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UPPER BOUNDS FOR DOUBLE EXPONENTIAL SUMS ALONG A SUBSEQUENCE

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ABSTRACT. We consider a class of double exponential sums studied in a paper of Sinai and Ulcigrai. They proved a linear bound for these sums along the sequence of denominators in the continued fraction expansion of α, provided α is badly-approximable. We provide a proof of a result, which includes a simple proof of their theorem, and which applies for all irrational α.

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1. Introduction

1.1. Some notation

Let $\alpha = [a_0; a_1, \ldots]$ denote the continued fraction expansion of $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. We write $||x||$ for the distance from x to the nearest integer. The *convergents* $p_n/q_n = [a_0; a_1, \ldots, a_n]$, where $(p_n, q_n) = 1$, give good approximations to α. We call $\{q_n\}_{n\in\mathbb{N}}$ the *sequence of denominators* of α. We say that an irrational number α is *badly-approximable* if there exists $\varepsilon_\alpha > 0$ such that for all $p, q \in \mathbb{Z}$, $(p, q) = 1$, we have

$$\left|\alpha - \frac{p}{q}\right| > \frac{\varepsilon_\alpha}{q^2}.$$

These correspond precisely with those numbers α for which there exists $N \in \mathbb{N}$ such that, $a_n(\alpha) \le N$ for all $n \in \mathbb{N}$. The set of all badly-approximable numbers is a set of Lebesgue measure zero.

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When α is badly approximable, we have the helpful bound that

$$||q_n\alpha|| > \frac{\varepsilon_\alpha}{q_n}.$$

Since convergents give the best approximations for the distance to the nearest integer (see [7]), this means that for $m \le q_{n+1} - 1$ we have the bound

$$||m\alpha|| > \frac{\varepsilon_\alpha}{q_n}.$$

We write $f(n) = O(g(n))$ to mean that there exists a constant C, (which does not depend on n), such that

$$f(n) \le C \cdot g(n) \quad \text{for all } \ n \in \mathbb{N}.$$

Finally, we define the discrepancy of a sequence.

Definition 1.1. Let (x_n) be a sequence of real numbers. For $N \in \mathbb{N}$ the *discrepancy* of (x_n) modulo one, $D_N(x_n)$, is defined as:

$$D_N(\{x_m\}) := \sup_{I \subseteq \mathbb{R}/\mathbb{Z}} \left| \sum_{m=1}^{N} \chi_I(x_m) - N \cdot |I| \right|,$$

where I denotes an interval and χ_I is the characteristic function of I.

1.2. Double exponential sums

In [9] Sinai and Ulcgrai studied double trigonometric sums of the form

$$T_M(\alpha) = \frac{1}{M} \sum_{m=0}^{M-1} \sum_{n=0}^{M-1} e(nm\alpha). \tag{1.2.1}$$

We want to determine when the absolute value of this sum is bounded uniformly (i.e., by a constant which depends only on α) over some subsequence

$$M \in \mathscr{A} \subseteq \mathbb{N}.$$

This will, obviously, depend on the Diophantine properties of α and the subsequence $\mathscr{A}$.

We will see that the problem of bounding this sum depends importantly on controlling sums such as

$$\left| \frac{1}{M} \sum_{m=1}^{M} \frac{1}{\{\{m\alpha\}\}} \right|. \tag{1.2.2}$$

Here

$$\{\{x\}\} := \begin{cases} \{x\}, & x \in [0, \frac{1}{2}], \\ \{x\} - 1, & x \in (-\frac{1}{2}, 0), \end{cases}$$

is the *signed fractional part* of $x \in \mathbb{R}$.

In [9] the following is proved

Theorem 1.2 (Sinai, Ulcigrai). *Let α be badly-approximable. Consider the following double trigonometric sum:*

$$T_M(\alpha) := \frac{1}{M} \sum_{m=0}^{M-1} \sum_{n=0}^{M-1} e(nm\alpha).$$

Then there exists a constant

$$C = C(\alpha) > 0$$

such that

$$|T_M| \leq C_\alpha \quad \textit{for all} \quad M \in \{q_n\}_{n \in \mathbb{N}}.$$

Remark 1.3. The sum here is an example of a 2-dimensional finite theta sum. In [2] Cosentino and Flaminio prove bounds for far more general g--dimensional finite theta sums. A special case of one of their results implies that the above theorem is true for all $M \in \mathbb{N}$.

Our main theorem generalises Theorem 1.2.

Theorem 1.4. *Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. Then there exists a constant*

$$C = C(\alpha) > 0$$

such that

$$|T_{q_n}| \leq C_\alpha \cdot \max\left\{ \frac{\log(2 \cdot \max_{i \leq n}\{a_i\})}{a_{n+1}}, 1 \right\} \quad \textit{for all } n \in \mathbb{N}. \tag{1.2.3}$$

Remark 1.5. By examining signs it appears that the upper bound here is close to best possible. Equation (1.13) in [1] gives a lower bound for the largest terms in a sum that we will consider. While it is true that we use the triangle inequality earlier in our calculation, it does not make our estimate so much larger.

2. Proof of main result

2.1. Reducing T_M

Following the methods in [9], we split T_M into two separate sums.

By summing the terms for $n = 0, \dots, M-1$, we can rewrite (1.2.1) as

$$T_M = 1 + \frac{1}{M} \sum_{m=1}^{M-1} \frac{e(Mm\alpha) - 1}{e(m\alpha) - 1}.$$

Then we can write $T_M = 1 + S'_M - S''_M$, where

$$S'_M := \frac{1}{M} \sum_{m=1}^{M-1} \frac{e(Mm\alpha)}{e(m\alpha) - 1} \tag{2.1.1}$$

and

$$S''_M := \frac{1}{M} \sum_{m=1}^{M-1} \frac{1}{e(m\alpha) - 1}. \tag{2.1.2}$$

We will prove that there exist constants $C', C'' \in \mathbb{R}$ such that

$$|S'_{q_n}| \leq C' \cdot \max\left\{ \frac{\log(2 \cdot \max_{i \leq n}\{a_i\})}{a_{n+1}}, 1 \right\} \tag{2.1.3}$$

and

$$|S''_{q_n}| \leq C'' \quad \text{for all } \; n \in \mathbb{N}.$$

These constants will depend only on α.

2.2. The sum S''_M (2.1.2)

Let us consider the '*less intimidating*' sum first. We want to show that there exists a $C'' \in \mathbb{R}$ such that $|S''_{q_n}| \leq C''$ for all $n \in \mathbb{N}$.

Note that in [3], Hardy and Littlewood prove a similar theorem.

Theorem 2.1 (Hardy, Littlewood). *Let α be badly-approximable. Then there exists $C^* > 0$ such that $|S''_M| \leq C^*$ for each $M \in \mathbb{N}^+$.*

We proceed by calculating real and imaginary parts.

$$\frac{1}{e(m\alpha) - 1} = -\frac{1}{2} - \frac{i}{2} \cot(\pi m \alpha).$$

The Taylor series expansion of $\cot x$ is

$$\cot x = \sum_{n=0}^{\infty} \frac{(-1)^n 4^n B_{2n}}{(2n)!} x^{2n-1} = \frac{1}{x} - \frac{x}{3} + \frac{x^3}{45} - \cdots$$

with radius of convergence $0 < |x| < \pi$. Here B_n is the nth *Bernoulli number*.

Note that due to the symmetry of $\cot x$,

$$\cot(\pi m\alpha) = \cot(\pi\{\{m\alpha\}\}).$$

So we can write

$$\cot(\pi m\alpha) = \frac{1}{\pi\{\{m\alpha\}\}}\left(1 + \sum_{n=1}^{\infty}\frac{(-1)^n 4^n B_{2n}}{(2n)!}\big(\pi\{\{m\alpha\}\}\big)^{2n}\right).$$

Now the series on the right is negative and it takes values strictly between 0 (when $\{\{m\alpha\}\}$ is close to 0) and -1 (when $\{\{m\alpha\}\}$ is close to $\pm\frac{1}{2}$).

Hence, in order to prove that $|S''_{q_n}|$ is bounded by a uniform constant for all $n \in \mathbb{N}$, we have to prove the following:

Lemma 2.2. *Let $\alpha \in \mathbb{R}$. Then there exists $C = C(\alpha) > 0$ such that,*

$$\left|\sum_{m=1}^{q_n-1}\frac{1}{q_n\{\{m\alpha\}\}}\right| \le C \quad \text{for all } n \in \mathbb{N}. \tag{2.2.1}$$

We will consider two different proofs of Lemma 2.2. The first one is simpler, while the latter one will be applicable to estimating S'_M as well. The second proof is also malleable to proving Theorem 2.1.

2.3. Koksma-Hlawka Proof of Lemma 2.2

Recall the Koksma-Hlawka inequality.

Lemma 2.3. *Let f be a real function with period 1 of bounded variation. Then for every sequence $\{x_m\}$ and every integer $N \ge 1$,*

$$\left|\frac{1}{N}\sum_{m=1}^{N} f(x_m) - \int_0^1 f(x)\mathrm{d}x\right| \le V(f)\frac{D_N(x_m)}{N},$$

where $V(f)$ is the total variation of the function.

We wish to apply this inequality with

$$f(x) = \frac{1}{\{\{x\}\}}, \quad x_m = \{m\alpha\}, \quad \text{and} \quad N = q_n - 1.$$

Therefore we have to restrict the domain on which we define our function, in order to ensure that it is integrable.

We are able to use the following from [8]

$$\left|\alpha - \frac{p_{n-1}}{q_{n-1}}\right| > \frac{1}{2q_{n-1}q_n}. \tag{2.3.1}$$

So, for all $m \leq N = q_n - 1$, we have

$$||m\alpha|| > \frac{1}{2q_n}.$$

Hence we can restrict the domain of f to the interval $[\frac{1}{2q_n}, 1 - \frac{1}{2q_n}]$. Furthermore, since f is anti-symmetric about $1/2$, the integral of f over this interval is equal to 0. The total variation, $V(f)$, of f is

$$\sup_{\mathcal{P}} \sum_{i=1}^{n_p} \left| \frac{1}{\{\{x_{i+1}\}\}} - \frac{1}{\{\{x_i\}\}} \right|,$$

where $\mathcal{P}$ is a partition of $[\frac{1}{2q_n}, 1 - \frac{1}{2q_n}]$. As f is monotone in this interval, the total variation is maximised when we take the trivial partition (that is the two endpoints). Therefore $V(f) = 4q_n$.

Finally, we move on to considering the Discrepancy.

Lemma 5.6 from [5] states that

$$D_N(m\alpha) \leq 3 \sum_{j=0}^{r} t_j,$$

where

$$N = \sum_{j=0}^{r} q_j t_j$$

is the *Ostrowski expansion* of N. This is defined in the next subsection (see Definition 2.5) but all we need to know here is that if $N = q_n$, then $t_n = 1$ and $t_i = 0$ for all $i \neq n$. So $D_{q_n}(m\alpha) \leq 3$.

Finally, we can apply all the estimates we have (with $N = q_n - 1$ and f & $\{x_m\}$ as above.)

$$\begin{aligned} \left| \sum_{m=1}^{q_n - 1} f(x_m) - (q_n - 1) \int_0^1 f(x)\,\mathrm{dx} \right| &\leq D_{q_n - 1}(x_m) V(f) \\ &\leq \big(D_{q_n}(x_m) + 1\big) V(f) \\ &\leq 4 \cdot 4q_n = 16q_n \end{aligned}$$

Hence,

$$\left| \frac{1}{q_n} \sum_{m=1}^{q_n - 1} \frac{1}{\{\{m\alpha\}\}} \right| \leq 16.$$

Here we used the obvious fact that $D_M(x_m) \leq D_{M+1}(x_m) + 1$.

2.4. The sum $S_M^{'}$ (2.1.1)

We move on to considering the sum

$$S'_M := \frac{1}{M} \sum_{m=1}^{M-1} \frac{e(Mm\alpha)}{e(m\alpha) - 1}.$$

We will write this sum as a telescoping series and then take advantage of some cancellation to reduce our situation to considering the sum $S_M^{''}$ (2.1.2). Firstly,

$$\sum_{m=1}^{M-1} \frac{e(Mm\alpha)}{e(m\alpha)-1} = \sum_{m=1}^{M-1} \Big(e(Mm\alpha) - e\big(M(m+1)\alpha\big)\Big) \sum_{k=1}^{m} \frac{1}{e(k\alpha)-1} \tag{2.4.1}$$

$$+ e(M^2\alpha) \sum_{k=1}^{M-1} \frac{1}{e(k\alpha)-1}. \tag{2.4.2}$$

We then consider the outer part of the sum on the right hand side of (2.4.1) (for $M = q_n$),

$$\begin{aligned} e(mq_n\alpha) - e\big((m+1)q_n\alpha\big) &= e(mq_n\alpha) - e\big((m+1)q_n\alpha\big) \\ &= e(mq_n\alpha) - e(mq_n\alpha)e(q_n\alpha) \\ &= \big(1 - e(q_n\alpha)\big)e(mq_n\alpha). \end{aligned}$$

In absolute value this is less than $2\pi/q_{n+1}$.

Now using the triangle inequality and Lemma 2.2, we see that (2.1.3) results from the following lemma.

Lemma 2.4. *For all $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and for all $m \leq q_n - 1$,*

$$\sum_{k=1}^{m} \frac{1}{\{\{k\alpha\}\}} = O\left(q_n \cdot \max_{i \leq n}\{1, \log a_i\}\right).$$

To prove this Lemma we will need to introduce some different techniques, which will also yield a new proof of Lemma 2.2.

2.5. The Ostrowski proof of Lemmas 2.2 and 2.4

Our alternative proof of Lemma 2.2 will involve decomposing the sum in (2.2.1) into segments where there is some obvious cancellation.

Definition 2.5. Let α be irrational. Then for every $n \in \mathbb{N}$ there exists a unique integer $M \geq 0$ and a unique sequence $\{c_{k+1}\}_{k=0}^{\infty}$ of integers such that

$$q_M \leq m < q_{M+1} \quad \text{and} \quad m = \sum_{k=0}^{\infty} c_{k+1} q_k,$$

with

$$\begin{array}{lllll} 0 \le c_1 < a_1, & 0 \le c_{k+1} \le a_{k+1} & \text{for} & k \ge 1, \\ c_k = 0 \quad \text{whenever} & c_{k+1} = a_{k+1} & \text{for some} & k \ge 1, \end{array}$$

and

$$c_{k+1} = 0 \quad \text{for} \quad k > M.$$

This is known as the *Ostrowski expansion*.

We will consider segments of our sum which 'spread out' in the unit interval. We take our inspiration from a set of intervals discussed in [6].

Definition 2.6 (Special intervals)**.** For fixed α define $\mathcal{A}(i)$ to be the collection of non-negative integers n with Ostrowski expansions of the form

$$n = \sum_{k=i}^{\infty} c_{k+1} q_k.$$

Then for each $i \in \mathbb{N}$ and for each $\gamma \in \mathbb{R}/\mathbb{Z}$ we define a subset $\mathcal{J}(i, \gamma)$ (which turns out to be an interval, see [6]) of $\mathbb{R}/\mathbb{Z}$ by

$$\mathcal{J}(i, \gamma) = \gamma + \overline{\{n\alpha : n \in \mathcal{A}(i)\}}.$$

These intervals have some very nice properties such as

$$\sup_{N \in \mathbb{N}} \sup_{\mathcal{J} \subseteq \mathbb{R}/\mathbb{Z}} \left| \sum_{n=1}^{N} \chi_{\mathcal{J}}(n\alpha) - N \cdot |\mathcal{J}| \right| \le K,$$

where K is a universal constant and the inner supremum is taken over all special intervals $\mathcal{J}$ for α. We will use what these intervals tell us about the distribution of $n\alpha$ on the unit interval to achieve cancellation in (2.2.1).

Let

$$m = \sum_{i=0}^{n-1} c_{i+1} q_i \le q_n - 1, \quad 0 \le c_{i+1} \le a_{i+1}$$

and

$$n(i, c) := \sum_{j=0}^{i-1} c_{j+1} q_j + c q_i.$$

We will use this decomposition to sum up to m.

$$\sum_{k=1}^{m} \frac{1}{\{\{k\alpha\}\}} = \sum_{i=0}^{n-1} \sum_{c=0}^{c_{i+1}-1} \sum_{l=n(i,c)+1}^{n(i,c)+q_i} \frac{1}{\{\{l\alpha\}\}}.$$

Note that

$$n(i, c) + q_i = n(i, c+1) \quad \text{and} \quad n(i, c_{i+1} - 1) + q_i = n(i+1, 0).$$

Let us consider a situation, where we are studying

$$\sum_{l=n(i,c)+1}^{n(i,c)+q_i} \frac{1}{\{\{l\alpha\}\}}. \tag{2.5.1}$$

We wish to approximate α by p_i/q_i and achieve (almost) complete cancellation in the main term that we get.

Obviously, problems can occur. Specifically, if $l \cdot p_i \equiv 0(q_i)$, then we do not want to divide by 0, so we want to isolate these terms and deal with them separately. Note that since $(p_i, q_i) = 1$, we have a complete set of residue classes modulo q_i, so in each sum (2.5.1) we will have exactly one term, $l = (c+1)q_i$, where this happens. Also, there exists $r \leq q_i$ such that

$$\begin{aligned} n(i,0) + r &= q_i, \\ n(i,1) + r &= 2q_i, \\ &\vdots \\ n(i, c_{i+1}-1) + r &= c_{i+1}q_i. \end{aligned}$$

So we can consider all of these terms separately.

Finally, we consider summing over a complete set of residue classes modulo q_i. We will first consider the simple case, $(1 \leq k \leq q_{i-1})$, which will give us a second proof of Lemma 2.2. We write

$$\alpha = \frac{p_i}{q_i} + \frac{\xi_i}{q_i q_{i+1}}, \quad \text{where} \quad \frac{1}{2} < |\xi_i| < 1.$$

Now

$$\sum_{k=1}^{q_i - 1} \frac{1}{\{\{k\alpha\}\}} = \sum_{k=1}^{q_i - 1} \frac{1}{\{\{k\frac{p_i}{q_i} + \frac{k\xi_i}{q_i q_{i+1}}\}\}}. \tag{2.5.2}$$

Now we use the fact that

$$\left\{\left\{\frac{kp_i}{q_i} + \frac{k\xi_i}{q_i q_{i+1}}\right\}\right\} = \left\{\left\{\frac{kp_i}{q_i}\right\}\right\} + \left\{\left\{\frac{k\xi_i}{q_i q_{i+1}}\right\}\right\},$$

unless perhaps if $kp_i \equiv \frac{q_i}{2}$ modulo q_i (when $2|q_i$), or if $kp_i \equiv \frac{q_i \pm 1}{2}$ (when $2|q_i+1$). Now (2.5.2) equals[1]

$$\sum_{k=1}^{q_{i-1}}{}' \frac{1}{\{\{k\frac{p_i}{q_i}\}\}} \left(\frac{1}{1 + \{\{\frac{kp_i}{q_i}\}\}^{-1}\frac{k\xi_i}{q_i q_{i+1}}}\right) + O(1).$$

[1]The one or two extra term/s mentioned just above have been removed from the sum and are accounted for by the $O(1)$ term.

Furthermore,

$$\left(\frac{1}{1+\{\{\frac{kp_i}{q_i}\}\}^{-1}\frac{k\xi_i}{q_iq_{i+1}}}\right) = 1 - \left\{\left\{\frac{kp_i}{q_i}\right\}\right\}^{-1}\frac{k\xi_i}{q_iq_{i+1}} + \left\{\left\{\frac{kp_i}{q_i}\right\}\right\}^{-2}\left(\frac{k\xi_i}{q_iq_{i+1}}\right)^2 - \cdots$$

There exists n_k such that $1 \leq n_k \leq q_i - 1$ and $n_k \equiv kp_i \mod q_i$. Now we define $n_k^{'}$ as follows

$$n_k^{'} := \begin{cases} n_k, & n_k \leq \frac{q_i}{2}, \\ n_k - q_i, & n_k > \frac{q_i}{2}. \end{cases}$$

Then,

$$\left\{\left\{\frac{kp_i}{q_i}\right\}\right\}^{-1}\frac{k\xi_i}{q_iq_{i+1}} = \frac{k\xi_i}{n_k^{'}q_{i+1}}.$$

We then know that for all k,

$$\left\{\left\{\frac{kp_i}{q_i}\right\}\right\}^{-1}\frac{k\xi_i}{q_iq_{i+1}} + \left\{\left\{\frac{kp_i}{q_i}\right\}\right\}^{-2}\left(\frac{k\xi_i}{q_iq_{i+1}}\right)^2 - \cdots \ = C_k\frac{k\xi_i}{n_k^{'}q_{i+1}}.$$

We need $|n_k^{'}| \geq 2$ in order to have a uniform bound over k for the constant C_k. When this is the case

$$-2 < C_k < -\frac{1}{2}$$

(apart from the one or two exceptions mentioned previously.) So we have to isolate another two terms. We write k_1, k_{-1} for the numbers, where $k_1p_i \equiv 1 \mod q_i$ and $k_{-1}p_i \equiv -1 \mod q_i$, respectively. So (2.5.2) becomes

$$\sum_{n_k=2}^{q_i-2}\left(\frac{1}{\{\{\frac{n_k}{q_l}\}\}} + C_k\left(\frac{k\xi_iq_i}{(n_k^{'})^2q_{i+1}}\right)\right) + \frac{1}{\{\{k_1\alpha\}\}} + \frac{1}{\{\{k_{-1}\alpha\}\}} + O(1) = \sum_{n_k=2}^{q_i-2}C_k\left(\frac{k\xi_iq_i}{(n_k^{'})^2q_{i+1}}\right) + O(q_i).$$

(Here we used the basic approximation from Khinchin (2.3.1) to deal with the two extra terms.)

By the rearrangement inequality, (see [4], Theorem 368), this first sum is smaller (in modulus) than

$$4q_i\left(\frac{1}{2^2}+\frac{1}{3^2}+\cdots\right),$$

which in turn is bounded above by $4q_i$. So

$$\sum_{k=1}^{q_i-1}\frac{1}{\{\{k\alpha\}\}}=O(q_i),$$

as required. Now, we move on to a proof of Lemma 2.4. We wish to prove, (for all i), that

$$\sum_{c=0}^{c_{i+1}-1}\sum_{l=n(i,c)+1}^{n(i,c)+q_i}\frac{1}{\{\{l\alpha\}\}}=O(q_{i+1}\log c_{i+1}).$$

Note that if we sum

$$\sum_{l=n(i,c)+1}^{n(i,c)+q_i}\frac{1}{\{\{l\alpha\}\}},$$

then a similar argument to the proof of Lemma 2.2 shows that this is equal to

$$O(q_i)+\frac{1}{\{\{k_{(1,c)}\alpha\}\}}+\frac{1}{\{\{k_{(-1,c)}\alpha\}\}}+\frac{1}{\{\{(c+1)q_i\alpha\}\}},$$

where

$$n(i,c)+1\le k_{(\pm 1,c)}\le n(i,c)+q_i,\quad\text{and}\quad k_{(\pm 1,c)}p_i\equiv\pm 1\mod q_i.$$

Clearly,

$$k_{(\pm 1,c)}=k_{(\pm 1,0)}+cq_i.$$

Furthermore, as $n(i,0)<q_i$,

$$k_{(\pm 1,c_{i+1}-r)}<n\big(i,(c_{i+1}-r)\big)+q_i<q_{i+1}-(r-1)q_i.$$

Now, we calculated earlier that

$$\frac{1}{\{\{k\alpha\}\}}=\frac{1}{\{\{k\frac{p_i}{q_i}\}\}}\left(\frac{1}{1+\{\{\frac{kp_i}{q_i}\}\}^{-1}\frac{k\xi_i}{q_iq_{i+1}}}\right).$$

Letting $k=k_{(1,0)}$,

$$\begin{aligned}\frac{1}{\{\{k_{(1,0)}\alpha\}\}}&=q_i\left(\frac{1}{1+\frac{k_{(1,0)}\xi_i}{q_{i+1}}}\right)\\&=\frac{q_iq_{i+1}}{q_{i+1}+k_{(1,0)}\xi_i}.\end{aligned}$$

Hence,

$$\frac{1}{\{\{k_{(1,c)}\alpha\}\}} = \frac{q_i q_{i+1}}{q_{i+1} + (k_{(1,0)} + cq_i)\xi_i}$$

and also

$$\frac{1}{\{\{k_{(-1,c)}\alpha\}\}} = \frac{-q_i q_{i+1}}{q_{i+1} - (k_{(-1,0)} + cq_i)\xi_i}.$$

Now, without loss of generality, assume that $\xi_i > 0$. Then

$$\frac{1}{\{\{k_{(1,c)}\alpha\}\}} < q_i \quad \text{for all } \ c.$$

Hence,

$$\begin{aligned}\sum_{c=0}^{c_{i+1}-1} \sum_{l=n(i,c)+1}^{n(i,c)+q_i} \frac{1}{\{\{l\alpha\}\}} &= O(q_{i+1}) \ + \sum_{c=0}^{c_{i+1}-1} \frac{1}{\{\{(c+1)q_i\alpha\}\}} \\ &\qquad + \sum_{c=0}^{c_{i+1}-1} \frac{-q_i q_{i+1}}{q_{i+1} - (k_{(-1,0)} + cq_i)\xi_i} \\ &= O(q_{i+1}) + O(q_{i+1}\log c_{i+1}) \\ &\qquad + \sum_{c=0}^{c_{i+1}-1} \frac{-q_i q_{i+1}}{q_{i+1} - (k_{(-1,0)} + cq_i)\xi_i}.\end{aligned}$$

Finally, (using $k_{(-1,0)} \leq q_i + q_{i-1}$ and $\xi < 1$),

$$\begin{aligned}\left|\sum_{c=0}^{c_{i+1}-1} \frac{-q_i q_{i+1}}{q_{i+1} - (k_{(-1,0)} + cq_i)\xi_i}\right| &\leq \frac{q_{i+1}}{c_{i+1}-1} + \cdots + \frac{q_{i+1}}{2} + q_{i+1} + 2q_{i+1} \\ &= O(q_{i+1}\log c_{i+1}).\end{aligned}$$

As this is true for all i, the condition for Lemma 2.4 follows.

Remark 2.7. Equation (1.13) in [1] tells us that the sum

$$\sum_{c=0}^{c_{i+1}-1} \frac{1}{\{\{(c+1)q_i\alpha\}\}}$$

can be no smaller than $O(q_{i+1}\log c_{i+1})$.

Remark 2.8. In our final calculation we have ignored the cancellation between the positive and negative terms. However, when $c_{i+1} \approx a_{i+1}/2$, for example, we get very little cancellation and our main term is

$$O(q_{i+1} \log a_{i+1}).$$

Remark 2.9. In fact since convergents p_i/q_i give lower bounds for α when i is even this implies that ξ_i is positive when i is even (and negative when i is odd). Now from the well-known formula for continued fractions

$$p_{i-1}q_i - p_i q_{i-1} = (-1)^i,$$

we see that for the negative terms we separated,

$$k_{(-1,c)} = cq_i + q_{i-1}.$$

These correspond to the *semiconvergents*

$$\frac{cp_i + p_{i-1}}{cq_i + q_{i-1}}$$

in the continued fraction expansion of α.

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