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WEAK UNIVERSALITY THEOREM ON THE APPROXIMATION OF ANALYTIC FUNCTIONS BY SHIFTS OF THE RIEMANN ZETA-FUNCTION FROM A BEATTY SEQUENCE

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ABSTRACT. In this paper, we prove a discrete analogue of Voronin's early finite-dimensional approximation result with respect to terms from a given Beatty sequence and make use of Taylor approximation in order to derive a weak universality statement.

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1. Introduction

Let $s = \sigma + it \in \mathbb{C}$ (where $\sigma = \mathrm{Re}(s)$ and $t = \mathrm{Im}(s)$) and $\zeta(s)$ the Riemann zeta-function. This function is usually defined first on the half-plane $\{s : \mathrm{Re}(s) > 1\}$ by the formula

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

and then extended to a meromorphic function on the whole complex plane, with one simple pole at $s = 1$ and no other singularity.

In 1914, H. Bohr and R. Courant [3] proved that, for any $\sigma_0 \in (\frac{1}{2}, 1)$ the set

$$\{\zeta(\sigma_0 + i\tau) : \tau \in \mathbb{R}\}$$

is dense in $\mathbb{C}$. In the next year Bohr [2] proved that the same result holds for $\log \zeta(\sigma_0 + i\tau)$. These results are called denseness theorems.

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Bohr's line of investigations appears to have been almost totally abandoned for some decades. Only in 1972, S. M. Voronin [8] obtained some significant generalizations of Bohr's denseness result.

Theorem 1. *Let m be a natural number and h a positive real number. For any fixed numbers $s_1, \ldots, s_m$ with $\frac{1}{2} < \mathrm{Re}(s_k) \leq 1$ for $1 \leq k \leq m$ and $s_k \neq s_\ell$ for $k \neq \ell$, the set*

$$\left\{\left(\zeta(s_1 + inh), \zeta(s_2 + inh), \ldots, \zeta(s_m + inh)\right) : n \in \mathbb{N}\right\}$$

is dense in $\mathbb{C}^m$. Moreover, for any fixed number s_0 in the strip $1/2 < \sigma \leq 1$, the set

$$\left\{\left(\zeta(s_0 + inh), \zeta'(s_0 + inh), \ldots, \zeta^{(m-1)}(s_0 + inh)\right) : n \in \mathbb{N}\right\}$$

is dense in $\mathbb{C}^m$.

However, Voronin did not stop there and in 1975 proved a remarkable universality theorem for $\zeta(s)$ which states, roughly speaking, that any non-vanishing analytic function can be approximated by certain purely imaginary shifts of the zeta-function in the critical strip.

Theorem 2. *Let $0 < r < 1/4$ and suppose that $g(s)$ is a non-vanishing continuous function on the disk $|s| \leq r$ which is analytic in the interior. Then, for any $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T} \mathrm{meas}\left\{\tau \in [0,T] : \max_{|s|\leq r}\left|\zeta\left(s + \frac{3}{4} + i\tau\right) - g(s)\right| < \varepsilon\right\} > 0\,.$$

Voronin called his universality theorem the *theorem about little disks.* A. Reich [7] and B. Bagchi [1] improved Voronin's result significantly in replacing the disk by an arbitrary compact set in the right half of the critical strip with connected complement and they even obtained a discrete analogue of it.

Theorem 3. *Suppose that K is a compact subset of the strip $1/2 < \mathrm{Re}(s) < 1$ with connected complement, and let $g(s)$ be a non-vanishing continuous function on K which is analytic in the interior of K. Then, for any $\varepsilon > 0$ and any $h > 0$,*

$$\liminf_{N\to\infty} \frac{1}{N} \mathrm{card}\left\{n \in \mathbb{N} \cap (0,N] : \max_{s\in K}|\zeta(s + inh) - g(s)| < \varepsilon\right\} > 0\,.$$

We note that Theorem 3 clearly implies both parts of Theorem 1 (except of $\mathrm{Re}(s_0) = 1$), since the truncated Taylor series of the target function $g(s)$ can be approximated by the truncated Taylor series of a certain shift of the zeta-function. Although Theorem 1 does not suffice to prove Theorem 3, we can derive from it a weak form of universality of the zeta-function as it was first indicated by R. Garunkštis, A. Laurinčikas, K. Matsumoto, J. Steuding and R. Steuding [5].

The aim of this note is to replace the arithmetical progression $(nh)_{n\in\mathbb{N}}$ in Theorem 1 by the sequence $(\lfloor n\alpha\rfloor h)_{n\in\mathbb{N}}$ with a fixed irrational number $\alpha > 0$. Here $\lfloor x\rfloor$ denotes the largest integer which is less or equal to x and for given $\alpha > 0$, the sequence $(\lfloor n\alpha\rfloor)_{n\in\mathbb{N}}$ is called Beatty sequence. We will consider only the case $\alpha > 1$ since for $\alpha < 1$ the discrete terms of the Beatty sequence is all the natural numbers and thus we get Theorem 1. Also, h will not be a random positive number but a number belonging to

$$L(\alpha) \cap [0, +\infty)\,,$$

where

$$L(\alpha) = \left\{ h \in \mathbb{R} : 1,\ \alpha^{-1},\ \frac{h}{2\pi}\ln p_1,\ \frac{h}{2\pi}\ln p_2,\ \ldots \ \text{ are linearly independent over } \mathbb{Q} \right\}.$$

We will show later on that $L(\alpha) \cap [0, +\infty) \neq \emptyset$ for every irrational number α.

Now, using the same arguments as Voronin did in [8], we will prove the following

Theorem 4 (Main theorem). *Let m be a natural number and $\alpha > 1$ an irrational number. Let also $s_0, s_1, \ldots, s_m$ be fixed numbers with*

$$\frac{1}{2} < \mathrm{Re}(s_k) \leq 1 \quad \textit{for } 0 \leq k \leq m \quad \textit{and} \quad s_k \neq s_\ell \quad \textit{for } k \neq \ell\,.$$

Then, for every $h \in L(\alpha) \cap [0, +\infty)$, the sets

$$\left\{ \left(\zeta(s_1 + i\lfloor n\alpha\rfloor h), \zeta(s_2 + i\lfloor n\alpha\rfloor h), \ldots, \zeta(s_m + i\lfloor n\alpha\rfloor h) \right) : n \in \mathbb{N} \right\}$$

and

$$\left\{ \left(\zeta(s_0 + i\lfloor n\alpha\rfloor h), \zeta'(s_0 + i\lfloor n\alpha\rfloor h), \ldots, \zeta^{(m-1)}(s_0 + i\lfloor n\alpha\rfloor h) \right) : n \in \mathbb{N} \right\}$$

are dense in $\mathbb{C}^m$.

Combining the preceding theorem and the method introduced in [5], we will also derive

Theorem 5 (Weak Universality). *Let $\sigma_0 \in (1/2, 1]$, $g : K = \overline{D(s_0, r)} \to \mathbb{C}$ continuous and analytic in the interior of K, and $\alpha > 1$ irrational. Then, for every $h \in L(\alpha) \cap [0, +\infty)$ and for every $\varepsilon > 0$, there exists*

$$n = n(\varepsilon, h) \in \mathbb{N} \quad \textit{and} \quad \delta = \delta(\varepsilon, h) \in (0, 1)$$

such that

$$\max_{|s - s_0| \leq \delta r} \left| \zeta\left(s + i\lfloor n\alpha\rfloor h\right) - g(s) \right| < \varepsilon.$$

2. Uniform distribution mod 1 and a set of full Lebesgue measure

Part of the proof that Voronin gave for Theorem 1 and that we will similarly give for Theorem 4, relies on the theory of uniformly distributed sequences. A beautiful monograph on this theory is [6]. The definition, theorems and corollaries that are stated below can be found there. But before that we introduce some notation. If $\mathtt{x} = (x_1, \ldots, x_\ell) \in \mathbb{R}^\ell$, then $\{\mathtt{x}\} = (\{x_1\}, \ldots, \{x_\ell\})$. Here $\{x_i\}$ denotes the fractional part of the real number x_i.

Definition 1. A sequence of points $(\mathtt{x}_n)_{n\in\mathbb{N}}$ belonging to $\mathbb{R}^\ell$ is said to be uniformly distributed mod 1 (u.d. mod 1) in $\mathbb{R}^\ell$ if for every box $B = I_1 \times \cdots \times I_\ell$ in $[0,1]^\ell$ (i.e., a cartesian product of intervals), the relation

$$\lim_{N\to\infty} \frac{\{1 \le n \le N : \{\mathtt{x}_n\} \in B\}}{N} = |I_1||I_2| \ldots |I_\ell| = \operatorname{meas}(B)$$

holds.

One of the many advantages when dealing with u.d. mod 1 sequences is a useful connection between sums and integrals, as the next theorem states.

Theorem 6. *A sequence $(\mathtt{x}_n)_{n\in\mathbb{N}}$ is u.d. mod 1 in $\mathbb{R}^\ell$ if and only if for every continuous complex-valued f on $[0,1]^\ell$, the relation*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(\{\mathtt{x}_n\}) = \int\limits_{[0,1]^\ell} f(\mathtt{x}) \mathrm{d}\mathtt{x}$$

holds.

P r o o f. For the proof, see [6], Chapter 1, Theorem 6.1. In fact, the condition of f being continuous can be relaxed to that of both $\mathrm{Re}f$ and $\mathrm{Im}f$ being Riemann integrable. □

Although the multi-dimensional definition complicates somewhat the study of whether a sequence is u.d. mod 1 or not, there exists a theorem that allows us to induce the process in the one-dimensional case.

Theorem 7. *A sequence $(\mathtt{x}_n)_{n\in\mathbb{N}}$ is u.d. mod 1 in $\mathbb{R}^\ell$ if and only if for every lattice point $\mathtt{k} \in \mathbb{Z}^\ell$, $\mathtt{k} \ne \mathbf{0}$, the sequence of real numbers $(\langle \mathtt{k}, \mathtt{x}_n \rangle)_{n\in\mathbb{N}}$ is u.d. mod 1 in $\mathbb{R}$. Here $\langle \cdot,\cdot \rangle$ denotes the inner product as it is usually defined on the vector space $\mathbb{R}^\ell$.*

P r o o f. For the proof, see [6], Chapter 1, Theorem 6.3. □

Corollary 1. *Let* $(\theta_k)_{k\in\mathbb{N}}$ *be a sequence of real numbers such that* $1, \theta_1, \theta_2, \dots$ *are linearly independent over* $\mathbb{Q}$*. Then, for any* $\ell \in \mathbb{N}$ *and any* $k_1, \dots, k_\ell \in \mathbb{N}$ *pairwise distinct, the sequence* $(n\theta_{k_1}, \dots, n\theta_{k_\ell})$, $n = 1, 2, \dots,$ *is u.d. mod 1 in* $\mathbb{R}^\ell$.

Proof. For the proof, see [6], Chapter 1, Example 6.1. □

It is desirable to substitute n from the above corollary with $\lfloor n\alpha \rfloor$ for given irrational $\alpha > 1$. D. Carlson [4] obtained a necessary and sufficient condition for that to happen in the one-dimensional case, and with the assistance of Theorem 7 we will be able to reformulate Corollary 1.

Theorem 8. *For rational* α*, the sequence* $(\lfloor n\alpha \rfloor \theta)_{n\in\mathbb{N}}$ *is u.d. mod 1 either for all irrationals* θ *or for no real number* θ*, depending on whether* $\alpha \neq 0$ *or* $\alpha = 0$*. If* α *is irrational, then* $(\lfloor n\alpha \rfloor \theta)_{n\in\mathbb{N}}$ *is u.d. mod 1 in* $\mathbb{R}$ *if and only if* $1, \alpha, \alpha\theta$ *are linearly independent over* $\mathbb{Q}$ *(or equivalently* $1, \alpha^{-1}, \theta$ *are linearly independent over* $\mathbb{Q}$*).*

Proof. For the proof, see [6], Chapter 5, Theorem 1.8. □

Corollary 2. *Let* α *be an irrational number and* $(\theta_k)_{k\in\mathbb{N}}$ *a sequence of real numbers. Then,* $1, \alpha^{-1}, \theta_1, \theta_2, \dots$ *are linearly independent over* $\mathbb{Q}$ *if and only if for any* $\ell \in \mathbb{N}$ *and any* $k_1, \dots, k_\ell \in \mathbb{N}$ *pairwise distinct, the sequence* $\mathtt{x}_n = (\lfloor n\alpha \rfloor \theta_{k_1}, \dots, \lfloor n\alpha \rfloor \theta_{k_\ell})$, $n = 1, 2, \dots,$ *is u.d. mod 1 in* $\mathbb{R}^\ell$.

Proof. The numbers $1, \alpha^{-1}, \theta_1, \theta_2, \dots$ are linearly independent over $\mathbb{Q}$ if and only if for any $\ell \in \mathbb{N}$, any $k_1, \dots, k_\ell \in \mathbb{N}$ pairwise distinct, and any $m_1, \dots, m_\ell \in \mathbb{Z}$ not all of them zero, the numbers $1, \alpha^{-1}, m_1\theta_{k_1} + \dots + m_\ell\theta_{k_\ell}$ are linearly independent over $\mathbb{Q}$. Combining Theorem 7 and Theorem 8, we see that the latter statement is equivalent to the one saying that for any $\ell \in \mathbb{N}$ and any $k_1, \dots, k_\ell \in \mathbb{N}$ pairwise distinct, the sequence $\mathtt{x}_n = (\lfloor n\alpha \rfloor \theta_{k_1}, \dots, \lfloor n\alpha \rfloor \theta_{k_\ell})$, $n = 1, 2, \dots$, is u.d. mod 1 in $\mathbb{R}^\ell$. □

The sequence of numbers that we are interested in is

$$\theta_k = \frac{h}{2\pi} \ln p_k, \quad k = 1, 2, \dots,$$

where p_k will denote from here on the kth prime and $h > 0$. We prove that for a given irrational α there exists $h > 0$ such that the necessary condition of Corollary 2 for the aforementioned sequence is fulfilled. In fact we prove the existence of a lot such h.

Theorem 9. *Let* α *be an irrational number and*

$$L(\alpha) = \left\{ h \in \mathbb{R} : 1,\ \alpha^{-1},\ \frac{h}{2\pi}\ln p_1,\ \frac{h}{2\pi}\ln p_2,\ \dots \ \textit{are linearly independent over } \mathbb{Q} \right\}.$$

The set $L(\alpha)$ has full Lebesgue measure in $\mathbb{R}$, *i.e.,* $\operatorname{meas}(\mathbb{R} \setminus L(\alpha)) = 0$.

P r o o f. Let $B = \mathbb{R} \setminus L(\alpha)$ and $h \in B$. Then, the numbers

$$1,\ \alpha^{-1},\quad \frac{h}{2\pi}\ln p_1,\quad \frac{h}{2\pi}\ln p_2,\ \dots$$

are linearly dependent over $\mathbb{Q}$ and consequently over $\mathbb{Z}$ as well. Thus, there exists integer $k \geq 1$ and integers $a_1, \dots, a_k, b, c$, where a_i are not all zeros, such that

$$a_1 \frac{h}{\pi}\ln p_1 + \dots + a_k \frac{h}{\pi}\ln p_k = b + c\alpha^{-1}. \tag{1}$$

Putting $A = p_1^{a_1} \dots p_k^{a_k}$, we observe that $A \in \mathbb{Q}^+ \setminus \{1\}$ and we can rewrite (1) as

$$h\ln A = b\pi + c\alpha^{-1}\pi.$$

Fix a vector $(A, b, c) \in (\mathbb{Q}^+ \setminus \{1\}) \times \mathbb{Z} \times \mathbb{Z} = \Gamma$. Consider the corresponding set

$$B(A, b, c) = \{h \in \mathbb{R} : h\ln A = b\pi + c\alpha^{-1}\pi\}.$$

The set $B(A, b, c)$ is clearly a singleton (since $\ln A \neq 0$) and thus of measure zero. Hence, the countable union of singletons

$$B = \bigcup_{(A,b,c)\in\Gamma} B(A, b, c)$$

is of measure zero. Therefore, its complement $\mathbb{R} \setminus B = L(\alpha)$ has full Lebesgue measure in $\mathbb{R}$. $\square$

3. Auxiliary lemmas

Before stating the lemmas needed for the proofs of Theorems 4 and 5, we introduce some notation. Let Ω denote the set of all sequences of real numbers indexed by the prime numbers in ascending order. Further, define for every finite subset M of the set of all primes, every $\omega = (\omega_2, \omega_3, \omega_5, \dots) \in \Omega$, and all complex numbers s, the truncated Euler product

$$\zeta_M(s, \omega) = \prod_{p\in M} \left(1 - \frac{\exp(-2\pi i\omega_p)}{p^s}\right)^{-1}.$$

Obviously, $\zeta_M(s, \omega)$ is a non-vanishing analytic function of s in the half-plane $\sigma > 0$. Observe also that for $M = \{p'_1, \dots, p'_\ell\}$ and constant s, $\zeta_M(s, \omega)$ can be treated as a continuous complex-valued function of ℓ variables $(\omega_{p'_1}, \dots, \omega_{p'_\ell})$ defined on $[0, 1]^\ell$. In such cases, where M and s are given, $\zeta_M(s, \omega)$ will be abbreviated as $\zeta_M(s, \omega_{p'_1}, \dots, \omega_{p'_\ell})$. Finally, $\operatorname{Log} z$ will denote the principal logarithm of z.

Lemma 1. *Let s_0 be complex number such that $\frac{1}{2} < \mathrm{Re}(s_0) \le 1$ and $k \in \mathbb{N}_0$. If we define $M_Q = \{p_1, p_2, \dots, p_Q\}$ to be the set of the first Q primes and $\mathbf{0} = (0, 0, \dots)$, then*

$$\lim_{Q\to\infty} \limsup_{T\to\infty} \frac{1}{T} \int_0^T \left|\zeta^{(k)}(s_0 + it) - \zeta^{(k)}_{M_Q}(s_0 + it, \mathbf{0})\right|^2 dt = 0.$$

P r o o f. For the proof, see [8], pages 164-166, 168. □

Lemma 2. *Suppose that $(a_1, \dots, a_m) \in \mathbb{C}^m$, $\varepsilon > 0$, $y \in \mathbb{N}$ and $x_{p_1}, \dots, x_{p_y}$ are real numbers, and $s_1, \dots, s_m$ are the numbers in the condition of Theorem 4, where $\mathrm{Im}(s_k) > 2$ for all $1 \le k \le m$. Then, there exists a finite set of primes $M = \{p'_1, p'_2, \dots, p'_\ell\}$ and a sequence $\omega \in \Omega$ such that*

$$M \supset \{p_1, \dots, p_y\}, \quad \omega_{p_r} = x_{p_r} \quad \textit{and} \quad |\zeta_M(s_k, \omega) - a_k| < \varepsilon,$$

for $1 \le r \le y$ and $1 \le k \le m$.

P r o o f. For the proof, see [8], Lemma 11. □

Lemma 3. *Suppose that $(a_0, \dots, a_{m-1}) \in \mathbb{C}^m$, $\varepsilon > 0$, $y \in \mathbb{N}$ and $x_{p_1}, \dots, x_{p_y}$ are real numbers, and s_0 is a number with $\frac{1}{2} < \mathrm{Re}(s_0) \le 1$ and $\mathrm{Im}(s_0) > 2$. Then, there exists a finite set of primes $M = \{p'_1, p'_2, \dots, p'_\ell\}$ and a sequence $\omega \in \Omega$ such that*

$$M \supset \{p_1, \dots, p_y\}, \quad \omega_{p_r} = x_{p_r} \quad \textit{and} \quad |\zeta^{(k)}_M(s_0, \omega) - a_k| < \varepsilon,$$

for $1 \le r \le y$ and $1 \le k \le m$.

P r o o f. For the proof, see [8], Lemma 12. □

Remark 1. Note that the condition for the imaginary parts of the complex numbers in Lemma 2 can be removed:

P r o o f. Let the assumptions of Lemma 2 hold without the restriction of the imaginary parts. There exists a number $c > 0$ such that $\mathrm{Im}(s_k) + 2\pi c > 2$ for $1 \le k \le m$. According to Lemma 2, for $\tilde{x}_{p_1} = x_{p_1} - c\ln p_1, \dots, \tilde{x}_{p_y} = x_{p_y} - c\ln p_y$, there exists a finite set of primes M and $\tilde{\omega} \in \Omega$ such that

$$M \supset \{p_1, \dots, p_y\}, \quad \tilde{\omega}_{p_r} = \tilde{x}_{p_r} \quad \text{and} \quad |\zeta_M(s_k + 2\pi i c, \tilde{\omega}) - a_k| < \varepsilon\,.$$

for $1 \le r \le y$ and $1 \le k \le m$.

Taking $\omega \in \Omega$ to be $\omega_p = \tilde{\omega}_p + c\ln p$ for all primes p, we observe that for $1 \le k \le m$,

$$\begin{aligned}\zeta_M(s_k + 2\pi ic, \tilde{\omega}) &= \prod_{p\in M}\left(1 - \frac{\exp(-2\pi i\tilde{\omega}_p)}{p^{s_k+2\pi ic}}\right)^{-1} \\ &= \prod_{p\in M}\left(1 - \frac{\exp(-2\pi i(\tilde{\omega}_p + c\ln p)}{p^{s_k}}\right)^{-1} \\ &= \prod_{p\in M}\left(1 - \frac{\exp(-2\pi i\omega_p)}{p^{s_k}}\right)^{-1} \\ &= \zeta_M(s_k, \omega),\end{aligned}$$

and of course, $\omega_{p_r} = x_{p_r}$ for $1 \le r \le y$. □

Remark 2. The same result as above can be obtained similarly for Lemma 3, since to prove it Voronin showed that the set of points

$$\Delta_{(M,\omega)} = \Big(\operatorname{Log}\zeta_M(s_0, \omega), [\operatorname{Log}\zeta_M(s_0, \omega)]', \dots, [\operatorname{Log}\zeta_M(s_0, \omega)]^{(m-1)}\Big) \in \mathbb{C}^m$$

is dense in $\mathbb{C}^m$ whenever (M, ω) runs through all possible finite sets of primes M and $\omega \in \Omega$ with the requirements $M \supset \{p_1, \dots, p_y\}$ and $\omega_{p_r} = x_{p_r}$ for $1 \le r \le y$.

Lemma 4. *Let $t_0, t_1, \dots, t_R$ be real numbers, where $t_0 < t_1 < \dots < t_R$. If $G(t)$ is a complex-valued function which is defined and continuously differentiable on the interval $[t_0, t_R]$, then*

$$\sum_{r=1}^{R}|G(t_r)|^2 \le \frac{1}{\delta}\int_{t_0}^{t_R}|G(t)|^2\mathrm{d}t + 2\left(\int_{t_0}^{t_R}|G(t)|^2\mathrm{d}t\right)^{\frac{1}{2}}\left(\int_{t_0}^{t_R}|G'(t)|^2\mathrm{d}t\right)^{\frac{1}{2}},$$

where

$$\delta = \min_{0\le r<R}|t_{r+1} - t_r|.$$

Proof. For the proof, see [8], Lemma 6. □

Lemma 5. *Let $s_1, \dots, s_\ell$ be numbers such that $\operatorname{Re}(s_j) > 0$ for $j = 1, \dots, \ell$, and $m \in \mathbb{N}$. Then, for every $\varepsilon > 0$, there exists an $N = N(\varepsilon) \in \mathbb{N}$ such that for every set S of prime numbers greater than p_N, every $k = 0, 1, \dots, m$, every $j = 1, \dots, \ell$ and every $\omega \in \Omega$, the inequality*

$$\left|(\zeta_S(s_j, \omega) - 1)^{(k)}\right| < \varepsilon$$

holds.

P r o o f. Let $\varepsilon > 0$ and

$$0 < t_0 < r_0 < \min_{1 \le j \le \ell} \operatorname{Re}(s_j).$$

If we set

$$\varepsilon' = \min_{0 \le k \le m} \frac{\varepsilon r_0^k}{k!},$$

then there exists a $\delta = \delta(\varepsilon) < 1$ such that $|e^z - 1| < \varepsilon'$ for every $|z| < \delta$. Since the series

$$\sum_{n=1}^{\infty} \frac{1}{p_n^{t_0} - 1}$$

converges, there exists an $N = N(\varepsilon)$ such that

$$\sum_{n=N}^{\infty} \frac{1}{p_n^{\sigma} - 1} < \frac{\delta}{2}$$

for every $\sigma > t_0$. Now let S be a set of prime numbers greater than p_N and $\omega \in \Omega$. Observe that whenever $|z| < \frac{1}{2}$, one can obtain

$$|\operatorname{Log}(1+z)| = \left| \int_1^{1+z} \frac{\mathrm{d}w}{w} \right| \le \int_1^{1+z} \frac{|\mathrm{d}w|}{|w|} \le 2|z|.$$

Keeping that in mind and taking advantage of the fact that for every $\operatorname{Re}(s) > t_0$ and $n \ge N$:

$$\begin{aligned} \left| \left(1 - \frac{\exp(-2\pi i \omega_{p_n})}{p_n^s}\right)^{-1} - 1 \right| &= \left| \frac{\exp(-2\pi i \omega_{p_n})}{p_n^s - \exp(-2\pi i \omega_{p_n})} \right| \\ &\le \frac{1}{p_n^{\sigma} - 1} < \frac{\delta}{2} < \frac{1}{2}, \end{aligned}$$

we can estimate

$$\begin{aligned} \left| \sum_{p \in S} \operatorname{Log}\left(1 - \frac{\exp(-2\pi i \omega_p)}{p^s}\right)^{-1} \right| &\le \sum_{p \in S} \left| \operatorname{Log}\left(1 - \frac{\exp(-2\pi i \omega_p)}{p^s}\right)^{-1} \right| \\ &\le 2 \sum_{n=N}^{\infty} \left| \left(1 - \frac{\exp(-2\pi i \omega_{p_n})}{p_n^s}\right)^{-1} - 1 \right| \\ &\le 2 \sum_{n=N}^{\infty} \frac{1}{p_n^{\sigma} - 1} < \delta. \end{aligned}$$

Thus, for every $\mathrm{Re}(s) > t_0$,

$$|\zeta_S(s,\omega) - 1| = \left|\exp\left(\sum_{p\in S}\mathrm{Log}\left(1 - \frac{\exp(-2\pi i\omega_p)}{p^s}\right)^{-1}\right) - 1\right| < \varepsilon'.$$

All inequalities

$$\left|(\zeta_S(s_j,\omega) - 1)^{(k)}\right| < \varepsilon,$$

for $k = 0,\ldots,m$ and $j = 1,\ldots,\ell$, can now be proved by computing the Cauchy's estimates of $\zeta_S(s,\omega)-1$ on the disks $D(s_j,r_0) \subset \{s : \mathrm{Re}(s) > t_0\}$, for $j = 1,\ldots,\ell$, respectively. □

4. Proofs of Theorem 4 and Theorem 5

Proof of Theorem 4. We prove the second part of Theorem 4 since the first part can be shown similarly. Let s_0 be a complex number with $\frac{1}{2} < \mathrm{Re}(s_0) \le 1$, $\alpha > 1$ irrational and $h \in L(\alpha) \cap [0,+\infty)$, where $L(\alpha)$ is the set defined in the first section and in Theorem 9.

To prove the theorem it suffices to show that any vector $(a_0,\ldots,a_{m-1}) \in \mathbb{C}^m$ can be approximated arbitrarily close by the vector

$$\Big(\zeta(s_0 + i\lfloor n\alpha\rfloor h),\ \ldots,\ \zeta^{(m-1)}(s_0 + i\lfloor n\alpha\rfloor h)\Big)$$

with a suitable natural number n. We fix any $(a_0,\ldots,a_{m-1})$. By Lemma 3, for every $\varepsilon > 0$ and every $y \in \mathbb{N}$, there exists $\zeta_M(s_0,\omega)$ such that $M \supset \{p_1,\ldots,p_y\}$, $\omega_{p_r} = 0$ for $1 \le r \le y$, and for $k = 0,\ldots,m-1$ we have

$$\left|\zeta_M^{(k)}(s_0,\omega) - a_k\right| < \varepsilon\,. \tag{2}$$

Let $M = \{p'_1,\ldots,p'_\ell\}$. By the continuity of $\zeta_M(s_0,\omega_{p'_1},\ldots,\omega_{p'_\ell})$ as a function of ℓ variables and (2), in $[0,1]^\ell$ there exists a subbox K with $\mathrm{meas}(K) > 0$ such that for $k = 0,\ldots,m-1$ all the points $(x_{p'_1},\ldots,x_{p'_\ell})$ belonging in K satisfy

$$\left|\zeta_M^{(k)}(s_0,x_{p'_1},\ \ldots,\ x_{p'_\ell}) - a_k\right| < 2\varepsilon\,. \tag{3}$$

Let $\sum{}'^{N}_{n=1}$ denote summation over those $n \in [1,N] \cap \mathbb{N}$ for which

$$\left(\left\{\frac{h\ln p'_1}{2\pi}\lfloor n\alpha\rfloor\right\},\ \ldots,\ \left\{\frac{h\ln p'_\ell}{2\pi}\lfloor n\alpha\rfloor\right\}\right) \in K.$$

We consider the expression

$$A_N = \frac{1}{N}\sum_{n=1}^{N}{}' \sum_{k=0}^{m-1} \left|\zeta^{(k)}(s_0 + i\lfloor n\alpha\rfloor h) - \zeta_M^{(k)}(s_0 + i\lfloor n\alpha\rfloor h, \mathbf{0})\right|^2.$$

We choose Q larger than any $p \in M$ and we define $M_Q = \{p_1, p_2, \ldots, p_Q\}$ to be the set of the first Q primes. Then,

$$A_N \leq \frac{2}{N}\sum_{n=1}^{N}{}' \sum_{k=0}^{m-1} \left|\zeta^{(k)}(s_0 + i\lfloor n\alpha\rfloor h) - \zeta_{M_Q}^{(k)}(s_0 + i\lfloor n\alpha\rfloor h, \mathbf{0})\right|^2$$

$$+\frac{2}{N}\sum_{n=1}^{N}{}' \sum_{k=0}^{m-1} \left|\zeta_{M_Q}^{(k)}(s_0 + i\lfloor n\alpha\rfloor h, \mathbf{0}) - \zeta_M^{(k)}(s_0 + i\lfloor n\alpha\rfloor h, \mathbf{0})\right|^2. \quad (4)$$

We denote the first double sum by S_1 and the second by S_2. Firstly, we estimate S_2. We make use of Leibniz's formula

$$\zeta_{M_Q}^{(k)} - \zeta_M^{(k)} = [\zeta_M(\zeta_{M_Q\setminus M} - 1)]^{(k)} = \sum_{j=0}^{k}\binom{k}{j}\zeta_M^{(j)}(\zeta_{M_Q\setminus M} - 1)^{(k-j)}.$$

Applying the Cauchy-Schwarz inequality, we obtain

$$\left|\zeta_{M_Q}^{(k)} - \zeta_M^{(k)}\right|^2 \leq (k+1)\sum_{j=0}^{k}\left|\binom{k}{j}\zeta_M^{(j)}(\zeta_{M_Q\setminus M} - 1)^{(k-j)}\right|^2.$$

Hence, putting in S_2 the summation over n on the inside, we get

$$S_2 \leq \sum_{k=0}^{m-1}(k+1)\sum_{j=0}^{k}\binom{k}{j}\sum_{n=1}^{N}{}' \Big|\zeta_M^{(j)}(s_0 + i\lfloor n\alpha\rfloor h, \mathbf{0}) \times$$
$$(\zeta_{M_Q\setminus M}(s_0 + i\lfloor n\alpha\rfloor h, \mathbf{0}) - 1)^{(k-j)}\Big|^2. \quad (5)$$

So it suffices to estimate the sums of the form

$$S_{k,j} = \sum_{n=1}^{N}{}' \left|\zeta_M^{(j)}(s_0 + i\lfloor n\alpha\rfloor h, \mathbf{0})(\zeta_{M_Q\setminus M}(s_0 + i\lfloor n\alpha\rfloor h, \mathbf{0}) - 1)^{(k-j)}\right|^2.$$

Note that if $M_Q \setminus M = \{p_1'', \ldots, p_{Q-\ell}''\}$, then a simple computation leads to

$$\zeta_M\left(s_0 + i\lfloor n\alpha\rfloor h, \mathbf{0}\right) = \zeta_M\left(s_0, \left\{\frac{h\ln p_1'}{2\pi}\lfloor n\alpha\rfloor\right\}, \ldots, \left\{\frac{h\ln p_\ell'}{2\pi}\lfloor n\alpha\rfloor\right\}\right)$$

and

$$\zeta_{M_Q\setminus M}\left(s_0 + i\lfloor n\alpha\rfloor h, \mathbf{0}\right) = \zeta_{M_Q\setminus M}\left(s_0, \left\{\frac{h\ln p_1''}{2\pi}\lfloor n\alpha\rfloor\right\}, \ldots, \left\{\frac{h\ln p_{Q-\ell}''}{2\pi}\lfloor n\alpha\rfloor\right\}\right).$$

We define $F : [0,1]^Q \to \mathbb{C}$ to be of the form

$$F(\omega_{p_1},\dots,\omega_{p_Q}) = \Big|\zeta_M^{(j)}\big(s_0,\omega_{p'_1},\dots,\omega_{p'_\ell}\big) \times \Big(\zeta_{M_Q\setminus M}\big(s_0,\omega_{p''_1},\dots,\omega_{p''_{Q-\ell}}\big)-1\Big)^{(k-j)}\Big|^2,$$

whenever $(\omega_{p'_1},\dots,\omega_{p'_\ell}) \in K$, and zero otherwise. If we set

$$\mathtt{x}_n = \left(\frac{h\ln p_1}{2\pi}\lfloor n\alpha\rfloor,\ \dots,\ \frac{h\ln p_Q}{2\pi}\lfloor n\alpha\rfloor\right),\quad n\in\mathbb{N},$$

then

$$S_{k,j} = {\sum_{n=1}^{N}}' F\left(\{\mathtt{x}_n\}\right) = \sum_{n=1}^{N} F\left(\{\mathtt{x}_n\}\right).$$

The last equality is true if we consider the definitions of $\sum'$ and F. Now recall that $h \in L(\alpha)$. Thus, according to Corollary 2, the sequence $(\mathtt{x}_n)_{n\in\mathbb{N}}$ is u.d. mod 1 in $\mathbb{R}^Q$. Using Theorem 6, we obtain

$$\begin{aligned}\lim_{N\to\infty}\frac{1}{N}S_{k,j} &= \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}F\left(\{\mathtt{x}_n\}\right) = \int_{[0,1]^Q} F(\mathtt{x})\,\mathrm{d}\mathtt{x} = \int_K\int_{[0,1]^{Q-\ell}} F(\mathtt{x})\,\mathrm{d}\mathtt{x}\\ &= \int_K \Big|\zeta_M^{(j)}\big(s_0,\omega_{p'_1},\dots,\omega_{p'_\ell}\big)\Big|^2 \mathrm{d}\omega_{p'_1}\dots\mathrm{d}\omega_{p'_\ell}\\ &\quad\times\int_{[0,1]^{Q-\ell}}\Big|\Big(\zeta_{M_Q\setminus M}\big(s_0,\omega_{p''_1},\dots,\omega_{p''_{Q-\ell}}\big)-1\Big)^{(k-j)}\Big|^2 \mathrm{d}\omega_{p''_1}\,\dots\,\mathrm{d}\omega_{p''_{Q-\ell}}.\end{aligned} \tag{6}$$

By (3), the first integral is bounded by $(|a_j|+2\varepsilon)^2\mathrm{meas}(K)$, and the second integral, in view of Lemma 5, approaches zero uniformly in Q as y increases. Hence, by (5) and (6), we may choose y sufficiently large so that for every Q larger than any $p\in M$, we can find an $N_0 = N_0(Q)$ with the property

$$S_2 < N\mathrm{meas}(K)\frac{\varepsilon^3}{2}\quad\text{for}\ \ N\geq N_0. \tag{7}$$

We estimate S_1,

$$S_1 = \sum_{k=0}^{m-1}{\sum_{n=1}^{N}}'\Big|\zeta^{(k)}(s_0+i\lfloor n\alpha\rfloor h)-\zeta_{M_Q}^{(k)}(s_0+i\lfloor n\alpha\rfloor h,\mathbf{0})\Big|^2 = \sum_{k=0}^{m-1}S'_k.$$

Let $k \in \{0, \dots, m-1\}$. We apply Lemma 4 for

$$G(t) = \zeta^{(k)}(s_0 + ith) - \zeta^{(k)}_{M_Q}(s_0 + ith, \mathbf{0}) :$$

$$\begin{aligned} S'_k &\leq \sum_{n=1}^{N} |G(\lfloor n\alpha \rfloor h)|^2 \\ &\leq \frac{1}{h(\alpha - 1)} \int_0^{N\alpha h} |G(t)|^2 \mathrm{d}t + 2 \left(\int_0^{N\alpha h} |G(t)|^2 \mathrm{d}t \right)^{\frac{1}{2}} \left(\int_0^{N\alpha h} |G'(t)|^2 \mathrm{d}t \right)^{\frac{1}{2}}. \end{aligned}$$

Using Lemma 1, we may choose Q sufficiently large such that

$$S_1 < N\mathrm{meas}(K)\frac{\varepsilon^3}{2} \quad \text{for } N \geq N_1 = N_1(Q). \tag{8}$$

Consequently, by (4), (7) and (8), we have

$$A_N < \mathrm{meas}(K)\varepsilon^3 \quad \text{for } N > N_2 = N_2(Q).$$

Since the sequence

$$\left(\left\{ \frac{h\ln p'_1}{2\pi} \lfloor n\alpha \rfloor \right\}, \dots, \left\{ \frac{h\ln p'_\ell}{2\pi} \lfloor n\alpha \rfloor \right\} \right)_{n\in\mathbb{N}}$$

is u.d. mod 1 in $\mathbb{R}^\ell$, A_N contains $\sim N\mathrm{meas}(K)$ terms in ${\sum'}_{n=1}^{N}$ as $N \to \infty$. Hence there exists an n such that

$$\left. \begin{array}{rcl} \displaystyle\sum_{k=0}^{m-1} \left| \zeta^{(k)}(s_0 + i\lfloor n\alpha \rfloor h) - \zeta^{(k)}_M(s_0 + i\lfloor n\alpha \rfloor h, \mathbf{0}) \right|^2 & < & \varepsilon^3, \\ \left(\left\{ \dfrac{h\ln p'_1}{2\pi} \lfloor n\alpha \rfloor \right\}, \dots, \left\{ \dfrac{h\ln p'_\ell}{2\pi} \lfloor n\alpha \rfloor \right\} \right) & \in & K. \end{array} \right\} \tag{9}$$

Combining (3) and (9) we showed that there exists an n such that

$$\left| \zeta^{(k)}(s_0 + i\lfloor n\alpha \rfloor h) - a_k \right| < 3\varepsilon,$$

for $k = 0, \dots, m-1$

The proof of the first part of Theorem 4 consists of the same arguments as we used until now. Instead of Lemma 3 we use Lemma 2, and there is no need to apply Leibniz's formula and the Cauchy-Schwarz inequality. □

Proof of Theorem 5. Let $h \in L(\alpha) \cap [0, +\infty)$ and $\varepsilon > 0$. Since the Taylor expansion of g is valid for all $s \in K$, there exists an $N = N(\varepsilon)$ such that

$$\max_{s\in K} \left| g(s) - \sum_{k=0}^{N-1} \frac{g^{(k)}(s_0)}{k!}(s-s_0)^k \right| < \frac{\varepsilon}{3}. \tag{10}$$

From Theorem 4, for the vector $\left(g(s_0), \ldots, g^{(N-1)}(s_0)\right)$ and $\varepsilon > 0$, there exists a sequence $(n_\ell)_{\ell\in\mathbb{N}}$ such that for every $\ell = 1, 2, \ldots$ and every $k = 0, \ldots, N-1$,

$$\left| \zeta^{(k)}\left(s_0 + i\lfloor n_\ell \alpha \rfloor h\right) - g^{(k)}(s_0) \right| < \varepsilon' = \frac{\varepsilon}{3N} \min_{0\le k\le N-1} \frac{k!}{r^k}.$$

We choose an $n_{\ell_0} = n_{\ell_0}(\varepsilon, h)$ such that $1 \notin K + i\lfloor n_{\ell_0}\alpha\rfloor h$. Then,

$$\max_{s\in K} \left| \sum_{k=0}^{N-1} \frac{\zeta^{(k)}(s_0 + i\lfloor n_{\ell_0}\alpha\rfloor h)}{k!}(s-s_0)^k - \sum_{k=0}^{N-1} \frac{g^{(k)}(s_0)}{k!}(s-s_0)^k \right| \le$$
$$\max_{s\in K} \varepsilon' \sum_{k=0}^{N-1} \frac{|s-s_0|^k}{k!} \le \frac{\varepsilon}{3}. \tag{11}$$

The choice of n_{ℓ_0} allows us to represent ζ in the disk $K + i\lfloor n_{\ell_0}\alpha\rfloor h$ as the sum of a Taylor series centered at $s_0 + i\lfloor n_{\ell_0}\alpha\rfloor h$,

$$\zeta(s + i\lfloor n_{\ell_0}\alpha\rfloor h) = \sum_{k=0}^{\infty} \frac{\zeta^{(k)}(s_0 + i\lfloor n_{\ell_0}\alpha\rfloor h)}{k!}(s-s_0)^k,$$

for all $s \in K$. If

$$M = M(\varepsilon, h) = \max_{s\in K} |\zeta(s + i\lfloor n_{\ell_0}\alpha\rfloor h)| \quad \text{and} \quad \delta \in (0,1),$$

then, using Cauchy's estimates, we get

$$\left| \frac{\zeta^{(k)}(s_0 + i\lfloor n_{\ell_0}\alpha\rfloor h)}{k!}(s-s_0)^k \right| \le \frac{Mk!}{r^k}\frac{|s-s_0|^k}{k!} \le M\delta^k,$$

for all $s \in \overline{D(s_0, \delta r)}$. Hence,

$$\left|\zeta(s + i\lfloor n_{\ell_0}\alpha\rfloor h) - \sum_{k=0}^{N-1} \frac{\zeta^{(k)}(s_0 + i\lfloor n_{\ell_0}\alpha\rfloor h)}{k!}(s-s_0)^k \right| =$$
$$\left| \sum_{k=N}^{\infty} \frac{\zeta^{(k)}(s_0 + i\lfloor n_{\ell_0}\alpha\rfloor h)}{k!}(s-s_0)^k \right| \le M\frac{\delta^N}{1-\delta}, \tag{12}$$

for all $s \in \overline{D(s_0, \delta r)}$. Combining relations (10), (11) and (12), we find

$$|\zeta(s + i\lfloor n_{\ell_0}\alpha\rfloor h) - g(s)| < M\frac{\delta^N}{1-\delta} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3},$$

for all $s \in \overline{D(s_0, \delta r)}$. Now choose $\delta = \delta(\varepsilon, h) \in (0,1)$ such that

$$M\frac{\delta^N}{1-\delta} = \frac{\varepsilon}{3}$$

This is possible since for the continuous function

$$F : (0,1) \to \mathbb{R} \quad \text{with} \quad F(t) = M\frac{t^N}{1-t}, \quad t \in (0,1)\,,$$

we have

$$\lim_{t\to 0} F(t) = 0 \quad \text{and} \quad \lim_{t\to 1} F(t) = +\infty.$$

We thus have shown

$$\max_{|s-s_0|\le\delta r} |\zeta(s + i\lfloor n_{\ell_0}\alpha\rfloor h) - g(s)| < \varepsilon$$

and this completes the proof. □

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