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uniform distribution theory

DOI: 10.1515/udt-2017–0011
Unif. Distrib. Theory **12** (2017), no.2, 1–10

DISTRIBUTION FUNCTIONS FOR SUBSEQUENCES OF GENERALIZED VAN DER CORPUT SEQUENCES

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ABSTRACT. For an integer $b>1$ let $(\phi_b(n))_{n\geq 0}$ denote the van der Corput sequence base in b in $[0,1)$. Answering a question of O. Strauch, C. Aistleitner and M. Hofer showed that the distribution function of

$$\big(\phi_b(n),\phi_b(n+1),\dots,\phi_b(n+s-1)\big)_{n\geq 0} \quad \text{on} \quad [0,1)^s$$

exists and is a copula. The first and third authors of the present paper showed that this phenomenon extends to a broad class of subsequences of the van der Corput sequence. In this result we extend this paper still further and show that this phenomenon is also true for more general numeration systems based on the beta expansion of W. Parry and A. Rényi.

Communicated by Oto Strauch

1. Introduction

For an integer $b>1$ we define the Kakutani-Von Neumann odometer

$$T_b:[0,1)\to[0,1)$$

by $\quad T_b(x)=x-1+b^{-k}+b^{-k-1} \quad$ for $x\in\big[1-b^{-k},1-b^{-k-1}\big)$, $\quad (k=0,1,\dots)$. The name arises from the fact that T_b is a 'Euclidean model' for the map $\tau(x)=x+1$ on the ring of b-adic integers. The sequence $\big(T_b^n(0)\big)_{n\geq 0}$ is called the base b van der Corput sequence and the sequence $\big(T_b^n(x)\big)_{n\geq 0}$ for arbitrary x is called the generalized van der Corput sequence. For pairwise coprime

2010 Mathematics Subject Classification: 11K31, 40A05.
Keywords: Generalised van der Corput sequences, beta-expansions, Hartman distributed sequences of integers, distribution functions.

$b_1, \dots, b_s > 1$, we call $\left(T_{b_1}^n(0), \dots, T_{b_s}^n(0)\right)_{n \geq 0}$ the Halton sequence. All three of these sequences are examples of low discrepancy sequences valuable in numerical integration [DP]. Henceforth we will write $\phi_b(n) = T_b^n(0)$ $(n \geq 0)$. For a real number y let $\{y\}$ denote its fractional part. Recall that we say a sequence $(x_n)_{n \geq 1}$ is *uniformly distributed modulo one* if for each interval $I \subseteq [0,1)$ of length $|I|$ we have

$$\lim_{N \to \infty} \frac{1}{N} \#\{1 \leq n \leq N : x_n \in I\} = |I|.$$

Following I. Niven [KN, p. 305] we say a sequence of integers $(k_n)_{n \geq 0}$ is *uniformly distributed on* $\mathbb{Z}$ if for every integer $m > 1$ and every residue class $j \bmod m$ for $j \in [0, m-1)$, we have

$$\lim_{N \to \infty} \frac{1}{N} \#\{1 \leq n \leq N : k_n \equiv j \bmod m\} = \frac{1}{m}.$$

We call a sequence of integers $(k_n)_{n \geq 0}$ *Hartman uniformly distributed* if for each irrational number α the sequence $(\{k_n \alpha\})_{n \geq 0}$ is uniformly distributed modulo one and the sequence $(k_n)_{n \geq 0}$ is uniformly distributed on $\mathbb{Z}$ [KN, p. 296, Ex. 5.11]. The concept can be generalized to second countable groups, though this degree of generality doesn't play any further role in our considerations. See [KN] for more details. A list of examples is given in Section 3.

Let $(X, \mathcal{B}, \mu)$ be a probability space and let $T : X \to X$ be a measurable map, that is also measure-preserving. That is, given $A \in \mathcal{B}$, we have $\mu(T^{-1}A) = \mu(A)$, where $T^{-1}A$ denotes the set $\{x \in X : Tx \in A\}$. We call $(X, \mathcal{B}, \mu, T)$ a *dynamical system* . We say the dynamical system is *ergodic* if $T^{-1}A = A$ for $A \in \mathcal{B}$ means that either $\mu(A)$ or $\mu(X \backslash A)$ is 0.

We say $(k_n)_{n \geq 0}$ is L^p *good universal* if for each dynamical system $(X, \mathcal{B}, \mu, T)$ and for each $f \in L^p(X, \mathcal{B}, \mu)$ the limit

$$\ell_{T,f}(x) = \lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} f(T^{k_n} x),$$

exists μ almost everywhere.

Recall that a function $C : [0,1]^s \to [0,1]$ is called a *copula* if:

(i) for each s-tuple u in $[0,1]^s$, we have $C(u) = 0$ if any one of the coordinates of u is 0;

(ii) for each $x \in [0,1]$, we have $C(1, 1, \dots, 1, x, 1, 1, \dots, 1) = x$ and

(iii) for each $B \subseteq [0,1]^s$ which is a product of intervals contained in $[0,1]$, we have $\int_B dC \geq 0$.

In [LN] a result is proved which implies the following

Theorem A. *Suppose that $(k_n)_{n\geq 0}$ is Hartman uniformly distributed and L^p-good universal for some $p \in [1,2]$. Also suppose that $(n_1,\dots,n_s)$ is an s-tuple of non-negative integers and that $b>1$ is an integer. Then the asymptotic distribution function of the sequence*

$$\big(\phi_b(k_n+n_1),\dots,\phi_b(k_n+n_s)\big)_{n\geq 0}$$

exists and is a copula.

In the case $k_n = n$ $(n = 1,2\dots)$ with $n_i = i-1$ $(i = 1,2,\dots s)$ this result appears in [AH] in response to a question of O. Strauch. The special case $s = 2$ of the result from [AH] appears in [FS]. Conditions for subsequences of the Halton sequence $\big(\phi_{b_1}(n),\dots,\phi_{b_s}(n)\big)_{n\geq 0}$ to be uniformly distributed modulo one for pairwise coprime $b_1,\dots,b_s$ appear in [HN]. See also [HKLP] for related results.

We now describe the extension of Theorem A proved in this paper.

Let $G = (G_n)_{n\geq 0}$ be an increasing sequence of positive integers with $G_0 = 1$. Then every natural number n can be written in the form

$$n = \sum_{k=0}^{\infty} g_k(n)G_k,$$

where $g_k(n) \in \{0,\dots,[G_{k+1}/G_k]\}$ and $[x]$ denotes the integer part of x. This expansion (called the G-expansion) is unique and finite, provided that for every finite $K>0$, we have

$$\sum_{k=0}^{K-1} g_k(n)G_k < G_K. \tag{1.1}$$

We call g_k the kth digit of the G-expansion. The digits $(g_k)_{k\geq 0}$ can be calculated using the greedy algorithm and $G = (G_k)_{n\geq 0}$ is called an enumeration system. We denote by $\mathcal{K}_G$ the subset of sequences satisfying (2.1). The elements of $\mathcal{K}_G$ are called G-admissible. To extend the addition-by-1 map from $\mathbb{N}$ to $\mathcal{K}_G$ we introduce

$$\mathcal{K}_G^0 = \left\{x \in \mathcal{K}_G : \exists M_x, \forall j \geq M_x, \sum_{k=0}^{j} g_k(x)G_k < G_{j+1}-1\right\} \subseteq \mathcal{K}_G.$$

Put $x_j = \sum_{k=0}^{j} g_kG_k$ and set

$$\tau(x) = \big(g_0(x_j+1),\dots g_j(x_j+1), g_{j+1}(x), g_{j+2}(x),\dots\big),$$

for every $x \in \mathcal{K}_G^0$ and $j \geq M_x$. This definition does not depend on the choice of $j \geq M_x$ and can be extended to $\mathcal{K}_G \backslash \mathcal{K}_G^0$ by setting $\tau(x) = 0 = (0^\infty)$. We have defined the G-odometer or G-adding machine.

In the sequel we restrict our attention to enumeration systems, where $G = (G_n)_{n\geq 0}$ is a linear recurrence, i.e., we require in addition that for each positive integer n we have

$$G_{n+d} = a_0 G_{n+d-1} + \cdots + a_{d-1} G_n. \tag{1.2}$$

To this linear recurrence we can associate the characteristic equation

$$x^d = a_0 x^{d-1} + \cdots + a_{d-1}.$$

Recall that we say a real algebraic number greater than one is a *Pisot-Vijayaraghavan number* if all its Galois conjugates are less than one in absolute number. We further restrict our attention to enumeration systems, with a characteristic equation having a Pisot-Vijayaraghavan (PV) number as a root. Note that this is always the case when

$$a_0 \geq a_1 \geq \cdots \geq a_{d-1} \geq 1.$$

See [B] for the details of this. W. Parry [P] showed that, under this hypothesis, if β is this PV root of this characteristic equation, then the β-expansion of β is finite, i.e.,

$$\beta = a_0 + \frac{a_1}{\beta} + \cdots + \frac{a_{d-1}}{\beta^{d-1}}, \quad \text{where } a_0 = [\beta]. \tag{1.3}$$

To enumeration systems, whose characteristic root β is a PV-number satisfying (2.3), a sum $\sum_{k=0}^{M} g_k G_k$ for finite M, is the expansion of an integer if and only if the digits of the G-expansion satisfy

$$(g_k, g_{k-1}, \ldots, g_0, 0^\infty) < (a_0, a_1, \ldots, a_{d-1})^\infty$$

for each k with $<$ denoting the lexicographic order. Representations $(g_k, \ldots, g_0)$ satisfying the condition are called admissible representations and thus belong to $\mathcal{K}_G$. We call a set of the form

$$Z = Z(n_1, \ldots, n_k, a_1, \ldots, a_k) = \{x \in \mathcal{K}_G : \tau^{n_1}(x) = a_1, \ldots, \tau^{n_k}(x) = a_k\}$$

for $n_1, \ldots, n_k \in \mathbb{N}$ with $a_1, \ldots, a_k \in \mathcal{K}_G$ and $0 \leq k < \infty$ a cylinder set of length k for the odometer $(\mathcal{K}_G, \tau)$ defined earlier. For such a cylinder set Z let

$$F_{k,r} = \#\{n < G_{k+r} : (g_0(n), g_1(n) \ldots) \in Z\}.$$

We can define a τ invariant measure μ on $\mathcal{K}_G$ by setting

$$\mu(Z) = \frac{F_{k,0}\beta^{d-1} + (F_{k,1} - a_0 F_{k,0}) + \cdots + (F_{k,d-1} - a_0 F_{k,d-2} - \cdots - a_{d-2} F_{k,0})}{\beta^k(\beta^{d-1} + \beta^{d-1} + \cdots + 1)},$$

for cylinder sets and then extending μ first to the σ-algebra generated by the cylinder sets and then to the bigger completion σ-algebra obtained by adding null sets. Here we are thinking of the cylinder sets as a basis of balls for the topology they generate on $\mathcal{K}_G$.

Now we turn to the definition of the Monna map ϕ_β for non-integer bases β as follows. Let $n = \sum_{j\geq 0} g_j(n)G_j$ be the G-expansion of a positive integer n. Then set

$$\phi_\beta(n) = \phi_\beta\left(\sum_{j\geq 0} g_j(n)G_j\right) = \sum_{j\geq 0} g_j(n)\beta^{-j-1}.$$

Furthermore, the restriction to $\mathcal{K}_G^0$ has a well defined inverse. In this context the β-adic van der Corput sequence is given as $\big(\phi_\beta(n)\big)_{n\geq 0}$, where β is the characteristic root of the G-expansion.

By the nature of its construction, as a consequence of Proposition 1.1 below, we see that $\phi_\beta(\mathbb{N}) \subset [0,1)$. Of course, this implies nothing about its distribution. The following gives conditions for the density of $\phi_\beta(\mathbb{N})$ [HIT]. Proposition 1.2 below is also proved in [HIT]. There is an extensive literature on this subject starting with [Ni]. See also [GHL] for further background.

Proposition 1.1. *Let $\mathbf{a} = (a_0, \ldots, a_{d-1})$, where $a_0, \ldots, a_{d-1} \geq 0$ are the coefficients defining the enumeration system G, and assume the corresponding characteristic root β is finite as defined by* (1.3)*. Furthermore, assume that there is no $\mathbf{b} = (b_1, \ldots, b_{k-1})$ with $k < d$ such that β is the characteristic root of the polynomial defined by $\mathbf{b}$. Then $\phi_\beta(\mathbb{N}) \subset [0,1)$ and $\phi_\beta(\mathbb{N})$ is not contained in $[0,x)$, for any $x \in (0,1)$ if and only if $\mathbf{a}$ can be written either as*

$$\mathbf{a} = (a_0, \ldots, a_0) \quad \text{or as} \quad \mathbf{a} = (a_0, a_0 - 1 \ldots, a_0 - 1, a_0),$$

where $a_0 > 0$.

Proposition 1.2. *For an enumeration system G of the form* (1.2)*, let us assume that the coefficients of the linear recurrence are given by $a_j = a$, where $j = 0, \ldots, d-1$, and a is a positive integer. Let β be the corresponding characteristic root. Then*

$$\mu(Z) = \lambda\big(\phi_\beta(Z)\big)$$

for all cylinder sets Z.

In this paper we prove the following theorem.

Theorem 1.3. *Let G be a unique finite enumeration system with characteristic root β. Suppose $(k_j)_{j\geq 1}$ is Hartman uniformly distributed and L^p-good universal for a specific $p \in [1, \infty]$. Let $(n_1, \ldots, n_s)$ be an s-tuple of non-negative integers. Then the sequence $\big(\phi_\beta(k_j + n_1), \ldots, \phi_\beta(k_j + n_s)\big)_{j\geq 1}$ has a distribution function on $[0,1)^s$.*

Remark 1. An explicit construction of the distribution function in Theorem 1.3 will appear via the proof.

Remark 2. Let ρ denote a polynomial mapping from $\mathbb{Z}$ to itself. In [LN] it is also shown that if instead of being Hartman uniformly distributed, we assume that $k_i = \rho(i)$ or $k_i = \rho(p_i)$ $(i = 1, 2, \ldots)$, where p_i is the i^{th} rational prime, the distribution of the sequence $\big(\phi_\beta(k_j + n_1), \ldots, \phi_\beta(k_j + n_s)\big)_{j \geq 1}$ also exists in the setting of Theorem A. The distribution function is not however known to be a copula. The proof of this result relies on harmonic analysis methods on the group of b-adic integers. These are not available in the setting of Theorem 1.3, and the analogous statements are open in this context.

Remark 3. See [FMS] for still more explicit information about the limit distribution in the special case, where $s = 3$ and the β are all rational integers.

2. Proof of Theorem 1.3

A central tool of ours is the following lemma [HJLN].

Lemma 2.1. *Suppose $(k_i)_{i=1}^\infty$ is Hartman uniformly distributed, and L^p-good universal for $p \in [1, 2]$ and that the dynamical system $(X, \mathcal{B}, \mu, T)$ is ergodic. Then the limit $\ell_{T,f}(x)$, defined in the introduction, exists and equals $\int_X f d\mu$ for μ almost all x.*

We have the following lemma, which in the case $k_n = n$ for $n \in \mathbb{N}$ is classical and due to J. C. Oxtoby [O]. A version of this lemma appears in [LN]. The proof of this contains a significant gap. This gap is filled in [JLN]. See Section 4 of [JLN] for an extensive list of sequences of natural numbers, that are both Hartman uniform distributed and L^p good universal for some $p \in [1, \infty)$. We say a dynamical system $(X, \mathcal{B}, \mu, T)$ is *uniquely ergodic* if there is no other measure μ' on $(X, \mathcal{B})$, such that $(X, \mathcal{B}, \mu', T)$ is also an ergodic dynamical system.

Lemma 2.2. *Suppose $(k_n)_{n \geq 0}$ is Hartman uniformly distributed and L^2-good universal. Let T be a continuous map of a compact metrizable space X. The following statements are equivalent:*

a) *the transformation $(X, \mathcal{B}, T)$ is uniquely ergodic;*
b) *for each continuous function f defined on X there is a constant C_f such that for all $x \in X$,*
$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} f(T^{k_n} x) = C_f;$$

c) *for each continuous function f defined on X there is a constant C_f such that for all $x \in X$ we have*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} f(T^{k_n} x) = C_f,$$

uniformly on X; and

d) *there is a T invariant measure μ on X, and whenever f is a continuous functions on X we have*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} f(T^{k_n} x) = \int_X f d\mu,$$

pointwise on X, i.e., for all $x \in X$.

Proof of Theorem 1.3. Using Proposition 1.2 and the definition of the Monna map, we get an isomorphism between the dynamical system (K_G, τ) and the dynamical system $\big([0,1), T\big)$ with $T : [0,1) \to [0,1)$ given by

$$T(x) = \phi_\beta \circ \tau \circ \phi_\beta^+(x),$$

where ϕ_β^+ is the inverse of ϕ_β from the subset of $\mathcal{K}_G$ on which bijection is well defined. The subset includes the natural numbers. In [HIT] it is proved that the dynamical system $(\mathcal{K}_G, \tau)$ is uniquely ergodic, and since it is also isomorphic to $\big([0,1), T\big)$, then $\big([0,1), T\big)$ is again uniquely ergodic. Following [AH] for $t \in [0,1)$ define

$$\gamma(t) = \big(T^{n_1}(t), \dots, T^{n_s}(t)\big) \in [0,1)^s,$$

and

$$\Gamma := \{\gamma(t) : t \in [0,1)\}.$$

The measure preserved by T which we denote by ℓ_β on $[0,1)$ is the push down of the unique ergodic measure of the dynamical system (K_G, τ) on to $[0,1)$. Note that ℓ_β induces a measure on Γ by setting

$$\nu(A) = \ell_\beta\big(\{t : \gamma(t) \in A\}\big) \quad \text{for } A \subset \Gamma.$$

Furthermore, ν induces a measure $\overline{\mu}$ on $[0,1)^s$ by embedding Γ in $[0,1)^s$ and for every Jordan-measurable $B \subseteq [0,1)^s$ setting $\overline{\mu}(B) = \nu(B \cap \Gamma)$.

We define the empirical measure of the first N points by

$$\mu_N(B) = \frac{1}{N} \# \left\{0 \le n \le N : \big(T^{k_n+n_1}(0), \dots, T^{k_n+n_s}(0)\big) \in B\right\}.$$

Then we have

$$\begin{aligned}\lim_{N\to\infty}\mu_N(B) &= \lim_{N\to\infty}\frac{1}{N}\#\left\{0\le n\le N : \left(T^{k_n+n_1}(0),\dots,T^{k_n+n_s}(0)\right)\in B\right\}\\ &= \lim_{N\to\infty}\frac{1}{N}\#\left\{0\le n\le N : \left(T^{k_n+n_1}(0),\dots,T^{k_n+n_s}(0)\right)\in B\cap\Gamma\right\}\\ &= \lim_{N\to\infty}\frac{1}{N}\#\left\{0\le n\le N : T^{k_n+n_1}(0)\in \mathrm{proj}_1(B\cap\Gamma)\right\}\\ &= \lim_{N\to\infty}\frac{1}{N}\#\left\{0\le n\le N : T^{k_n}(0)\in \mathrm{proj}_1(B\cap\Gamma)\right\},\end{aligned}$$

where proj_1 denotes the projection onto the first coordinate of $[0,1)^s$. Note the fact that $\left(T^{k_n}(0)\right)_{n\ge 0}$ is asymptotically distribution with respect to μ as a consequence of Lemmas 2.1 and 2.2. Using this and the fact that $t\to T(t)$ is bijective we have that

$$\ell_\beta\left(\mathrm{proj}_1(B\cap\Gamma)\right)=\nu(B\cap\Gamma)=\overline{\mu}(B).$$

This $\overline{\mu}$ is the required distribution function. □

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Received September 23, 2015
Accepted August 23, 2016

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