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DISCREPANCY OF GENERALIZED LS-SEQUENCES

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ABSTRACT. The LS-sequences are a parametric family of sequences of points in the unit interval. They were introduced by Carbone [4], who also proved that under an appropriate choice of the parameters L and S, such sequences are low-discrepancy. The aim of the present paper is to provide explicit constants in the bounds of the discrepancy of LS-sequences. Further, we generalize the construction of Carbone [4] and construct a new class of sequences of points in the unit interval, the generalized LS-sequences.

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1. Introduction

A sequence $(x_n)_{n\in\mathbb{N}}$ of points in $[0,1)$ is called *uniformly distributed modulo 1* (u.d. mod 1) if

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\mathbf{1}_{[a,b)}(x_n)=\lambda([a,b))$$

for all intervals $[a,b)\subseteq[0,1)$. A further characterization of u.d. is given by the following well-known result of Weyl [14]: a sequence $(x_n)_{n\in\mathbb{N}}$ of points in $[0,1)$ is u.d. mod 1 if and only if for every continuous function f on $[0,1)$ the relation

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}f(x_n)=\int_{[0,1)}f(x)\mathrm{dx}$$

holds.

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An important quantity called *discrepancy* is introduced when dealing with u.d. sequences. It measures the maximal deviation between the empirical distribution of a sequence and the uniform distribution. Let $\omega_N = \{x_1, \dots, x_N\}$ be a finite set of real numbers in $[0,1]$. The quantity

$$D_N(\omega_N) = \sup_{0 \le a < b \le 1} \left| \frac{1}{N} \sum_{n=1}^{N} \mathbf{1}_{[a,b)}(x_n) - (b-a) \right|$$

is called the discrepancy of the given set ω_N. This definition naturally extends to infinite sequences $(x_n)_{n \in \mathbb{N}}$ by associating to it the sequence of positive real numbers $D_N(\{x_1, x_2, \dots, x_N\})$. We denote by $D_N(x_n)$ the discrepancy of the initial segment $\{x_1, x_2, \dots, x_N\}$ of the infinite sequence.

Sometimes it is also useful to restrict the family of intervals considered in the definition of discrepancy to intervals of the form $[0, a)$ with $0 < a \le 1$. This leads to the following definition of star-discrepancy

$$D_N^*(\omega_N) = \sup_{0 < a \le 1} \left| \frac{1}{N} \sum_{n=1}^{N} \mathbf{1}_{[0,a)}(x_n) - a \right| .$$

It is a well known result that a sequence $(x_n)_{n \in \mathbb{N}}$ of points in $[0,1)$ is u.d. if and only if

$$\lim_{N \to \infty} D_N^*(x_n) = 0 .$$

Sequences whose discrepancy is of order $\mathcal{O}(N^{-1} \log N)$ are called *low-discrepancy* sequences. These sequences are of particular interest in the theory of numerical integration and are used in the *Quasi-Monte Carlo (QMC)* integration. For more information on discrepancy theory, low-discrepancy sequences and QMC integration see [7, 11].

We now recall the splitting procedure that is closely related to the construction of the generalized LS-sequences as sequences of points in the unit interval.

Definition 1 (Kakutani splitting procedure)**.** If $\alpha \in (0,1)$ and $\pi = \{[t_{i-1}, t_i) : 1 \le i \le k\}$ is any partition of $[0,1)$, then $\alpha\pi$ denotes its so-called α-refinement, which is obtained by subdividing all intervals of π having maximal length into two parts, proportional to α and $1-\alpha$, respectively. The so-called Kakutani's sequence of partitions $(\alpha^n \omega)_{n \in \mathbb{N}}$ is obtained as the successive α-refinement of the trivial partition $\omega = \{[0,1)\}$.

The notion of α-refinements can be generalized in a natural way to so-called ρ-refinements.

Definition 2 (ρ-refinement)**.** Let ρ denote a non-trivial finite partition of $[0,1)$. Then the ρ-refinement of a partition π of $[0,1)$, denoted by $\rho\pi$, is given by subdividing all intervals of maximal length positively homothetically to ρ. Note that the α-refinement is a special case with $\rho = \{[0,\alpha),[\alpha,1)\}$.

By a classical result due to Kakutani [10], for any α the sequence of partitions $(\alpha^n\omega)_{n\in\mathbb{N}}$ is uniformly distributed, which means that for every interval $[a,b] \subset [0,1]$,

$$\lim_{n\to\infty} \frac{1}{k(n)} \sum_{i=1}^{k(n)} \mathbf{1}_{[a,b]}(t_i^n) = b - a,$$

where $k(n)$ denotes the number of intervals in $\alpha^n\omega = \{[t_{i-1}^n, t_i^n),\ 1 \le i \le k(n)\}$. The same result holds for any sequence of ρ-refinements of ω, due to a result of Volčič [13] (see also [1, 6]).

The generalized LS-sequence of partitions, already introduced in [6], represent a special case of a ρ-refinement.

Definition 3 (Generalized LS-sequence of partitions)**.** Let $L_1,\dots,L_k$ be non-negative integers, with $L_1L_k \neq 0$. We define the generalized LS-sequence of partitions $(\rho^n_{L_1,\dots,L_k}\omega)_{n\in\mathbb{N}}$ as the successive ρ-refinement of the trivial partition ω, where $\rho_{L_1,\dots,L_k}$ consists of $L_1+L_2+\cdots+L_k$ intervals such that L_1 has length β, L_2 has length β^2 and so on up to L_k having length β^k.

Note that necessarily $L_1\beta+\cdots+L_k\beta^k = 1$ holds, and consequently for each k-tuple $(L_1,\dots,L_k)$ of parameters satisfying the assumptions made in Definition 3 there exists exactly one positive real number β satisfying $L_1\beta+\cdots+L_k\beta^k = 1$.

If $k = 2$, then we obtain the definition of the classical LS-sequence of partitions introduced by Carbone [4]. Given a sequence of partitions we can assign a sequence of points by ordering the left endpoints of the intervals in the partition. Of course, changing the ordering of the points may lead to other discrepancy bounds. In particular, the discrepancy of the sequence might get rather large if we do not pay attention. One issue of this paper is to find an ordering that yields good upper bounds for the discrepancy of the sequence. In the case that $k = 2$ the corresponding LS-sequences of points, denoted by $(\xi^n_{L,S})_{n\in\mathbb{N}}$, have been introduced by Carbone [4] who proved that whenever $L \ge S$ there exists a positive constant k_1 such that

$$D_N(\xi^1_{L,S},\xi^2_{L,S},\dots,\xi^N_{L,S}) \le k_1 \frac{\log N}{N}\,. \tag{1}$$

One purpose of the present paper is to give an estimate for k_1. This problem still open in [4] can be solved by the numeration approach presented by

Aistleitner et.al. [2]. However the main focus of the present paper is to generalize the construction of classical LS-sequences of points by Carbone [4] in the following sense: we introduce a numeration system for the integers which is the generalization of the numeration system introduced by Aistleitner et.al. [2] and, by a radical inverse construction, similar as the construction of the van der Corput sequence, we study the obtained sequence of points. In particular, the choice of this numeration system yields a point sequence whose first t_n points are exactly the points of the nth step of the Kakutani splitting procedure, already ordered as in [4], and not by increasing magnitude. This definition by a numeration system allows us to prove several auxiliary results on the distribution of generalized LS-sequences which were proved in the case that $k = 2$ by Carbone [4] and Aistleitner et.al. [2]. Moreover, let us point out that a generalization of the LS-sequences of partitions is already contained in [6], but with our approach we can directly obtain the LS-sequences of points without relating them to the sequences of partitions. All this is the content of the next section. In Section 3 we explicitly compute the constant k_1 in (1) and obtain explicit upper bounds for the discrepancy of the classical LS-sequences (see Theorem 1). The computation of the discrepancy in the classical case, i.e., the case that $k = 2$ will pave the way to compute explicit bounds for the discrepancy of generalized LS-sequences, which we compute in the final section of this paper.

2. Generalized LS-sequences

Let us consider the generalized LS-sequence of partitions. Then the partition $\rho^n_{L_1,\dots,L_k}\omega$ consists of intervals having lengths $\beta^n, \dots, \beta^{n+k-1}$ and this fact makes the analysis of the generalized LS-sequences more complicated, compared to the analysis of the classical ones, where only two lengths are considered. We denote by t_n the total number of intervals of $\rho^n_{L_1,\dots,L_k}\omega$, and correspondingly by $l_{n,1}, \dots, l_{n,k}$ the number of intervals of the nth partition having length $\beta^n, \dots, \beta^{n+k-1}$ respectively.

From a general point of view the only canonical restrictions to the k-tuple $(L_1, \dots, L_k)$ are that the L_i are non-negative integers for all $i = 1, \dots, k$ such that $L_1 L_k \neq 0$. However, we are interested in low-discrepancy sequences and we will see (Remark 2) that we obtain low discrepancy sequences if and only if all roots but one (counted with multiplicity) of the polynomial $L_k X^k + \dots + L_1 X - 1$ have absolute value smaller than one. Therefore we will assume from now on that the polynomial $L_k X^k + \dots + L_1 X - 1$ has no double zeros and that there is exactly one root $0 < \beta < 1$ and all other roots have absolute value less than 1.

Let us remark that excluding multiple zeros is mainly to avoid technical issues and similar results would be obtained if we only assume that the unique root $0 < \beta < 1$ is simple.

Furthermore, let us note that, if we assume that

$$L_1 \geq L_2 \geq \cdots \geq L_k > 0, \tag{2}$$

then $L_k X^k + \cdots + L_1 X - 1$ is irreducible and is the minimal polynomial of the reciprocal of a Pisot-number (see [3]), i.e., a k-tuple $(L_1, \ldots, L_k)$ which satisfies (2) also meets our assumptions made above.

Let us write $\beta_1 = \beta$ and let $\beta_2, \ldots, \beta_k$ be the other roots of $L_k X^k + \cdots + L_1 X - 1$. When the coefficients of the polynomial fullfill the condition (2), then the $\beta_2, \ldots, \beta_k$ are the Galois conjugates of β in some order. Note that since $1/\beta$ is a Pisot-number by our assumptions we have that $|\beta_i| > 1$ for $i = 2, \ldots, k$.

In the first step we consider the quantities t_n and $l_{n,i}$ for $n \geq 0$ and $i = 1, \ldots, k$. First we observe that in the nth partition step the longest $l_{n-1,1}$ intervals are divided into $L_1 + \cdots + L_k$ intervals, where L_1 intervals have the length β^n, L_2 intervals have the length β^{n+1} and so on. Hence we have that

$$\begin{aligned} t_n &= l_{n-1,1}(L_1 + \cdots + L_k) + l_{n-1,2} + \cdots + l_{n-1,k} \\ &= t_{n-1} + l_{n-1,1}(L_1 + \cdots + L_k - 1), \\ l_{n,1} &= l_{n-1,2} + L_1 l_{n-1,1}\,, \\ &\vdots \\ l_{n,k-1} &= l_{n-1,k} + L_{k-1} l_{n-1,1}\,, \\ l_{n,k} &= L_k l_{n-1,1} \qquad\qquad \text{for all } \ n > 0\,. \end{aligned} \tag{3}$$

Of course, the interval $[0,1)$ yields the initial conditions $t_0 = l_{0,1} = 1$ and $l_{0,2} = \cdots = l_{0,k} = 0$ and therefore we can recursively compute the quantities $t_n, l_{n,1}, \ldots, l_{n,k}$ for all $n > 0$. Since from the recursion point of view the sequence $(t_n)_{n \geq 0}$ is closely related to the sequences $(l_{n,i})_{n \geq 0}$, we define $l_{n,0} = t_n$ in order to state several of our results in a compact way. Moreover, we put $l_{n,j} = 0$ if $j > k$ or $n < 0$. For our purposes we desire an explicit formula for the quantities $l_{n,i}$:

Lemma 1. *The sequences $(l_{n,i})_{n \geq 0}$, with $i = 0, \ldots, k$ satisfy the recursion*

$$l_{n,i} = L_1 l_{n-1,i} + \cdots + L_k l_{n-k,i}, \qquad n \geq k.$$

In particular, there exist explicit computable constants $\lambda_{j,i}$ for $0 \leq i \leq k$ and $1 \leq j \leq k$ such that

$$l_{n,i} = \sum_{j=1}^{k} \lambda_{j,i} \beta_j^{-n}\,. \tag{4}$$

P r o o f. First, we observe that by inserting the last line of (3) into the second to last line we obtain

$$l_{n,k-1} = L_k l_{n-2,1} + L_{k-1} l_{n-1,1}.$$

Now inserting this identity into the third to last line of (3) and going on we end up with

$$l_{n,1} = L_1 l_{n-1,1} + \cdots + L_k l_{n-k,1},$$

hence we proved the first part of the lemma for $i = 1$. Since the characteristic polynomial of this recursion is

$$X^k - L_1 X^{k-1} - \cdots - L_k$$

there exist constants $\lambda_{j,1}$, with $j = 1, \ldots, k$ such that

$$l_{n,1} = \sum_{j=1}^{k} \lambda_{j,1} \beta_j^{-n}.$$

Note that the $1/\beta_j$ for $j = 1, \ldots, k$ are the roots of the characteristic polynomial.

Since $l_{n,i} = l_{n+1,1} - L_i l_{n,1}$ we obtain (4), with $\lambda_{j,i} = (\beta_j^{-1} - L_i)\lambda_{j,1}$ for $i = 2, \ldots, k$ and $j = 1, \ldots, k$. And since $t_n = l_{n,0} = l_{n,1} + \cdots + l_{n,k}$ we see that (4) also holds for $i = 0$, with $\lambda_{j,0} = \lambda_{j,1} + \cdots + \lambda_{j,k}$ for $j = 1, \ldots, k$. On the other hand every sequence of the form (4) fulfills a recursion of the required form and the proof of the lemma is complete. □

Let us note that the explicit computation of the λ's is easy for a given k-tuple $(L_1, \ldots, L_k)$. Indeed one can compute the values of $l_{n,i}$ for all $i = 0, 1, \ldots, k$ and $n = 0, 1, \ldots, k-1$ by using the recursion (3). Therefore (4) gives for each $i = 0, 1, \ldots, k$ a linear inhomogeneous system with unknowns $\lambda_{j,i}$. Solving for the λ's by Cramer's rule we obtain

$$\lambda_{j,i} = \frac{\begin{vmatrix} \beta_1^0 & \cdots & \beta_{j-1}^0 & l_{0,i} & \beta_{j+1}^0 & \cdots & \beta_k^0 \\ \beta_1^{-1} & \cdots & \beta_{j-1}^{-1} & l_{1,i} & \beta_{j+1}^{-1} & \cdots & \beta_k^{-1} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \beta_1^{1-k} & \cdots & \beta_{j-1}^{1-k} & l_{k-1,i} & \beta_{j+1}^{1-k} & \cdots & \beta_k^{1-k} \end{vmatrix}}{\begin{vmatrix} \beta_1^0 & \cdots & \beta_{j-1}^0 & \beta_j^0 & \beta_{j+1}^0 & \cdots & \beta_k^0 \\ \beta_1^{-1} & \cdots & \beta_{j-1}^{-1} & \beta_j^{-1} & \beta_{j+1}^{-1} & \cdots & \beta_k^{-1} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \beta_1^{1-k} & \cdots & \beta_{j-1}^{1-k} & \beta_j^{1-k} & \beta_{j+1}^{-k} & \cdots & \beta_k^{1-k} \end{vmatrix}}. \tag{5}$$

Let us also state another property of the sequences $(l_{n,i})_{n \geq 0}$ for $i = 1, \ldots, k$, which we need at several places in the construction of generalized LS-sequences:

Lemma 2. *We have $l_{n,1} + \cdots + l_{n,m} \geq l_{n-1,1} + \cdots + l_{n-1,m+1}$ for all $m \geq 1$.*

Proof. If $m \geq k$ this is clear since the sequences $l_{n,m}$ are all strictly monotone increasing with n for $m = 1, \ldots, k$ and $l_{n,m} = 0$ if $m > k$. In case that $m = 1, \ldots, k-1$ we have

$$\begin{aligned} & l_{i-1,1} + \cdots + l_{i-1,m+1} \\ & = l_{i-1,1} + (l_{i,1} - L_1 l_{i-1,1}) + \cdots + (l_{i,m} - L_m l_{i-1,1}) \\ & \leq l_{i,1} + \cdots + l_{i,m}. \end{aligned}$$ □

Our next step is to introduce the numeration system which will be the basis for our construction of the generalized LS-sequence. Let $N \geq 0$ be a fixed integer and choose n such that $t_n \leq N < t_{n+1}$. We construct finite sequences

$$(\epsilon_m)_{0 \leq m \leq n}, \quad (\eta_m)_{0 \leq m \leq n}, \quad (N_m)_{0 \leq m \leq n} \quad \text{and} \quad (T_m)_{0 \leq m \leq n},$$

recursively, in the following way:

- First we put $N_n = N$, $T_n = t_n$, $\epsilon_n = 1$ and $\eta_n = \lfloor (N - T_n)/l_{n,1} \rfloor$.

Assume that we have computed the quantities N_i, T_i, ϵ_i and η_i for all indices $n \geq i > m \geq 0$.

- We denote by j the unique integer such that $\epsilon_{m+1} = \cdots = \epsilon_{m+j} = 0$ and $\epsilon_{m+j+1} \neq 0$, in particular, if $\epsilon_{m+1} \neq 0$, we put $j = 0$.
- Now we compute N_m and T_m:

$$\begin{aligned} N_m &= N_{m+1} - \epsilon_{m+1} T_{m+1} - \eta_{m+1} l_{m+1,1}\,, \\ T_m &= l_{m,1} + \cdots + l_{m,j+2}\,. \end{aligned}$$

- If $N_m < T_m$ we put $\epsilon_m = \eta_m = 0$. Otherwise we put $\epsilon_m = 1$ and

$$\eta_m = \lfloor (N_m - T_m)/l_{m,1} \rfloor.$$

Thus we have after the $n - m$th step

$$N = \sum_{i=m+1}^{n} (\epsilon_i T_i + \eta_i l_{i,1}) + N_m\,.$$

Since $T_0 = l_{0,1} = 1$ in the nth step either $N_0 = 0$ or $\epsilon_0 = 1$ and $\eta_0 = N_0 - 1$. In any case we obtain by the definition of our numeration system a representation for N of the form

$$N = \sum_{i=0}^{n} (\epsilon_i T_i + \eta_i l_{i,1}). \tag{6}$$

Obviously, $\epsilon_i \in \{0, 1\}$ for all $i = 0, 1, \ldots, n$. Let us note that we also have $0 \leq \eta_i \leq L_1 + \cdots + L_k - 2$ for all $i = 0, 1, \ldots, n$. Indeed assume to the contrary that $\eta_m \geq L_1 + \cdots + L_k - 1$ and that $\epsilon_{m+1} = \cdots = \epsilon_{m+j} = 0$ but $\epsilon_{m+j+1} \neq 0$. Then we would obtain in case that $j > 0$,

$$\begin{aligned} T_{m+1} > N_{m+1} &\geq T_m + (L_1 + \cdots + L_k - 1) l_{m,1} \\ &= l_{m,1} + l_{m,2} + \cdots + l_{m+j+2} + (L_1 + \cdots + L_k - 1) l_{m,1} \\ &\geq (L_1 l_{m,1} + l_{m,2}) + \cdots + (L_{j+1} l_{m,1} + l_{m,j+2}) \\ &= l_{m+1,1} + \cdots + l_{m+1,j+1} = T_{m+1} \end{aligned}$$

a contradiction. In case that $j = 0$ we similarly have

$$\begin{aligned} T_{m+1} &\geq l_{m+1,1} + l_{m+1,2} > l_{m+1,1} \\ &\geq N_m \geq T_m + (L_1 + \cdots + L_k - 1) l_{m,1} \\ &= l_{m,1} + l_{m,2} + \cdots + l_{m+j+2} + (L_1 + \cdots + L_k - 1) l_{m,1} \\ &\geq (L_1 l_{m,1} + l_{m,2}) + \cdots + (L_{j+1} l_{m,1} + l_{m,j+2}) \\ &= l_{m+1,1} + \cdots + l_{m+1,j+1} = T_{m+1}, \end{aligned}$$

again a contradiction.

The following lemma gives a bijection between the integers and digit-expansions of the form given above. Note that the following lemma is a generalization of a result due to A i s t l e i t n e r et.al. for the classical LS-sequences [2, Lemma 3].

Lemma 3. *There is a bijection between positive integers and finite sequences of the form*

$$\mathcal{D} = \big((\epsilon_n, \eta_n), \ldots, (\epsilon_0, \eta_0)\big)$$

such that $\epsilon_i \in \{0,1\}$, $\epsilon_n = 1$, $0 \leq \eta_i \leq L_1 + \cdots + L_k - 2$ for all $0 \leq i \leq n$, $\epsilon_i = 0$ implies $\eta_i = 0$ and for all $1 \leq m \leq k-1$ we have that $\eta_i \geq L_1 + \cdots + L_m - 1$ implies $\epsilon_{i+m} = 0$.

This bijection is given by

$$\Psi(\mathcal{D}) = \sum_{i=0}^{n} (\epsilon_i T_i + \eta_i l_{i,1})$$

and its inverse

$$\Phi(N) = \big((\epsilon_n, \eta_n), \ldots, (\epsilon_0, \eta_0)\big),$$

where the T_i, ϵ_i and η_i are computed by the algorithm described above.

P r o o f. In order to prove bijectivity we have to show that for every integer N and every finite sequence $\mathcal{D}$ we have $\Psi\big(\Phi(N)\big) = N$ and $\Phi\big(\Psi(\mathcal{D})\big) = \mathcal{D}$. The first equation is evident from the presented algorithm, i.e., Φ is injective. Thus we are left to prove that $\Phi\big(\Psi(\mathcal{D})\big) = \mathcal{D}$, i.e., Φ is surjective. The proof is technical and we proceed in several steps:

Step I: We show that
$$\Phi(N) = \big((\epsilon_n, \eta_n), \dots, (\epsilon_0, \eta_0)\big)$$
such that the ϵ_i and η_i for all $i \geq 0$ satisfy the conditions of the lemma. Thus we prove that $\Phi(N)$ is well defined.

Note that from the algorithm it is evident that $\epsilon_i \in \{0, 1\}$, $\epsilon_n = 1$ and that $\epsilon_i = 0$ implies $\eta_i = 0$. Moreover, in the discussion after (6) we have shown that $0 \leq \eta_i \leq L_1 + \dots + L_k - 2$ for all $0 \leq i \leq n$. Therefore we have to show that $\eta_i \geq L_1 + \dots + L_m - 1$ implies $\epsilon_{i+m} = 0$.

We proceed by induction on m. We start with the induction basis $m = 1$, i.e., we assume that $\eta_i \geq L_1 - 1$. Let us assume for the moment that $\epsilon_{i+1} \neq 0$. Then we get that
$$N_i \geq \epsilon_i T_i + \eta_i l_{i,1} \geq l_{i,1} + l_{i,2} + (L_1 - 1)l_{i,1} = l_{i+1,1} > N_i$$
a contradiction and therefore we conclude that $\epsilon_{i+1} = 0$, i.e., we have proved the induction basis.

Now, let us assume that $\eta_i \geq L_1 + \dots + L_{m-1} - 1$ implies $\epsilon_{i+m-1} = 0$ for all $1 \leq m \leq M - 1$ and assume that $\eta_i \geq L_1 + \dots + L_M - 1$. We aim to show that $\epsilon_{i+M} \neq 0$ yields a contradiction. By the induction basis, see the paragraph above, we may assume that $M \geq 2$. Moreover, by induction we may assume that $\epsilon_{i+m} = 0$ for all $1 \leq m < M$. Assuming $\epsilon_{i+M} \neq 0$ implies
$$T_i = l_{i,1} + \dots + l_{i,M+1} \quad \text{and} \quad T_{i+1} = l_{i+1,1} + \dots + l_{i+1,M}.$$
Therefore we deduce
$$\begin{aligned} T_{i+1} > N_{i+1} = N_i &\geq T_i + \eta_i l_{i,1} \\ &\geq l_{i,1} + \dots + l_{i,M+1} + (L_1 + \dots + L_M - 1)l_{i,1} \\ &= (L_1 l_{i,1} + l_{i,2}) + \dots + (L_M l_{i,1} + l_{i,M+1}) \\ &= l_{i+1,1} + \dots + l_{i+1,M} = T_{i+1} \end{aligned}$$
a contradiction, i.e., $\epsilon_{i+M} = 0$.

Step II: It is enough to show that there are exactly $t_n - 1$ sequences of length $\leq n$ satisfying the conditions of the Lemma.

Indeed, we have already seen that Φ is injective. Therefore we have to show that Φ is surjective. In particular, it is enough to prove that Φ induces a surjective map between the positive integers $< t_n$ and sequences of length $\leq n$, which satisfy the restrictions of the Lemma. Therefore we have to prove the following claim:

Claim 1. There are exactly $t_n - 1$ sequences of length $\leq n$ satisfying the conditions of the Lemma. Moreover, there are exactly $l_{n-1,1}$ such sequences of length n of the form $\mathcal{D} = \big((1, 0), \dots\big)$.

We will prove that claim be induction.

Step III: The induction basis is evident, since there are exactly

$$t_1 - 1 = L_1 + \cdots + L_k - 1$$

sequences of length 1. Moreover there is only one sequence of length 1 starting with the pair $(1,0)$.

Step IV: Suppose the claim is true for all integers $M < n+1$. We show that it is also true for $M = n+1$.

First, let us show that there are exactly $l_{n,1}$ sequences of length $n+1$ starting with $(1,0)$. Let $1 \leq m \leq k$, then there are exactly

$$(L_1 + \cdots + L_m - 1) l_{n-m,1}$$

sequences of the form

$$\big((1,0), \overbrace{(0,0), \ldots, (0,0)}^{m-1 \text{ times}}, (1,\eta), \ldots\big)$$

by induction hypothesis and the assumptions of the lemma. Similarly there are exactly t_{n-k} sequences of the form

$$\big((1,0), \overbrace{(0,0), \ldots, (0,0)}^{\text{at least } k \text{ times}}, (1,\eta), \ldots\big).$$

Therefore the number of sequences of length $n+1$ starting with $(1,0)$ is

$$\begin{aligned}
&(L_1 - 1) l_{n-1,1} + (L_1 + L_2 - 1) l_{n-2,1} + \cdots \\
&\qquad\qquad \cdots + (L_1 + \cdots + L_k - 1) l_{n-k,1} + t_{n-k} = \\
&(L_1 - 1) l_{n-1,1} + (L_1 + L_2 - 1) l_{n-2,1} + \cdots \\
&\qquad\qquad \cdots + (L_1 + \cdots + L_{k-1} - 1) l_{n-k+1,1} + t_{n-k+1} = \\
&(L_1 - 1) l_{n-1,1} + (L_1 + L_2 - 1) l_{n-2,1} + \cdots \\
&\qquad\qquad \cdots + (L_1 + \cdots + L_{k-1} - 1) l_{n-k+1,1} + l_{n-k+1,1} + \cdots + l_{n-k+1,k}\,.
\end{aligned}$$

Now note that

$$\begin{aligned}
&(L_1 + \cdots + L_{m-1} - 1) l_{n,1} + l_{n,1} + \cdots + l_{n,m} \\
&= (L_1 l_{n,1} + l_{n,2}) + \cdots + (L_{m-1} l_{n,1} + l_{n,m}) \\
&= l_{n+1,1} + \cdots + l_{n+1,m-1}
\end{aligned}$$

applying this identity to the equation above we obtain that there are

$$\begin{aligned}
&(L_1-1)l_{n-1,1}+(L_1+L_2-1)l_{n-2,1}+\cdots\\
&\qquad\cdots+(L_1+\cdots+L_{k-1}-1)l_{n-k+1,1}+l_{n-k+1,1}+\cdots+l_{n-k+1,k}=\\
&(L_1-1)l_{n-1,1}+(L_1+L_2-1)l_{n-2,1}+\dots\\
&\qquad\cdots+(L_1+\cdots+L_{k-2}-1)l_{n-k+1,1}+l_{n-k+2,1}+\cdots+l_{n-k+2,k-1}=\\
&\qquad\vdots\\
&(L_1-1)l_{n-1,1}+l_{n-1,1}+l_{n-1,2}=L_1l_{n-1,1}+l_{n-1,2}=l_{n,1}
\end{aligned}$$

sequences of length $n+1$ starting with $(0,1)$. Therefore there are

$$(L_1+\cdots+L_k-1)l_{n,1}$$

sequences of lenght $n+1$ starting with $(1,\eta)$ and $0\le\eta\le L_1+\cdots+L_k-2$ and by induction there are t_n-1 sequences of length $< n+1$. Therefore all togetter there are

$$(L_1+\cdots+L_k-1)l_{n,1}+t_n-1=t_{n+1}-1$$

sequences of length $\le n+1$, satisfying the conditions of the lemma. □

The numeration system (6) allows us to define the generalized LS-sequences of points in a direct way, as clarified in the following definition

Definition 4. Let N be an integer with representation given in (6). Then

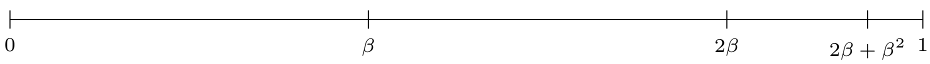
$$\begin{aligned}
\xi^N_{L_1,\dots,L_k}=&\sum_{i=0}^{n}\big(\beta^{i+1}\min\{L_1,\epsilon_i+\eta_i\}\\
&+\beta^{i+2}(\max\{\epsilon_i+\eta_i-L_1,0\}-\max\{\epsilon_i+\eta_i-L_1-L_2,0\})\\
&+\beta^{i+3}(\max\{\epsilon_i+\eta_i-L_1-L_2,0\}-\max\{\epsilon_i+\eta_i-L_1-L_2-L_3,0\})\\
&\vdots\\
&+\beta^{i+k-1}(\max\{\epsilon_i+\eta_i-L_1-\cdots-L_{k-2},0\}\\
&-\max\{\epsilon_i+\eta_i-L_1-\cdots-L_{k-1},0\})\\
&+\beta^{i+k}(\max\{\epsilon_i+\eta_i-L_1-\cdots-L_{k-1},0\})\big)\ .
\end{aligned}$$

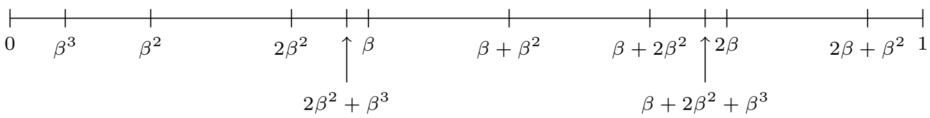
By the construction of our sequence $\xi^N_{L_1,\dots,L_k}$ it is not hard to see that $\xi^N_{L_1,\dots,L_k}$ is the left endpoint of an interval of the generalized LS-sequence of partitions $(\rho^n_{L_1,\dots,L_k}\omega)_{n\in\mathbb{N}}$ with $t_n\le N<t_{n+1}$. Indeed, the quantitiy $\epsilon_0+\eta_0$ tells us in which interval I^0 of $\rho_{L_1,\dots,L_k}$ the point $\xi^N_{L_1,\dots,L_k}$ lies. Assume that I_0 is a longest interval in the ith splitting step, then the quantity $\epsilon_i+\eta_i$ tells us in which interval of $I_0\rho_{L_1,\dots,L_k}$ the point $\xi^N_{L_1,\dots,L_k}$ lies. By construction of the ϵ_i we know

that in this case $\epsilon_\ell = 0$ for all $0 < \ell < i$. Proceeding in this manner we deduce that $(\xi^n_{L_1,\dots,L_k})_{n\in\mathbb{N}}$ is indeed the left endpoint of an interval of $\rho^n_{L_1,\dots,L_k}\omega$. Let us point out once more that the order of the points $\xi^N_{L_1,\dots,L_k}$ is given by the radical inverse construction of our numeration system (6).

The easiest example of a generalized LS-sequence is obtained by considering only three possible lengths for the intervals determining the partition. We call such sequences LMS-sequences.

Example 1. Consider the LMS-sequence of partitions $(\rho^n_{L,M,S}\omega)_{n\in\mathbb{N}}$ defined by the equation $\beta^3 + \beta^2 + 2\beta - 1 = 0$. In the following we describe the sequence of partitions and the associated sequence of points obtained by the procedure introduced above.





The sequence of partitions is obtained by splitting at each step the longest interval into 2 long intervals followed by an interval of medium length and by a short interval. So at the first partition we split the unit interval into two intervals of length β, followed by one of length β^2 and one of length β^3. At the second step we split only the first two intervals proportionally into two intervals of length β^2, one of lenght β^3 and one of length β^4, respectively. The procedure goes on in this way as for the classical LS-sequences.

The associated sequence of points is the sequence of left endpoints of the intervals determining the sequence of partitions, where the order is determined by Definition 4.

Let us list the digit expansion of the first 10 positive integers and the corresponding points of the sequence.

$$\begin{array}{lcll}
\Phi(1) & = & ((1,0)) & \to \ \beta; \\
\Phi(2) & = & ((1,1)) & \to \ 2\beta; \\
\Phi(3) & = & ((1,2)) & \to \ 2\beta+\beta^2; \\
\Phi(4) & = & ((1,0),(0,0)) & \to \ \beta^2; \\
\Phi(5) & = & ((1,0),(1,0)) & \to \ \beta+\beta^2; \\
\Phi(6) & = & ((1,1),(0,0)) & \to \ 2\beta^2; \\
\Phi(7) & = & ((1,1),(1,0)) & \to \ \beta+2\beta^2; \\
\Phi(8) & = & ((1,2),(0,0)) & \to \ 2\beta^2+\beta^3; \\
\Phi(9) & = & ((1,2),(1,0)) & \to \ \beta+2\beta^2+\beta^3; \\
\Phi(10) & = & ((1,0),(0,0),(0,0)) & \to \ \beta^3.
\end{array}$$

Remark 1. Let us point out the connection of the generalized LS-sequences to the classical LS-sequences and van der Corput sequences:

The case $k = 1$: In the case $k = 1$ we have $\beta = 1/L_1$ and the numeration introduced above is the usual L_1-adic numeration. Indeed, let us note that $l_n = t_n = T_n = L_1^n$ and in particular we obtain

$$N = \sum_{i=0}^{n} (\epsilon_i T_i + \eta_i l_{i,1}) = \sum_{i=0}^{n} d_i L_1^i,$$

where $d_i = \epsilon_i + \eta_i \in \{0, 1, \ldots, L_1 - 1\}$. A close look on Definition 4 reveals that in this case the generalized LS-sequence coincides with the van der Corput sequence.

The case $k = 2$: In this case the generalized LS-sequence coincides with the classical LS-sequence. Let us note that in the case $k = 2$ we have that $T_n = l_{n,1} + l_{n,2} = t_n$, i.e.,

$$N = \sum_{i=0}^{n} (\epsilon_i T_i + \eta_i l_{i,1}) = \sum_{i=0}^{n} (\epsilon_i t_i + \eta_i l_{i,1})$$

which corresponds to the numeration system introduced by Aistleitner et.al. [2].

In order to give an estimate for the discrepancy, it is necessary to introduce the notion of elementary intervals. An interval is called *elementary* if it is an element of $\rho^n_{L_1,\ldots,L_k}\omega$ for some n. Equivalently we can define elementary intervals as all intervals of the form $I_x^{(m)} = [\xi^x_{L_1,\ldots,L_k}, \xi^x_{L_1,\ldots,L_k} + \beta^m)$ for some m, where

$$\Phi(x + l_{m-1,1}) = \big((1, \eta_{m-1}), \ldots, (\epsilon_0, \eta_0)\big)$$

with $\eta_{m-1} < L_1$. In particular, there exists an integer $y < t_m$ such that $\xi^x_{L_1,\ldots,L_k} + \beta^m = \xi^y_{L_1,\ldots,L_k}$. Obviously, we may choose $y = x + l_{m-1,1} < t_m$.

The next step consists in finding a method to decide whether a point $\xi^N_{L_1,\dots,L_k}$ is contained in some given elementary interval or not.

Lemma 4. *Let N be a positive integer represented as in (6) and let*

$$I_x^{(m)} = [\xi^x_{L_1,\dots,L_k}, \xi^x_{L_1,\dots,L_k} + \beta^m)$$

be an elementary interval. Then

$$\xi^N_{L_1,\dots,L_k} \in I_x^{(m)} \quad \text{if and only if} \quad x = \sum_{i=0}^{m-1} (\epsilon_i T_i + \eta_i l_{i,1})$$

is the truncated representation of N.

In addition let $A_x^{(m)}(N) = \sharp \left\{ l \,:\, l \le N, \xi^l_{L_1,\dots,L_k} \in I_x^{(m)} \right\}$ and assume that

$$N = x + \sum_{i=m}^{n} (\epsilon_i T_i + \eta_i l_{i,1}).$$

Then

$$A_x^{(m)}(N) = \sum_{i=0}^{n-m} (\epsilon_{i+m} \tilde{T}_i + \eta_{i+m} l_{i,1}) + 1,$$

where $\tilde{T}_{i-m} = l_{i-m,1} + \dots + l_{i-m,j}$ such that j is the integer such that

$$\epsilon_{i+1} = \dots = \epsilon_{i+j} = 0 \quad \text{and} \quad \epsilon_{i+j+1} \neq 0.$$

Proof. Let N be of the form (6) and let $x = \sum_{i=0}^{m-1} \epsilon_i T_i + \eta_i l_{i,1}$ and $N_2 = \sum_{i=0}^{n-m} \epsilon_{i+m} \tilde{T}_i + \eta_{i+m} l_{i,1}$. Then by Definition 4, we have that

$$\xi^N_{L_1,\dots,L_k} = \xi^x_{L_1,\dots,L_k} + \beta^m \xi^{N_2}_{L_1,\dots,L_k}.$$

Since the generalized LS-sequences of points only take value in the unit interval and since two distinct points of a generalized LS-sequence with index strictly less than $l_{m,1}$ differ at least by β^m, this implies the first statement of the lemma. More precisely, it implies that

$$N = x + \sum_{i=m}^{n} (\epsilon_i T_i + \eta_i l_{i,1})$$

has the same first m digits as $\Phi(x)$, i.e., x is such that

$$\Phi(x + l_{m-1,1}) = \big((1, \eta_{m-1}), \dots, (\epsilon_0, \eta_0)\big)$$

with $\eta_{m-1} \le L_1 - 1$. Finally, due to the restrictions on the digits ϵ_i and η_i in Lemma 3, we have that

$$\tilde{N}_2 = \sum_{i=0}^{n-m} (\epsilon_{i+m} \tilde{T}_i + \eta_{i+m} l_{i,1})$$

with $\tilde{T}_{i-m} = l_{i-m,1} + \cdots + l_{i-m,j}$ such that j is the integer such that $\epsilon_{i+1} = \cdots = \epsilon_{i+j} = 0$ and $\epsilon_{i+j+1} \neq 0$, counts all points $\xi^i_{L_1,\dots L_k}$ in the interval $I^{(m)}_x$ with $x < i \leq N$. Since $\xi^x_{L_1,\dots L_k}$ is the first point that hits the interval $I^{(m)}_x$, the last statement of the lemma is established □

Next we are interested in an accurate formula for $\frac{A^{(m)}_x(N)}{N}$, where

$$\xi^N_{L_1,\dots,L_k} \in I^{(m)}_x.$$

Lemma 5. *Let us assume that N has a representation of the form* (6)*, and assume that $\xi^N_{L_1,\dots,L_k} \in I^{(m)}_x$. Then we have*

$$\frac{A^{(m)}_x(N)}{N} = \beta^m + \frac{R}{N}, \tag{7}$$

where

$$R \leq 1 + |\lambda_{1,0}| + \sum_{j=2}^{k} (2\Lambda_j + |\lambda_{j,0}|).$$

and

$$\Lambda_j = \max_{\ell=2,\dots,k} \left\{ \frac{|\sum_{i=1}^{\ell} |\lambda_{j,i}| + (L_1 + \cdots + L_k - 2)|\lambda_{j,1}|}{1 - |\beta_j|^{-1}} \right\}.$$

Remind that we assume that the polynomial $L_k X^k + \cdots + L_1 X - 1$ has no double roots and has only one positive root $\beta < 1$.

P r o o f. Using our assumptions and Lemma 4 we can calculate the exact values of $A^{(m)}_x(N)$ and N. In fact, we have

$$N = x + \sum_{i=m}^{n} (\epsilon_i T_i + \eta_i l_{i,1}) \quad \text{and} \quad A^{(m)}_x(N) = \sum_{i=m}^{n} (\epsilon_i \tilde{T}_{i-m} + \eta_i l_{i-m,1}) + 1,$$

In particular, if we write

$$T_i = \sum_{j=1}^{k} \beta_j^{-i} \Lambda_{j,i}\,,$$

then we have

$$\tilde{T}_{i-m} = \sum_{j=1}^{k} \beta_j^{-i+m} \Lambda_{j,i}\,,$$

where $\Lambda_{j,i} = \lambda_{j,1} + \cdots + \lambda_{j,\ell_i}$ for some integer $2 \leq \ell_i \leq k$ depending on i.

This yields

$$\begin{aligned}\frac{A_x^{(m)}(N)}{N} &= \frac{\sum_{i=m}^{n}(\epsilon_i \tilde{T}_{i-m} + \eta_i l_{i-m,1}) + 1}{\sum_{i=m}^{n}(\epsilon_i T_i + \eta_i l_{i,1}) + x} \\ &= \frac{\sum_{i=m}^{n}(\epsilon_i \Lambda_{1,i} + \eta_i \lambda_{1,1})\beta^{-i+m} + \sum_{j=2}^{k} \tilde{R}_j + 1}{\beta^{-m}\sum_{i=m}^{n}(\epsilon_i \Lambda_{1,i} + \eta_i \lambda_{1,1})\beta^{-i+m} + \sum_{j=2}^{k} \tilde{R}_j \beta_j^{-m} + x},\end{aligned}$$

where

$$\tilde{R}_j = \sum_{i=m}^{n}(\epsilon_i \Lambda_{j,i} + \eta_i \lambda_{j,1})\beta_j^{-i+m} \leq \Lambda_j \quad \text{for } j = 2, \ldots, k.$$

Further, note that

$$N = \beta^{-m}\sum_{i=m}^{n}(\epsilon_i \Lambda_{1,i} + \eta_i \lambda_{1,i})\beta^{-i+m} + \sum_{j=2}^{k} \tilde{R}_j \beta_j^{-m} + x\,.$$

Therefore we obtain

$$\begin{aligned}\frac{A_x^{(m)}(N)}{N} &= \beta^m + \frac{\sum_{j=2}^{k} \tilde{R}_j + 1}{N} + \frac{\sum_{i=m}^{n}(\epsilon_i \Lambda_{1,i} + \eta_i \lambda_{1,1})\beta^{-i+m} - N\beta^m}{N} \\ &= \beta^m + \frac{\sum_{j=2}^{k} \tilde{R}_j + 1}{N} + \frac{\sum_{i=m}^{n}(\epsilon_i \Lambda_{1,i} + \eta_i \lambda_{1,1})\beta^{-i+m}}{N} \\ &\quad - \frac{\sum_{i=m}^{n}(\epsilon_i \Lambda_{1,i} + \eta_i \lambda_{1,1})\beta^{-i+m} + \sum_{j=2}^{k} \tilde{R}_j \left(\frac{\beta}{\beta_j}\right)^m + x\beta^m}{N} \\ &= \beta^m + \frac{1 + \sum_{j=2}^{k} \tilde{R}_j \left(1 - \left(\frac{\beta}{\beta_j}\right)^m\right) - x\beta^m}{N}.\end{aligned}$$

Finally, we want to estimate $x\beta^m$. Since we assume that

$$\Psi^{-1}(x + l_{m-1,1}) = \big((1, \eta_{m-1}), \ldots, (\epsilon_0, \eta_0)\big), \quad \text{with} \quad \eta_{m-1} \leq L_1 - 1$$

we obviously have that

$$x < t_{m-1} + l_{m-1,1}(L_1 - 1) < t_m,$$

and therefore,

$$x\beta^m \leq |\lambda_{1,0}| + \sum_{j=2}^{k} |\lambda_{j,0}| \left|\frac{\beta}{\beta_j}\right|^m \leq \sum_{j=1}^{k} |\lambda_{j,0}|.$$

If we put all our results together and note that $|1 - (\beta/\beta_j)^m| \leq 2$ for each positive integer m, we obtain the statement of the Lemma. □

Remark 2. We note that the only place, where we used the fact that the polynomial $L_k X^k + \cdots + L_1 X - 1$ has no double zeros and all roots but one have absolute value smaller than 1 is in the proof of Lemma 5. Let us note that dropping the assumption that there exists no double zeros would not change the result. But, it would result in a slightly worse estimate for R and more technical difficulties in the course of the proof. For the sake of simplicity we stick with the case of simple zeros.

Further, let us note that assuming that all roots but one have an absolute value ≤ 1 would result in an estimate of the form

$$|R| \leq (\log N)^A C_{L_1,\dots,L_k},$$

where A is the number of roots with absolute value $= 1$ counted with multiplicities and $C_{L_1,\dots,L_k}$ is some constant depending on the k-tuple $(L_1,\dots,L_k)$. Plugging this bound into the proof of Theorem 2 we would obtain

$$D_N(\xi^1_{L_1,\dots,L_k},\xi^2_{L_1,\dots,L_k},\dots,\xi^N_{L_1,\dots,L_k}) \leq C_{L_1,\dots,L_K}\frac{(\log N)^{A+1}}{N}.$$

In case that $L_kX^k+\cdots+L_1X-1$ has more than one root with an absolute value >1 the bound of the discrepancy would be only of the order N^{-B} for some $B<1$. Let us note that similar observations have been made by Carbone [4] in the case that $k=2$.

In the next section we compute the discrepancy of classical LS-sequences. To do so we need the following Lemma proved in [2, Lemma 6] which is more precise than Lemma 5.

Lemma 6. *Assume that N has a representation of the form (6), and assume that $\xi^N_{L,S}\in I^{(m)}_x$. Then we have*

$$\frac{A^{(m)}_x(N)}{N} = \beta^m + \frac{R(1-(-S\beta)^m)+1-x\beta^m}{N}, \tag{8}$$

where

$$R = \sum_{i=m}^{n}(\epsilon_i\tau_1+\eta_i\lambda_1)(-S\beta)^{i-m}$$

with

$$\tau_1 = \frac{-L-2S+\sqrt{L^2+4S}}{2\sqrt{L^2+4S}}, \qquad \lambda_1 = \frac{-L+\sqrt{L^2+4S}}{2\sqrt{L^2+4S}}\ .$$

Moreover, R can be estimated by

$$|R| < \max\{|\tau_1|,|\tau_1+(L+S-2)\lambda_1|\}\frac{1-(S\beta)^{n-m+1}}{1-S\beta}$$

if $S\beta\neq 1$ and

$$|R| < \max\{|\tau_1|,|\tau_1+(L+S-2)\lambda_1|\}\max\{n-m+1,0\}$$

if $S\beta = 1$.

3. Discrepancy bounds for classical LS-sequences

This section is devoted to the explicit computation of the discrepancy of classical LS-sequences with the aim proving

Theorem 1. *Let $(\xi^n_{L,S})_{n\in\mathbb{N}}$ be a classical LS-sequence of points with $L \geq S$. Then*

$$D_N(\xi^1_{L,S}, \xi^2_{L,S}, \dots, \xi^N_{L,S}) \leq \frac{\log N}{N|\log\beta|}(2L+S-2)\left(\frac{\tilde{R}}{1-S\beta}+1\right)+\frac{B}{N} \tag{9}$$

with

$$\tilde{R} = \max\{|\tau_1|, |\tau_1+(L+S-2)\lambda_1|\}$$

and

$$B = (2L+S-2)\left(\frac{\tilde{R}}{1-S\beta}+1\right)+2.$$

Proof. Let $\xi^1_{L,S}, \xi^2_{L,S}, \dots, \xi^N_{L,S}$ be the first N points of the sequence $(\xi^n_{L,S})_{n\in\mathbb{N}}$ and take n such that $t_n \leq N < t_{n+1}$. Consider an arbitrary subinterval $[x, y)$ in $[0, 1[$. We want to estimate the number of points among $\xi^1_{L,S}, \xi^2_{L,S}, \dots, \xi^N_{L,S}$ which lie in the interval. To do so we approximate the interval $[x, y)$ from above by elementary intervals of length at least β^{n+1} and apply (8) to each interval. The points x and y belong to some interval determined by the nth partition, respectively. In particular, let us assume that x lies in an elementary interval $[x'', x')$ and y lies in an elementary interval $[y', y'')$ of length β^n, respectively.

We approximate the interval $[x, y)$ from above, i.e., we try to cover the interval $[x', y') \subseteq [x, y)$ by as few as possible, disjoint elementary intervals of length at least β^{n+1}. Starting from the point x' we move to the right in order to reach the point y' with steps of variable length. By the definition of our LS-sequence there is an integer $0 \leq \ell \leq L-1$ such that $x'_1 = x' + \ell\beta^n + S\beta^{n+1}$ is the left endpoint of an elementary interval of length at least $\beta^{n_1} \geq \beta^{n-1}$. In case that $y' \leq x'_1$ we have found our covering. On the other hand there exists an integer $0 \leq \ell \leq L-1$ such that $y'_1 = y' - \ell\beta^n$ is the right endpoint of an elementary interval of length at least β^{n-1}. Hence we are left by the problem to cover the interval $[x'_1, y'_1)$ by as few as possible, disjoint elementary intervals of length at least β^n. It is now easy to see by induction on n that $[x', y')$ is covered by at most $2L+S-2$ disjoint elementary intervals of length $\beta^2, \dots, \beta^n$, respectively and at most S intervals of length β^{n+1} and L intervals of length β.

Let $A_{x,y}(N) = \sharp\{l : l \le N, \xi^l_{L,S} \in [x, y)\}$. Since by construction there are no $\xi^l_{L,S} \in [x', x) \cup [y, y')$ with $l \le N$ we obtain by Lemma 6

$$\left|(y-x) - \frac{A_{x,y}(N)}{N}\right| \le |x-x'| + |y-y'| + \left|(y'-x') - \frac{A_{x',y'}(N)}{N}\right| \\ \le 2\beta^n + (2L+S-2)\sum_{m=0}^{n+1}\left|\frac{R(1-(-S\beta)^m)+1}{N}\right|, \tag{10}$$

where we used the inequality

$$1 \ge 1 - x\beta^m \ge 1 - l_m\beta^m = 1 - \lambda_0 - \lambda_1(-S\beta^2)^m > 1 - \lambda_0 - \lambda_1 = 0,$$

where

$$l_m = l_{m,1} = \lambda_0\beta^m + \lambda_1(-S\beta)^m \quad \textit{and} \quad \lambda_0 = \frac{L+\sqrt{L^2+4S}}{2\sqrt{L^2+4S}}.$$

In order to establish Theorem 1 we are left to estimate the sum in (10). Since $|R| \le \tilde{R}\frac{1-(S\beta)^{n-m+1}}{1-(S\beta)}$ we find

$$\left|\sum_{m=0}^{n+1}|R(1-(-S\beta)^m)|\right| \le \tilde{R}\sum_{m=0}^{n+1}\frac{(1-(-S\beta)^m)(1-(S\beta)^{n-m+1})}{1-(S\beta)} \\ \le \tilde{R}\sum_{m=0}^{n+1}\frac{(1+(S\beta)^m)(1-(S\beta)^{n-m+1})}{1-(S\beta)} \\ \le \frac{(n+2)\tilde{R}}{1-S\beta}.$$

Since $\beta^{-n} \le t_n \le N$, we get $n \le \frac{\log N}{|\log\beta|}$ and we obtain Theorem 1. □

As an example we can compute the discrepancy of a particular LS-sequence obtained by taking $L = S = 1$. This sequence, also called Kakutani-Fibonacci sequence, has been also analysed in detail in [5, 9] in the frame of ergodic theory where it has been shown that it can be obtained as the orbit of an ergodic transformation.

Take $L = S = 1$, then by Theorem 1 we have that

$$D_N(\xi^1_{1,1}, \xi^2_{1,1}, \ldots, \xi^N_{1,1}) \le 2.366\frac{\log N}{N} + \frac{3.139}{N}.$$

In the case that $L = 10$ and $S = 1$ we obtain

$$D_N(\xi^1_{10,1}, \xi^2_{10,1}, \ldots, \xi^N_{10,1}) \le 8.66\frac{\log N}{N} + \frac{22.02}{N}.$$

In particular we obtain

Corollary 1. *Let S be fixed and assume that L is large. Then we obtain*

$$\lim_{N\to\infty} \frac{ND_N(\xi^1_{L,S},\dots,\xi^N_{L,S})}{\log N} \sim \frac{2L}{\log L}$$

as $L\to\infty$.

Proof. By the formula given for $\tilde{R}$ in Lemma 6 and since $\beta = \frac{-L+\sqrt{L^2+4S}}{2S}$ we obtain that $\tilde{R} \sim S\beta \sim S/L$ hence

$$\frac{2L+S}{|\log\beta|}\left(\frac{\tilde{R}}{1-S\beta}+1\right) \sim \frac{2L}{\log L}$$

as $L\to\infty$ and S is fixed. □

Remark 3. One can rather easily improve our bound for the discrepancy by a factor $1/2$. This can be done by balancing our choice of the intervals that cover $[x,y)$. Let us assume that x' is the right endpoint of an interval of length β^n and let us assume that x' lies nearer to the right endpoint x'_1 than to the left end point. We go from x' to the right instead to the left and proceed in this manner. We need fewer intervals (roughly $1/2$-times fewer) to cover the interval $[x',y')$. We did not work out the details for this improvement, since the paper is already rather technical and this approach would further increase the technical difficulties.

Remark 4. Faure [8] (see also [12, page 25]) obtains the upper bound

$$\lim_{N\to\infty} \frac{ND_N\left(\xi^1_L,\dots,\xi^N_L\right)}{\log N} \sim \frac{L}{4\log L}$$

if L is large, where ξ_L denotes the van der Corput sequence for base L. This shows that our approach gives up to a factor 8 (respectively 4 considering the remark above) a similar main term for the discrepancy as one obtains for the van der Corput sequence.

At the end of this section we want to compare our approach to obtain explicit bounds for the discrepancy of LS-sequences to the approach due to Carbone [4]. Therefore we give explicit bounds for the star-discrepancy of the Kakutani-Fibonacci sequence of partitions $D^*_n(\rho^n_{1,1})$ and of points $D^*_N(\xi^n_{1,1})$, where

$$D^*_n(\rho^n_{1,1}) = \sup_{0\le b\le 1}\left|\frac{1}{t_n}\sum_{i=1}^{t_n}\mathbf{1}_{[0,b)}(\xi^i_{1,1}) - b\right|.$$

Following [4], in order to find upper and lower bounds for $D_n^*(\rho_{1,1}^n)$, we need to estimate

$$\frac{1}{t_n}\sum_{i=1}^{t_n}\mathbf{1}_{[0,b)}(\xi_{1,1}^i)-b,$$

where $b\in(0,1)$. Let $[b_1^{(n-1)},b_2^{(n-1)})$ be the interval of $\rho_{1,1}^{n-1}$ containing b. Since the number of points of $\rho_{1,1}^{n-1}$ contained in $[b_1^{(n-1)},b_2^{(n-1)})$ is 1 if this interval is a short one, and it is 2 if it is a long one, we have

$$\begin{aligned}\frac{1}{t_n}\sum_{i=1}^{t_n}\mathbf{1}_{[0,b)}(\xi_{1,1}^i)-b &\le \frac{1}{t_n}\sum_{i=1}^{t_n}\mathbf{1}_{[0,b_2^{(n-1)})}(\xi_{1,1}^i)-b_1^{(n-1)}\\ &\le \frac{1}{t_n}\sum_{i=1}^{t_n}\mathbf{1}_{[0,b_1^{(n-1)})}(\xi_{1,1}^i)-b_1^{(n-1)}+\frac{2}{t_n}\end{aligned}\tag{11}$$

and

$$\begin{aligned}\frac{1}{t_n}\sum_{i=1}^{t_n}\mathbf{1}_{[0,b)}(\xi_{1,1}^i)-b &\ge \frac{1}{t_n}\sum_{i=1}^{t_n}\mathbf{1}_{[0,b_1^{(n-1)})}(\xi_{1,1}^i)-b_2^{(n-1)}\\ &\ge \frac{1}{t_n}\sum_{i=1}^{t_n}\mathbf{1}_{[0,b_2^{(n-1)})}(\xi_{1,1}^i)-b_2^{(n-1)}-\frac{2}{t_n}\ .\end{aligned}\tag{12}$$

We will estimate $D_n^*(\rho_{1,1}^n)$ evaluating

$$\sum_{i=1}^{t_n}\mathbf{1}_{[0,b_k^{(n-1)})}(\xi_{1,1}^i)-b_k^{(n-1)}\quad \text{for}\ \ k=1,2,$$

using (11) and (12). For this purpose, it will be convenient to represent $[0,b_k^{(n-1)})$ (denoted by $[0,b^{(n-1)})$ from now on) as union of consecutive intervals of the partitions $\rho_{1,1}^p$ for $p\le n-1$. Consider $[0,b)$ as a union of intervals defining the pth partitions $\rho_{1,1}^p$, for $p\le n-1$. Therefore let us count how many consecutive intervals $I_1^1,I_2^1,\dots I_{m_1}^1$ of $\rho_{1,1}^1$ are contained in $[0,b)$. Now, we count how many consecutive intervals $I_1^2,I_2^2,\dots I_{m_2}^2$ of $\rho_{1,1}^2$ are contained in $[0,b)\setminus\bigcup_{i=1}^{m_1}I_i^1$ and so on. Of course, it may happen that $m_k=0$ for some k. Going on with this procedure we get

$$\bigcup_{i=1}^{m_{n-1}} I_i^{n-1}\subset[0,b)\setminus\bigcup_{p=1}^{n-2}\left(\bigcup_{i=1}^{m_p}I_i^p\right)\ .$$

Thus

$$[0,b)=\bigcup_{p=1}^{n-1}\left(\bigcup_{i=1}^{m_p}I_i^p\right)\ .\tag{13}$$

In particular, we need to compute m_p. Since $L = S = 1$, we have that if $I_1^p = L^{(p)} = I_2^p$, then $I_1^p \cup I_2^p = I_i^{p-1}$ for some i, and if $I_1^p = I_2^p = L^{(p)}$, then $I_1^p = S^{(p-1)}$. We denote by $L^{(p)}$, respectively $S^{(p)}$ some long respectively some short interval in the pth partition step. In particular, this shows that $m_p = 1$. Thus there are no consecutive intervals of the same partition. Recall [4, Equation 5-6], that is for $n \geq p$,

$$\frac{1}{t_n}\sum_{i=1}^{t_n} \mathbf{1}_{L_p}(\xi_{1,1}^i) - \lambda(L_p) = \frac{t_{n-p}}{t_n} - \beta^p = -\frac{B}{t_n}(-S\beta)^n\big((-S\beta)^{-p} - \beta^p\big)\,,$$

and

$$\frac{1}{t_n}\sum_{i=1}^{t_n} \mathbf{1}_{S_p}(\xi_{1,1}^i) - \lambda(S_p) = \frac{t_{n-p-1}}{t_n} - \beta^{p+1} = -\frac{B}{t_n}(-S\beta)^n\big((-S\beta)^{-p-1} - \beta^{p+1}\big)\,,$$

where

$$B = \frac{S\beta - S\beta^2}{1 + S\beta^2} \quad \text{and} \quad \beta = \frac{\sqrt{5}-1}{2}$$

Thus, we can write:

$$\begin{aligned}
&\frac{1}{t_n}\sum_{i=1}^{t_n} \mathbf{1}_{[0,b^{(n-1)})}(\xi_{1,1}^i) - b^{(n-1)} \\
&= \frac{1}{t_n}\sum_{i=1}^{t_n} \mathbf{1}_{\cup_{p=1}^{n-1} I^p}(\xi_{1,1}^i) - \lambda\left(\cup_{p=1}^{n-1} I^p\right) \\
&= \sum_{p=1}^{n-1}\left(\frac{1}{t_n}\sum_{i=1}^{t_n} \mathbf{1}_{I^p}(\xi_{1,1}^i) - \lambda(I^p)\right) \\
&= \sum_{p=1}^{n-1}\left(\frac{1}{t_n}\underbrace{\sum_{i=1}^{t_n} \mathbf{1}_{L^{(p)}}(\xi_{1,1}^i)}_{t_{n-p}} - \underbrace{\lambda(L^{(p)})}_{\beta^p}\right) \\
&= \frac{1}{t_n}\frac{\beta^2 - \beta}{1+\beta^2}(-\beta)^n \sum_{p=1}^{n-1}[(-\beta)^{-p} - \beta^p] \\
&= \frac{1}{t_n}\frac{\beta^2 - \beta}{1+\beta^2}(-\beta)^n \left[\frac{\beta(-\beta)^n - \beta}{(-\beta)^n(1+\beta)} - \frac{1-\beta^n}{1-\beta}\right] \\
&= \frac{1}{t_n}\frac{(\beta-1)}{(1+\beta^2)}\Big(\beta^2\big(1-(-\beta)^n\big) - \beta + (-\beta)^n\beta^n(1+\beta) - (-\beta)^n\Big).
\end{aligned}$$

By considering separately the case that n is even and the case that n is odd, one gets the following upper and lower bounds:

$n = 2h,\ h \geq 1$

$$\begin{aligned}
&\frac{1}{t_{2h}} \frac{(\beta-1)}{(1+\beta^2)} \left(\beta^2 - \beta - \beta^{2h+2} - \beta^{2h}\right) \\
&\leq \frac{1}{t_{2h}} \sum_{i=1}^{t_{2h}} \mathbf{1}_{[0,b^{(2h-1)})}(\xi^i_{1,1}) - b^{(2h-1)} \\
&\leq \frac{1}{t_{2h}} \frac{(\beta-1)}{(1+\beta^2)} \left(\beta^2 - \beta + \beta^{4h} + \beta^{4h+1}\right).
\end{aligned}$$

$n = 2h+1,\ h \geq 1$

$$\begin{aligned}
&\frac{1}{t_{2h+1}} \frac{(\beta-1)}{(1+\beta^2)} \left(\beta^2 - \beta - \beta^{4h+2} - \beta^{4h+3}\right) \\
&\leq \frac{1}{t_{2h+1}} \sum_{i=1}^{t_{2h+1}} \mathbf{1}_{[0,b^{(2h)})}(\xi^i_{1,1}) - b^{(2h)} \\
&\leq \frac{1}{t_{2h+1}} \frac{(\beta-1)}{(1+\beta^2)} \left(\beta^2 - \beta + \beta^{2h+3} + \beta^{2h+1}\right).
\end{aligned}$$

Now, from (11) and (12), we have

$$\begin{aligned}
&-2 + \frac{(\beta-1)}{(1+\beta^2)} \left(\beta^2 - \beta - \beta^{2h+2} - \beta^{2h}\right) \\
&\leq t_{2h} \left[\sup_{0<b\leq 1} \sum_{i=1}^{t_{2h}} \mathbf{1}_{[0,b)}(\xi^i_{1,1}) - b \right] \\
&\leq 2 + \frac{(\beta-1)}{(1+\beta^2)} \left(\beta^2 - \beta + \beta^{4h} + \beta^{4h+1}\right),
\end{aligned}$$

and

$$\begin{aligned}
&-2 + \frac{(\beta-1)}{(1+\beta^2)} \left(\beta^2 - \beta - \beta^{4h+2} - \beta^{4h+3}\right) \\
&\leq t_{2h+1} \left[\sup_{0<b\leq 1} \sum_{i=1}^{t_{2h+1}} \mathbf{1}_{[0,b)}(\xi^i_{1,1}) - b \right] \\
&\leq 2 + \frac{(\beta-1)}{(1+\beta^2)} \left(\beta^2 - \beta + \beta^{2h+3} + \beta^{2h+1}\right).
\end{aligned}$$

Since $\beta < 1$, the following limits exist:

$$\begin{aligned}&\lim_{h\to\infty}\left(\beta^2-\beta-\beta^{2h+2}-\beta^{2h}\right)\\&=\lim_{h\to\infty}\left(\beta^2-\beta-\beta^{4h+2}-\beta^{4h+3}\right)=\beta^2-\beta,\end{aligned}$$

and

$$\begin{aligned}&\lim_{h\to\infty}\left(\beta^2-\beta+\beta^{4h}+\beta^{4h+1}\right)\\&=\lim_{h\to\infty}\left(\beta^2-\beta-\beta^{2h+3}+\beta^{2h+1}\right)=\beta^2-\beta,\end{aligned}$$

so that the sequences of upper and lower bounds are bounded. Therefore, we conclude that

$$\frac{c_1}{t_n}\le D_n^*(\rho_{1,1}^n)\le\frac{c_2}{t_n},$$

with

$$c_1=-1.9348 \quad\text{and}\quad c_2=2.0652.$$

Now, let $(\tilde\rho_{L,S}^n)_{n\in\mathbb{N}}$ be the sequence of long intervals l_n of $\rho_{L,S}^n$. Then it is possible to bound its discrepancy (see [4, Proposition 3.4]) and obtain for $S<L+1$ and every $n\in\mathbb{N}$

$$\frac{\tilde c_1}{l_n}\le D_n(\tilde\rho_{L,S}^n)\le\frac{\tilde c_2}{l_n}, \tag{14}$$

where $\tilde c_1$ and $\tilde c_2$ are constants independent of n. Furthermore, it can be shown (see [4, page 843]) that if $S<L+1$, then

$$D_N^*\left(\xi_{L,S}^1,\dots,\xi_{L,S}^N\right)\le 2\frac{c_2+(L+S-2)\tilde c_2}{N|\log\beta|}\log N+(L+S-2),$$

where c_2 is the constant in the upper bound of $D_n(\rho_{L,S}^n)$ (see [4, Equation 29]).

Finally, plugging in $L=S=1$, we can give an estimate for the discrepancy of the sequence $(\xi_{1,1}^n)_{n\in\mathbb{N}}$.

$$D_N^*(\xi_{1,1}^n)\le 2\frac{c_2}{|\log\beta|}\frac{\log N}{N}=2\frac{2.0652}{|\log\beta|}\frac{\log N}{N}\sim 8.5833\frac{\log N}{N}\ .$$

Let us point out that this result is in accordance with the following relation

$$D_N^*(x_n)\le D_N(x_n)\le 2D_N^*(x_n)\ ,$$

which holds true for every point set $x_1,\dots,x_N$.

4. The Discrepancy of generalized LS-sequences

The aim of the present section is to provide bounds for the generalized-LS-sequences:

Theorem 2. *Let $(\xi^n_{L_1,\dots,L_k})_{n\in\mathbb{N}}$ be a generalized-LS-sequence of points. Then*

$$D_N(\xi^1_{L_1,\dots,L_k},\dots,\xi^N_{L_1,\dots,L_k}) \le \frac{\log(N+1)-\log|\lambda_{1,0}\beta^k|}{N|\log\beta|}(2L_1+L_2+\cdots+L_k-2)\tilde{R}, \tag{15}$$

where

$$\tilde{R} = 1+|\lambda_{1,0}|+\sum_{j=2}^{k}(2\Lambda_j+|\lambda_{j,0}|)$$

and

$$\Lambda_j = \max_{\ell=2,\dots,k}\left\{\frac{|\sum_{i=1}^{\ell}|\lambda_{j,i}|+(L_1+\cdots+L_k-2)|\lambda_{j,1}|}{1-|\beta_j|^{-1}}\right\}$$

provided N is large enough.

P r o o f. The proof of Theorem 2 runs along the same lines as the proof of Theorem 1.

As above we put n such that $t_n \le N < t_{n+1}$ and we consider an arbitrary subinterval $[x,y)\subset[0,1)$ and approximate it from above by elementary intervals. As in the LS-case we assume that $x\in[x'',x')$ and $y\in[y',y'')$ lie in elementary intervals of length β^n, respectively.

Let x'_1 be the next left endpoint of an elementary interval of length at least β^{n-1} from x'. We need at most $L_1+\cdots+L_k-1$ intervals of variable length to cover the interval $[x',x'_1)$. Similarly we proceed for the interval $[y'_1,y')$, where y'_1 is the nearest right endpoint of an elementary interval of length at least β^{n-1} lying left of y'. Obviously, we need at most L_1-1 elementary intervals of length β^n to cover $[y'_1,y')$. Similarly as in the proof of Theorem 1 we conclude that we need at most $2L_1+L_2+\cdots+L_k-2$ elementary intervals of each length β^ℓ with $\ell=1,\dots,n+k$ to cover $[x',y')$.

Using similar notations as in the proof of Theorem 1 and by Lemma 5 we obtain

$$\begin{aligned}\left|(y-x)-\frac{A_{x,y}(N)}{N}\right| &\le |x-x'|+|y-y'|+\left|(y'-x')-\frac{A_{x',y'}(N)}{N}\right| \\ &\le 2\beta^n+(2L_1+\cdots+L_k-2)(n+k)\tilde{R}\,,\end{aligned} \tag{16}$$

where

$$\tilde{R} = 1+|\lambda_{1,0}|+\sum_{j=2}^{k}(2\Lambda_j+|\lambda_{j,0}|)$$

and

$$\Lambda_j = \max_{\ell=2,\dots,k}\left\{\frac{|\sum_{i=1}^{\ell}|\lambda_{j,i}| + (L_1+\cdots+L_k-2)|\lambda_{j,1}|}{1-|\beta_j|^{-1}}\right\}.$$

Since $|\lambda_{1,0}|\beta^{-n} - 1 \le t_n \le N$, provided N is large enough, we get

$$n \le \frac{\log(N+1) - \log|\lambda_{1,0}|}{|\log\beta|}$$

and we obtain Theorem 2. □

Let us compute the discrepancy for a concrete example.

Example 2. Let us consider the case of the LMS-sequence considered above (see Example 1), with $L = 2$ and $M = S = 1$. To ease the notations we write $l_n = l_{n,1}, m_n = l_{n,2}$ and $s_n = l_{n,3}$. First we note that the roots or the polynomial $X^3 + X^2 + 2X - 1$ are $\beta \simeq 0.393$, $\beta_{2,3} \simeq -0.696 \pm i1.436$. Moreover, we can apply Cramer's rule as in (5) to compute the $\lambda_{j,i}$'s, for $1 \le j \le 3$ and $0 \le i \le 3$. Then by Theorem 2 we have that

$$D_N\left(\xi^1_{2,1,1}, \xi^2_{2,1,1}, \dots, \xi^N_{2,1,1}\right) \le 51.4562\frac{\log(N+1)}{N} + \frac{122.5173}{N}\ .$$

Finally, let us state the following remark.

Remark 5. Let us note that the proof of Theorem 2 leaves a lot of room for improvement. First, as already explained in Remark 3 by more carefully choosing the intervals which cover $[x', y')$ we might replace the factor $2L_1+L_2+\cdots+L_k-2$ by something like $L_1 + L_2 + \cdots + L_k$. Moreover, the estimates for Λ_j and R in Lemma 5 are rather rough and can be certainly improved in concrete cases.

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