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PALINDROMIC CLOSURES AND THUE-MORSE SUBSTITUTION FOR MARKOFF NUMBERS

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Dedicated to the memory of Professor Pierre Liardet

ABSTRACT. We state a new formula to compute the Markoff numbers using iterated palindromic closure and the Thue-Morse substitution. The main theorem shows that for each Markoff number m, there exists a word $v \in \{a, b\}^*$ such that $m - 2$ is equal to the length of the iterated palindromic closure of the iterated antipalindromic closure of the word av. This construction gives a new recursive construction of the Markoff numbers by the lengths of the words involved in the palindromic closure. This construction interpolates between the Fibonacci numbers and the Pell numbers.

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1. Introduction

Markoff numbers are fascinating integers; the reader may use the recent book by Martin Aigner [A] for studying them. These numbers are related to number theory, hyperbolic geometry, continued fractions and Christoffel words [A, M1, M2, F, Re1, Re2]. Many great mathematicians have worked on these numbers and the famous uniqueness conjecture by Frobenius is still unsolved [B, M1, M2, F, C1, C2]. Markoff numbers are positive integers that appear in the solution of the Diophantine equation

$$x^2 + y^2 + z^2 = 3xyz.$$

The first Markoff numbers are 1, 2, 5, 13, 29, 34, 89, 169, 194, 233, 433, 610, 985, 1325, 1597, 2897, 4181, 5741, 6466, 7561, 9077, 10946, 14701, 28657, 33461,

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37666, 43261, 51641; they are listed in the Sloane Encyclopedia of Integer Sequences (sequence number A002559). One shows that if a *Markoff triple* (x, z, y), that is, a triple satisfying the previous Diophantine equation, has maximum z, then the triple gives birth to two others, which are $(x, 3xy - z, z)$ and $(z, 3zy - x, y)$ (see [A] Section 3.1). One can construct a binary tree using these computations, were each node is a Markoff triple (see [A]). The Frobenius conjecture asserts that each Markoff number is the maximum of a unique Markoff triple ([A, Re2]). In the work of Markoff [M1, M2], one find implicitly combinatorics on words and construction of balanced sequences [CF, BS, V, BdLR] on the alphabet $\{11, 22\}$. The Markoff numbers are also linked with approximation theory and continued fractions [BRS, B].

In this article, we find a new relation between Markoff numbers and combinatorics on words. The main theorem shows that for each Markoff number m there exists a word $v \in \{a, b\}^*$ such that $m - 2$ is equal to the length of the iterated palindromic closure of the iterated antipalindromic closure of the word av.

The *iterated palindromic closure* (due to Aldo de Luca) is used in combinatorics on words in order to generate standard Sturmian words and central words [dL, J, BS]. One defines first the *palindromic closure* $w^{(+)}$ of a word w: it is the shortest palindrome having w as prefix (it exists and is unique). The iterated palindromic closure $\mathrm{Pal}(u)$ is then defined recursively by $\mathrm{Pal}(1) = 1$ (the empty word), and $\mathrm{Pal}(va) = (\mathrm{Pal}(v)a)^{(+)}$ for any word v and any letter a.

The *iterated antipalindromic closure* appears in the literature in order to construct antipalindromes and to generalize the iterated palindromic closure [dLDL, BPTV]. In fact, when the alphabet is binary, the iterated antipalindromic closure of a word u is obtained by applying the Thue-Morse substitution to the iterated palindromic closure of u [dLDL].

As an application of the main theorem, we give a new computation of Markoff numbers by a recursive construction on the lengths of the words involved in the iterated palindromic closure. The lengths of these words allow us to state a recursive formula using a *directive sequence* $d = d_1 d_2 \dots d_j$ with d_i on the alphabet $\{a, b\}$. One interesting property is to recover the usual Fibonacci recursive construction if $d_j \neq d_{j-1} \neq d_{j-2}$ and the usual Pell recursive construction if $d_j \neq d_{j-1} = d_{j-2}$ [C3, BRS].

Note that in the articles [F, P] we find two other decompositions of the Markoff numbers as sums of positive integers: using properties of continued fractions in the work of Frobenius and properties of snake graphs in the work of Propp et al. (see also [A]).

2. Iterated palindromic closures

In the sequel we work with the usual notations in combinatorics on words [BS]. Let $\mathcal{A}$ be a finite alphabet.

The *reversal* of a word $x = x_1x_2\dots x_n$ with $x_i \in \mathcal{A}$ is the word $\widetilde{x} = x_nx_{n-1}\dots x_1$.

A word p is a *palindrome* if it is equal to its reversal (that is $p = \widetilde{p}$).

The length of a word $u = u_1u_2\dots u_m$, where $u_i \in \mathcal{A}$, is equal to m and is denoted $|u|$.

The *concatenation* of two words $u = u_1u_2\dots u_m$ and $v = v_1v_2\dots v_n$ is the word of the length $m+n$ given by $u \cdot v = u_1u_2\dots u_mv_1v_2\dots v_n$.

In this article we use the *palindromic closure*, introduced by Aldo de Luca [dL] (more precisely, it is the *right palindromic closure*): the palindromic closure of a word x is the shortest palindrome having x as a prefix; it exists and is unique; it is denoted by $x^{(+)}$. For example, if

$$x = ab, \quad \text{then } x^{(+)} = aba.$$

It is known that

$$x^{(+)} = x'y\widetilde{x'},$$

where $x = x'y$ with y the longest palindrome suffix of x. We consider the iterated palindromic closure (also introduced in [dL]), denoted by $\mathrm{Pal}(d)$: it is a mapping from the free monoid on $\mathcal{A}$ into itself, defined recursively by

$$\mathrm{Pal}(d_1d_2\dots d_n) = \big(\mathrm{Pal}(d_1d_2\dots d_{n-1})d_n\big)^{(+)}, \quad d_i \in \mathcal{A},$$

with the initial condition $\mathrm{Pal}(1) = 1$, where 1 denotes the empty word. This mapping is injective and w is called the *directive word* of $\mathrm{Pal}(w)$. For example, $\mathrm{Pal}(aba) = abaaba$: indeed,

$$\mathrm{Pal}(a) = a \quad \text{and} \quad \mathrm{Pal}(ab) = \big(\mathrm{Pal}(a)b\big)^{(+)} = (ab)^{(+)} = aba$$

and then

$$\mathrm{Pal}(aba) = \big(\mathrm{Pal}(ab)a\big)^{(+)} = (abaa)^{(+)} = abaaba.$$

We also use the *Thue-Morse substitution*, denoted by $\theta = (ab, ba)$: it is an endomorphism of the free monoid $\{a,b\}^*$ that maps the letter a to ab and the letter b to ba.

3. Main theorem

From now on, we work with the binary alphabet $\mathcal{A} = \{a, b\}$. We give a link between the computation of Markoff numbers and the length of words computed by iterated palindromic closure and Thue-Morse substitution:

Theorem 1. *For each word $v \in \{a, b\}^*$, the number $|\mathrm{Pal} \circ \theta \circ \mathrm{Pal}(\mathrm{av})| + 2$ is a Markoff number $\neq 1, 2$. The mapping defined in this way from $\{a, b\}^*$ into the set of Markoff numbers different from $1, 2$ is surjective. Injectivity of this mapping is equivalent to the Frobenius conjecture.*

Remark. If v' is obtained from v by interchanging a and b, one finds that $\mathrm{Pal} \circ \theta \circ \mathrm{Pal}(av)$ and $\mathrm{Pal} \circ \theta \circ \mathrm{Pal}(bv')$ have the same length. In other words, since the roles of a and b are symmetric, starting the word with b would give exactly symmetric words of the same length, so that we can consider the word av without loss of generality.

Examples. The first Markoff numbers (not equal to 1 or 2) are 5, 13 and 29. The Markoff number $m = 5$ is given by $v = 1$: indeed, $\mathrm{Pal}(a) = a$, thus $\theta \circ \mathrm{Pal}(a) = ab$ and then $\mathrm{Pal} \circ \theta \circ \mathrm{Pal}(a) = aba$, which is of length 3.

The Markoff number $m = 13$ is given by $v = a$: indeed, $\mathrm{Pal}(aa) = aa$, thus $\theta \circ \mathrm{Pal}(aa) = abab$ and then $\mathrm{Pal} \circ \theta \circ \mathrm{Pal}(aa) = abaababaaba$, which is of length 11.

The Markoff number $m = 29$ is given by $v = b$ indeed, $\mathrm{Pal}(ab) = aba$, thus $\theta \circ \mathrm{Pal}(aba) = abbaab$ and then $\mathrm{Pal} \circ \theta \circ \mathrm{Pal}(ab) = ababaababaabababaababaababa$, which is of length 27.

Proof. Define the monoid homomorphism μ from the free monoid $\{a, b\}^*$ into $\mathrm{SL}_2(\mathbb{Z})$ by

$$\mu(a) = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad \mu(b) = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}.$$

It is known that $\mu(u)_{12}$ is a Markoff number for each word lower Christoffel word u, and that each Markoff number m is equal to $\mu(u)_{12}$ for some lower Christoffel word u, see [BLRS, Th. 8.10]. Moreover, the uniqueness of u is equivalent to the Frobenius conjecture.

If $m \neq 1, 2$, then $u \neq a, b$; in this case $u = apb$, and it is known that $p = \mathrm{Pal}(v)$ for some word v in $\{a, b\}^*$; moreover, the mapping $v \mapsto a\mathrm{Pal}(v)b$ is a bijection from $\{a, b\}^*$ onto the set of proper lower Christoffel words (this well-known result follows for example from [BdLR, Corollary 3.1]).

Consider the monoid homomorphism α from the free monoid $\{a, b\}^*$ into $\mathrm{SL}_2(\mathbb{Z})$ defined by

$$\alpha(a) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad \alpha(b) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

We have
$$\alpha(ab) = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = \mu(a), \qquad \alpha(aabb) = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix} = \mu(b).$$
Consider $\phi = (ab, aabb)$. Then $\mu = \alpha\phi$.

Note that for each word m, $b\phi(m) = \psi(m)b$, where $\psi = (ba, baab) = (ba, ab)G$ with $G = (a, ab)$. Using [BdLR, Corollary 3.2], we see that the length of the Christoffel word $a\mathrm{Pal}(w)b$ is equal to $h+i+j+k$, where $\alpha(w) = \begin{pmatrix} h & i \\ j & k \end{pmatrix}$.

The word w is defined as follows: we have $\phi(u) = \phi(apb) = ab\phi(p)aabb$ and we define $w = b\phi(p)a$. Thus $\phi(u) = awabb$. Then we have
$$\begin{aligned} \mu(u) = \alpha\phi(u) = \alpha(awabb) &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} h & i \\ j & k \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \\ &= \begin{pmatrix} * & h+i+j+k \\ * & * \end{pmatrix} \end{aligned}$$
and therefore $m = \mu(u)_{12} = h+i+j+k = |a\mathrm{Pal}(w)b|$.

Furthermore,
$$w = b\phi(p)a = \psi(p)ba = \big((ba, ab)G(p)\big)\big((ba, ab)G(a)\big) = (ba, ab)G(pa).$$
Since $p = \mathrm{Pal}(v)$, we obtain
$$G(pa) = G(\mathrm{Pal}(v)a) = \mathrm{Pal}(av),$$
by Justin's formula [Be, J]. Thus $w = (ba, ab) \circ \mathrm{Pal}(av)$. The computation of m gives
$$\begin{aligned} m &= 2 + |\mathrm{Pal}(w)| \\ &= 2 + |\mathrm{Pal} \circ (ba, ab) \circ \mathrm{Pal}(av)| \\ &= 2 + |E \circ \mathrm{Pal} \circ (ba, ab) \circ \mathrm{Pal}(av)| \\ &= 2 + |\mathrm{Pal} \circ E \circ (ba, ab) \circ \mathrm{Pal}(av)| \\ &= 2 + |\mathrm{Pal} \circ (ab, ba) \circ \mathrm{Pal}(av)| \\ &= 2 + |\mathrm{Pal} \circ \theta \circ \mathrm{Pal}(av)|. \end{aligned}$$
□

A word z is an *antipalindrome* if it is equal to the exchange of its reversal (that is $z = E(\widetilde{z})$). For example, $z = aababb$ is an antipalindrome because its reversal is $\widetilde{z} = bbabaa$ and the exchange gives $E(\widetilde{z}) = aababb$.

As for the palindromic case, we use the antipalindromic closure and the iterated antipalindromic closure which are defined in the work of de Luca and De Luca [dLDL]. The *antipalindromic closure* of a word x is the shortest antipalindrome having x as a prefix; it is denoted by $x^{\oplus}$. For example, if $x = ab$, then $x^{\oplus} = ab$ because ab is already an antipalindrome and if $x = aa$, then $x^{\oplus} = aabb$. The iterated antipalindromic closure noted $\mathrm{AntiPal}(d)$ is defined by the recursive formula $\mathrm{AntiPal}(d_1 d_2 \cdots d_n) = (\mathrm{AntiPal}(d_1 d_2 \cdots d_{n-1}) d_n)^{\oplus}$ and the initial condition $\mathrm{AntiPal}(1) = 1$. For example, $\mathrm{AntiPal}(aba) = abbaababbaab$; indeed, $\mathrm{AntiPal}(a) = ab$, thus $\mathrm{AntiPal}(ab) = (\mathrm{AntiPal}(a)b)^{\oplus} = (abb)^{\oplus} = abbaab$ and then $\mathrm{AntiPal}(aba) = (\mathrm{AntiPal}(ab)a)^{\oplus} = (abbaaba)^{\oplus} = abbaababbaab$. We see that

$$\mathrm{AntiPal}(aba) = ab \cdot ba \cdot ab \cdot ab \cdot ba \cdot ab = (ab, ba) \circ abaaba = (ab, ba) \circ \mathrm{Pal}(aba).$$

This is a general fact, as shown in [dLDL] Theorem 7.6.

Theorem 2 (de Luca, De Luca)**.** *Let v be a word on the alphabet $\mathcal{A} = \{a, b\}$ and $\theta = (ab, ba)$ be the Thue-Morse substitution. Then*

$$\mathrm{AntiPal}(v) = \theta \circ \mathrm{Pal}(v).$$

Corollary 3. *For each word $v \in \{a, b\}^*$, the number $|\mathring{Pal} \circ \mathrm{AntiPal}(av)| + 2$ is a Markoff number $\neq 1, 2$. The mapping defined in this way from $\{a, b\}^*$ into the set of Markoff numbers different from $1, 2$ is surjective. Injectivity of this mapping is equivalent to the Frobenius conjecture.*

4. Computation of Markoff numbers

The previous corollary gives a new way to compute the Markoff numbers by using iterated antipalindromic closures and iterated palindromic closures. We now give a recursive formula for computing the Markoff numbers.

Theorem 4. *Consider $d = \mathrm{AntiPal}(av)$ with $v \in \{a, b\}^*$. We write $d = d_1 d_2 \ldots d_{|d|}$ with $d_i \in \{a, b\}$. We let $L_0 = L_1 = 1$ and $L_2 = L_1 + L_0 = 2$. For $j \geq 3$ we define recursively the L_j:*

$$L_j = \begin{cases} L_{j-1} & \text{if} \quad d_j = d_{j-1}, \\ L_{j-1} + L_{j-2} & \text{if} \quad d_j \neq d_{j-1} \neq d_{j-2}, \\ L_{j-1} + L_{j-2} + L_{j-3} & \text{if} \quad d_j \neq d_{j-1} = d_{j-2}. \end{cases}$$

Then the Markoff number m_v is given by

$$m_v = 1 + \sum_{j=0}^{|d|} L_j.$$

Consider the example $v = ab$.
We have $d = \mathrm{AntiPal}(aab) = \theta\big(\mathrm{Pal}(aab)\big) = \theta(aabaa) = ababbaabab$ and then $L_0 = 1$; $L_1 = 1$; $L_2 = L_1 + L_0 = 1 + 1 = 2$; $L_3 = L_2 + L_1 = 2 + 1 = 3$ (because $d_3 = a \neq d_2 = b \neq d_1 = a$); $L_4 = L_3 + L_2 = 3 + 2 = 5$

(because $d_4 = b \neq d_3 = a \neq d_2 = b$); $L_5 = L_4$ (because $d_5 = d_4 = b$). $L_6 = L_5 + L_4 + L_3 = 5 + 5 + 3 = 13$ (because $d_6 = a \neq d_5 = d_4 = b$); and so on. $L_7 = 13$; $L_8 = 31$; $L_9 = 44$; $L_{10} = 75$. Thus the sum of the L_j is 193 and if we add 1, we find the Markoff number 194.

A more compact way of writing the L_i's is to write d and above each letter the L_i :

$$\begin{array}{ccccccccccc} d_v = & a & b & a & b & b & a & a & b & a & b \\ & 1 & 1 & 2 & 3 & 5 & 5 & 13 & 13 & 31 & 44 & 75 \end{array}$$

P r o o f. To prove the theorem, we use Justin's Formula [J, Be]

$$\mathrm{Pal}(d'd'') = \psi_{d'}(d'') \cdot \mathrm{Pal}(d')$$

with d' a word on $\{a,b\}^*$ and d'' a letter.

We recall that

$$\psi_{d'}(a) = \psi_{d'_1}\Big(\psi_{d'_2}\big(\cdots\big(\psi_{d'_{|d'|}}(a)\big)\big)\Big) \quad \text{with } \psi_a(a) = a, \ \ \psi_a(b) = ab$$

(ψ_a was previously denoted G) and $\psi_b(a) = ba, \psi_b(b) = b$.

In our construction, we use $d = \mathrm{AntiPal}(av)$ with $v \in \{a,b\}^*$ and we have to study $\mathrm{Pal}(d) = \mathrm{Pal}(d_1 d_2 \ldots d_{|d|-1} d_{|d|})$. By successive applications of Justin's Formula we find

$$\mathrm{Pal}(d_1 d_2 \ldots d_{|d|}) = \psi_{d_1 d_2 \ldots d_{|d|-1}}(d_{|d|}) \cdot \mathrm{Pal}(d_1 d_2 \ldots d_{|d|-1}),$$
$$\mathrm{Pal}(d) = \mathrm{Pal}(d_1 d_2 \ldots d_{|d|}) = \psi_{d_1 d_2 \ldots d_{|d|-1}}(d_{|d|}) \cdots \psi_{d_1 d_2}(d_3) \cdot \psi_{d_1}(d_2) \cdot d_1.$$

We define

$$W_j = \psi_{d_1 d_2 \ldots d_{j-1}}(d_j)$$

and

$$L'_j = |W_j| \qquad \text{for } j = 1, \ldots, |d|.$$

Thus we have $W_1 = d_1$, $W_2 = \psi_{d_1}(d_2), \ldots,$ $W_{|d|} = \psi_{d_1 d_2 \ldots d_{|d|-1}}(d_{|d|})$ and the L'_j are the length of each W_j. We will prove that the L'_i's satisfy the same recursive formula as the L_i's; thus it will follow that $L_j = L'_j$. The recursive formula for the L'_j's is constructed on the prefixes of d.

We investigate the base cases. Note that d begins by ab: indeed,

$$\theta\big(\mathrm{Pal}(av)\big) = \theta(av') = abd''.$$

For the prefix of length one of d we find $W_1 = a$, in accordance with the base case $L'_1 = |W_1| = 1$. For a prefix of length two of d we have $W_2 = \psi_a(b) = ab$, and thus $L'_2 = |ab| = 2$.

Now we compute the recursive formula for prefixes of length at least three of d. We have six cases to consider, indeed, we use the directive word $d = (ab, ba) \circ \mathrm{Pal}(av)$ and thus aaa and bbb are forbidden words in the directive word d. It is sufficient to use Justin's formula for the following prefixes of d:

$$d'aba, \quad d'bab, \quad d'bba, \quad d'aab, \quad d'baa \quad \text{and} \quad d'abb \quad \text{with} \quad d' \in \{a,b\}^*.$$

The first case of the recursive formula is given by the prefixes of d of the form $d'baa$. We write $d = d_1 d_2 \dots d_{j-1} d_j = d'baa$ for a given j and we are in the case $d_j = d_{j-1} = a$. Thus we have by the definition

$$W_j = \psi_{d'ba}(a) \quad \text{and} \quad W_{j-1} = \psi_{d'b}(a).$$

We have

$$W_j = \psi_{d'ba}(a) = \psi_{d'b}\big(\psi_a(a)\big) = \psi_{d'b}(a) = W_{j-1}, \quad \text{thus } W_j = W_{j-1}.$$

We find $L'_j = L'_{j-1}$ for $d_j = d_{j-1} = a$. Similarly, by exchanging the roles of a and b that is by considering the prefixes of the form $d'abb$ we find $W_j = W_{j-1}$; thus $L'_j = L'_{j-1}$ for $d_j = d_{j-1} = b$.

The second case of the recursive formula is given by the prefixes of d of the form $d'aba$. We write $d_1 d_2 \dots d_{j-1} d_j = d'aba$ for a some j and we are in the case $d_j = a \neq d_{j-1} = b \neq d_{j-2} = a$. By the definition

$$W_j = \psi_{d'ab}(a) \quad \text{and} \quad W_{j-1} = \psi_{d'a}(b) \quad \text{and} \quad W_{j-2} = \psi_{d'}(a),$$

we have

$$\begin{aligned} W_j &= \psi_{d'ab}(a) = \psi_{d'a}\big(\psi_b(a)\big) = \psi_{d'a}(ba) \\ &= \psi_{d'a}(b) \cdot \psi_{d'a}(a) \\ &= \psi_{d'a}(b) \cdot \psi_{d'}\big(\psi_a(a)\big) \\ &= \psi_{d'a}(b) \cdot \psi_{d'}(a) \\ &= W_{j-1} \cdot W_{j-2}. \end{aligned}$$

Thus we have

$$W_j = W_{j-1} \cdot W_{j-2} \quad \text{and} \quad L'_j = L'_{j-1} + L'_{j-2}$$

for $d_j = a \neq d_{j-1} = b \neq d_{j-2} = a$. Similarly, for the prefixes of the form $d'bab$ by exchanging the roles of a and b we have

$$W_j = \psi_{d'ba}(b) = W_{j-1} W_{j-2} \quad \text{and thus} \quad L'_j = L'_{j-1} + L'_{j-2}$$

for $d_j = b \neq d_{j-1} = a \neq d_{j-2} = b$.

The third case is given by the prefixes of d of the form $d'''bba$. As bbb is forbidden in d, thus we write

$$d_1 d_2 \dots d_{j-1} d_j = d'abba \quad \text{and we have} \quad d_j = a \neq d_{j-1} = d_{j-2} = b.$$

By the definition

$$W_j = \psi_{d'abb}(a), \quad W_{j-1} = \psi_{d'ab}(b), \quad W_{j-2} = \psi_{d'a}(b) \quad \text{and} \quad W_{j-3} = \psi_{d'}(a)$$

for a given j we have

$$\begin{aligned} W_j &= \psi_{d'abb}(a) = \psi_{d'a}\Big(\psi_b\big(\psi_b(a)\big)\Big) = \psi_{d'a}\big(\psi_b(ba)\big) = \psi_{d'a}(bba) \\ &= \psi_{d'a}(b)\cdot\psi_{d'a}(b)\cdot\psi_{d'a}(a) = \psi_{d'ab}(b)\cdot\psi_{d'a}(b)\cdot\psi_{d'a}(a) \\ &= \psi_{d'ab}(b)\cdot\psi_{d'a}(b)\cdot\psi_{d'}(a) \\ &= W_{j-1}\cdot W_{j-2}\cdot W_{j-3} \end{aligned}$$

Thus we have

$$W_j = \psi_{d'abb}(a) = W_{j-1}\cdot W_{j-2}\cdot W_{j-3}$$

and thus

$$L'_j = L'_{j-1} + L'_{j-2} + L'_{j-3} \quad \text{for } d_j = a \neq d_{j-1} = d_{j-2} = b,$$

And similarly, for the prefixes of the form $d'baab$ we find

$$W_j = \psi_{d'baa}(b), = W_{j-1}\cdot W_{j-2}\cdot W_{j-3}$$

and

$$L'_j = L'_{j-1} + L'_{j-2} + L'_{j-3} \quad \text{for } d_j = b \neq d_{j-1} = d_{j-2} = a.$$

Finally, we have to compute the Markoff numbers by using Corollary 3:

$$m = |\mathrm{Pal}(d)| + 2 \quad \text{with} \quad d = \mathrm{AntiPal}(av).$$

Thus by Justin's Formula

$$\begin{aligned} m &= |\psi_{d_1 d_2 \cdots d_{|d|-1}}(d_{|d|}) \cdots \psi_{d_1 d_2}(d_3)\cdot\psi_{d_1}(d_2)\cdot d_1| + 2 \\ &= |W_{|d|}| + |W_{|d|-1}| + \cdots + |W_2| + |W_1| + 2 \\ &= 2 + \sum_{j=1}^{|d|} L_j = 1 + \sum_{j=0}^{|d|} L_j. \end{aligned}$$

□

Note that in the second case of the recursive formula, we have a Fibonacci recurrence

$$L_j = L_{j-1} + L_{j-2}.$$

In the third case of the recursive formula we have

$$L_j = L_{j-1} + L_{j-2} + L_{j-3} \quad \text{and} \quad d_{j-1} = d_{j-2}.$$

By application of the first case of the recursive formula we find

$$L_{j-1} = L_{j-2}$$

and then a Pell recurrence

$$L_j = 2L_{j-2} + L_{j-3}.$$

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