdots + U_{n-r+1} + 1)2^{r-3} - U_n\left(\frac{2^{r-3}-1}{3}\right)}{U_n + 2^{r-2}}.$$

On the other hand, when r is odd,

$$U_{2r+2} = 2U_{2r+1} - U_{r+1} = 2^{r+2} - 8 \equiv 0 \pmod 3.$$

We decompose $\{0, 1, 2, \ldots, U_{2r+2}\}$ into the sets $\{3i, 3i+1, 3i+2\}$ and $\{U_{2r+2}\}$, where $0 \le i \le \frac{U_{2r+2}-3}{3}$. Let S be an U-set with $0 \in S$. Then it is clear that $|S \cap \{3i, 3i+1, 3i+2\}| \le 1$ and $U_{2r+2} \notin S$. Thus, using Lemma 1.2, for $n > 2r+1$, we have $\mu(U) \le \mu\{U_r, U_{r+1}, \ldots, U_{2r+2}\} \le \frac{U_{2r+2}}{3(U_{2r+2}+1)} < \frac{1}{3}$. Therefore,

$$\frac{1}{3} > \mu(U) \ge \kappa(U) \ge \frac{(U_{n-2} + U_{n-4} + \cdots + U_{n-r+1} + 1)2^{r-3} - U_n\left(\frac{2^{r-3}-1}{3}\right)}{U_n + 2^{r-2}}.$$

This completes the proof of the theorem. □

The following theorem directly follows from the above theorem.

Theorem 3.2. *Let $U = \{U_r, U_{r+1}, \ldots, U_n\}$ and $n > 2r+1$. Then*

$$\kappa(U) \ge \frac{1}{3} - 2^{r-3}\left(\frac{1}{3} - \frac{\alpha-1}{\alpha+1}\right) > \frac{1}{4},$$

where α is the real root of the polynomial $f(x) = x^r - x^{r-1} - x^{r-2} - \cdots - x - 1$.

P r o o f. We have that $\left(\frac{U_{n-2}+U_{n-4}+\cdots+U_{n-r+1}+\frac{2^{r-2}+1}{3}}{U_n+2^{r-2}}\right)$ is a decreasing sequence, by Lemma 2.4. Now,

$$\begin{aligned}
&\frac{\left(U_{n-2} + U_{n-4} + \cdots + U_{n-r+1} + \frac{2^{r-2}+1}{3}\right)2^{r-3} - (U_n + 2^{r-2})\frac{2^{r-3}-1}{3}}{U_n + 2^{r-2}} \\
&= \frac{(U_{n-2} + U_{n-4} + \cdots + U_{n-r+1} + 1)2^{r-3} - U_n\left(\frac{2^{r-3}-1}{3}\right)}{U_n + 2^{r-2}} \\
&= z_n, \quad \text{say.}
\end{aligned}$$

Then (z_n) is also a decreasing sequence. We find the limit of the sequence (z_n). Note that (U_n) is an increasing sequence. Hence, $U_n \to \infty$ as $n \to \infty$. Let

$$\begin{aligned}
z &= \lim_{n\to\infty} z_n \\
&= \lim_{n\to\infty} \frac{(U_{n-2} + U_{n-4} + \cdots + U_{n-r+1} + 1)2^{r-3} - U_n\left(\frac{2^{r-3}-1}{3}\right)}{U_n + 2^{r-2}} \\
&= \lim_{n\to\infty} \frac{\left(\frac{U_{n-2}}{U_n} + \frac{U_{n-4}}{U_n} + \cdots + \frac{U_{n-r+1}}{U_n} + \frac{1}{U_n}\right)2^{r-3} - \left(\frac{2^{r-3}-1}{3}\right)}{1 + \frac{2^{r-2}}{U_n}}.
\end{aligned}$$

Since

$$\lim_{n\to\infty} \frac{U_n}{U_{n-1}} = \alpha, \quad \text{we have} \quad \lim_{n\to\infty} \frac{U_n}{U_{n-k}} = \lim_{n\to\infty} \frac{U_n}{U_{n-1}} \cdot \frac{U_{n-1}}{U_{n-2}} \cdots \frac{U_{n-k+1}}{U_{n-k}} = \alpha^k.$$

Therefore,
$$z=\left(\frac{1}{\alpha^2}+\frac{1}{\alpha^4}+\cdots+\frac{1}{\alpha^{r-1}}\right)2^{r-3}-\left(\frac{2^{r-3}-1}{3}\right).$$

Letting α the real positive root of $f(x)$, we get

$$\begin{aligned}
\alpha^r-\alpha^{r-1}-\alpha^{r-2}-\cdots\alpha-1 &= 0\\
\Rightarrow 1-\frac{1}{\alpha}-\frac{1}{\alpha^2}-\cdots-\frac{1}{\alpha^r} &= 0\\
\Rightarrow 1-\frac{1}{\alpha}-\frac{1}{\alpha^3}-\cdots-\frac{1}{\alpha^r} &= \frac{1}{\alpha^2}+\frac{1}{\alpha^4}+\cdots+\frac{1}{\alpha^{r-1}}\\
\Rightarrow 1-\frac{1}{\alpha}-\frac{1}{\alpha}\left(\frac{1}{\alpha^2}+\frac{1}{\alpha^4}+\cdots+\frac{1}{\alpha^{r-1}}\right) &= \frac{1}{\alpha^2}+\frac{1}{\alpha^4}+\cdots+\frac{1}{\alpha^{r-1}}\\
\Rightarrow \frac{1}{\alpha^2}+\frac{1}{\alpha^4}+\cdots+\frac{1}{\alpha^{r-1}} &= \frac{\alpha-1}{\alpha+1}.
\end{aligned}$$

Therefore,
$$z=\left(\frac{\alpha-1}{\alpha+1}\right)2^{r-3}-\left(\frac{2^{r-3}-1}{3}\right)=\frac{1}{3}-2^{r-3}\left(\frac{1}{3}-\frac{\alpha-1}{\alpha+1}\right).$$

However, z is the limit of the decreasing sequence (z_n), where
$$z_n=\frac{(U_{n-2}+U_{n-4}+\cdots+U_{n-r+1}+1)2^{r-3}-U_n\left(\frac{2^{r-3}-1}{3}\right)}{U_n+2^{r-2}}.$$

This implies that for every n, $z_n\geq z$. Now using Theorem 3.1, we have
$$\begin{aligned}
\kappa(U) &\geq \frac{(U_{n-2}+U_{n-4}+\cdots+U_{n-r+1}+1)2^{r-3}-U_n\left(\frac{2^{r-3}-1}{3}\right)}{U_n+2^{r-2}}\\
&\geq \frac{1}{3}-2^{r-3}\left(\frac{1}{3}-\frac{\alpha-1}{\alpha+1}\right).
\end{aligned}$$

Let $g(x)=(x-1)f(x)=x^{r+1}-2x^r+1$. Since $\frac{2^r+1}{3}<2^{r-1}$, we have

$$\begin{aligned}
g\left(2-\frac{3}{2^r+1}\right) &= \left(2-\frac{3}{2^r+1}\right)^{r+1}-2\left(2-\frac{3}{2^r+1}\right)^r+1\\
&< \left(2-\frac{1}{2^{r-1}}\right)^{r+1}-2\left(2-\frac{1}{2^{r-1}}\right)^r+1\\
&= -2\left(1-\frac{1}{2^r}\right)^r+1<0.
\end{aligned}$$

Now as $g(2) = 1 > 0$, $g(x)$ has at least one root in the interval $(2 - \frac{3}{2^r+1}, 2)$. But using Descartes' rule of signs, $g(x)$ has only two positive roots. Therefore, positive root of $g(x)$ other that 1 is α. Hence, $\alpha > 2 - \frac{3}{2^r+1} = \frac{2^{r-1}-1}{2^r+1}$, which implies that $\frac{\alpha-1}{\alpha+1} > \frac{2^{r-1}-1}{32^{r-1}}$. Therefore,

$$\kappa(U) \geq \frac{1}{3} - 2^{r-3}\left(\frac{1}{3} - \frac{\alpha-1}{\alpha+1}\right) > \frac{1}{4}.$$

This completes the proof of the theorem. □

4. Chromatic number of the distance graph $G(Z,U)$

Using Theorems 3.1 and 3.2, we determine below the chromatic number of the distance graph $G(Z,U)$.

Theorem 4.1. *Let* $U = \{U_r, U_{r+1}, \dots, U_n\}$. *Then*

(i) *if* $r+1 \leq n \leq 2r+1$, *then*

$$\frac{1}{\mu(U)} = \chi_f\big(G(Z,U)\big) = \chi_c\big(G(Z,U)\big) = \chi\big(G(Z,U)\big) = \frac{1}{\kappa(U)} = 3;$$

(ii) *if* $n > 2r+1$, *then*

$$3 < \frac{1}{\mu(U)} = \chi_f\big(G(Z,U)\big) \leq \chi_c\big(G(Z,U)\big) \leq \frac{1}{\kappa(U)} < 4,$$

and

$$\chi\big(G(Z,U)\big) = 4.$$

Proof. Using Theorems 1.1 and 1.2 for a distance set D, we have

$$\frac{1}{\mu(D)} = \chi_f\big(G(Z,D)\big) \leq \chi_c\big(G(Z,D)\big) \leq \frac{1}{\kappa(D)},$$

and

$$\lceil \chi_c\big(G(Z,D)\big) \rceil = \chi\big(G(Z,D)\big).$$

Therefore, by Theorem 3.1, if $r+1 \leq n \leq 2r+1$, then

$$\frac{1}{\mu(U)} = \chi_f\big(G(Z,U)\big) = \chi_c\big(G(Z,U)\big) = \chi\big(G(Z,U)\big) = \frac{1}{\kappa(U)} = 3.$$

Next, if $n > 2r+1$, then using Theorem 3.1 and Theorem 3.2,

$$3 < \frac{1}{\mu(U)} = \chi_f\big(G(Z,U)\big) \leq \chi_c\big(G(Z,U)\big) \leq \frac{1}{\kappa(U)} < 4,$$

and

$$\chi\big(G(Z,U)\big) = \lceil \chi_c\big(G(Z,U)\big) \rceil = 4.$$ □

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