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# TRACTABILITY OF MULTIVARIATE INTEGRATION USING LOW-DISCREPANCY SEQUENCES

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*Dedicated to the memory of Professor Pierre Liardet*

ABSTRACT. We propose a notion of $(\mathbf{t}, \mathbf{e}, s)$-sequences in multiple bases, which unifies the Halton sequence and $(t, s)$-sequences under one roof, and obtain an upper bound of their discrepancy consisting only of the leading term. By using this upper bound, we improve the tractability results currently known for the Halton sequence, the Niederreiter sequence, the Sobol' sequence, and the generalized Faure sequence, and also give tractability results for the Xing-Niederreiter sequence and the Hofer-Niederreiter sequence, for which no results have been known so far.

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## 1. Introduction

Information based complexity (IBC) is one of the most exciting fields in theoretical computer science. Since the remarkable success of quasi-Monte Carlo methods for financial applications in the mid '90s (see, e.g., [8, 14, 19, 23, 24, 27]), tractability theory of multivariate problems, a frontier of IBC, quickly emerged and has been developing in many different directions in the last two decades [2, 13, 15, 28]. In particular, tractability of multivariate integration using low-discrepancy sequences, which form a mainstay of quasi-Monte Carlo methods, has been an important topic and many interesting results were already obtained (see, e.g., [3, 6, 16, 25, 26]).

The aim of this paper is to improve the currently known tractability results on low-discrepancy sequences. The key idea for the improvement is briefly explained by using the case of the Halton sequence. Hickernell and Wang [6]

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proved the strong tractability with the exponent 1 for the Halton sequence under a condition, $\sum_{i=1}^{\infty} \gamma_i^{1/2} i \log i < \infty$, where $\gamma_1 \geq \gamma_2 \geq \cdots \geq 0$ are weights for the weighted Sobolev space of integrands. Their analysis is based on the upper bound by Niederreiter [11] for the star discrepancy $D_s^*(P_N)$ of the first $N$ points, denoted by $P_N$, of the $s$-dimensional Halton sequence, which is given as

$$\begin{aligned} D_s^*(P_N) &\leq \frac{1}{N}\left(s + \prod_{i=1}^{s}\left(\frac{(p_i-1)\log N}{2\log p_i} + \frac{p_i+1}{2}\right)\right) \\ &= \prod_{i=1}^{s}\left(\frac{p_i-1}{2\log p_i}\right)\frac{(\log N)^s}{N} + O\left(\frac{(\log N)^{s-1}}{N}\right) \end{aligned}$$

for all $N > 1$, where $p_i$ is the $i$th smallest prime for $i = 1, 2, \ldots$ On the other hand, what happens if we use the upper bound by Halton [5] published more than half a century ago, which is given as

$$D_s^*(P_N) \leq \prod_{i=1}^{s}\left(\frac{3p_i-2}{\log p_i}\right)\frac{(\log N)^s}{N} \tag{1}$$

for all $N > 1$? We should notice that Halton's bound has only the leading term without any additional terms, while the leading constant is bigger than that of Niederreiter. According to the prime number theorem, $p_i = O(i \log i)$, the right-hand side of (1) is further bounded by $N^{-1}\prod_{i=1}^{s}(C\, i \log N)$, where $C$ is a positive constant. In fact, this bound eventually leads us to a weaker condition, $\sum_{i=1}^{\infty} \gamma_i^{1/2} i < \infty$, under which the strong tractability with the exponent 1 holds.

The organization of this paper is as follows. In Section 2, we introduce a general concept of $(\mathbf{t}, \mathbf{e}, s)$-sequences in multiple bases, which unifies the Halton sequence and $(t, s)$-sequences under one roof, and give an upper bound consisting only of the leading term like (1) for the discrepancy of their first $N(> 1)$ points. This result yields upper bounds for the discrepancies of the Halton sequence, of the Niederreiter sequence, of the Sobol' sequence, of the generalized Faure sequence, of the Xing-Niederreiter sequence, and of the Hofer-Niederreiter sequence. In Section 3, based on the discrepancy upper bounds obtained in Section 2, we analyze the tractability for finite-order weights using points from the Halton sequence, the Niederreiter sequence, the Sobol' sequence, and the generalized Faure sequence, and improve the exising results. Then, we analyze the tractability using points from the Xing-Niederreiter sequence and the Hofer-Niederreiter sequence, for which no results have been known so far, and present new results for finite-order weights. In the final section, we discuss the tractability using points from the Niederreiter-Xing sequence.

## 2. $(\mathbf{t}, \mathbf{e}, s)$-sequences in multiple bases

First, we introduce the definition of discrepancy [2, 10, 11]. For a point set $P_N = \{X_0, X_1, \ldots, X_{N-1}\}$ of $N$ points in $[0,1]^s$ and an interval $J \subseteq [0,1]^s$, we define $A_N(J)$ as the number of $n$, $0 \le n \le N-1$, with $X_n \in J$ and $\mu(J)$ is the volume of $J$. Then the star discrepancy of $P_N$ is defined by

$$D_s^*(P_N) = \sup_J \left| \frac{A_N(J)}{N} - \mu(J) \right|,$$

where the supremum is taken over all intervals $J$ of the form $\prod_{i=1}^s [0, \alpha_i)$ for $0 \le \alpha_i \le 1$.

Let $\mathbf{b} = (b_1, \ldots, b_s)$ be an integer vector with $b_i \ge 2$ $(1 \le i \le s)$. An elementary interval in multiple bases $(b_1, \ldots, b_s)$ is an interval of the form

$$E_{\mathbf{b}}(j_1, \ldots, j_s) = \prod_{i=1}^s \left[ \frac{a_i}{b_i^{j_i}}, \frac{a_i + 1}{b_i^{j_i}} \right), \quad \left( 0 \le a_i < b_i^{j_i},\ j_i \ge 0 \right),$$

where $a_i$ and $j_i$ are integers for $i = 1, \ldots, s$.

**Definition 1.** Let $\mathbf{t} = (t_1, \ldots, t_s)$, $\mathbf{m} = (m_1, \ldots, m_s)$, $\mathbf{e} = (e_1, \ldots, e_s)$ and $\mathbf{j} = (j_1, \ldots, j_s)$ be integer vectors, where $t_i \ge 0$, $m_i \ge 0$, $e_i \ge 1$ and $j_i \ge 0$ for $i = 1, \ldots, s$. Let $\mathbf{t}$ and $\mathbf{m}$ be vectors with $t_i \le m_i$ $(1 \le i \le s)$ such that $\mathbf{m} - \mathbf{t} \in M(\mathbf{e}) := \{(e_1 j_1, \ldots, e_s j_s) \mid j_i \ge 0 \ (1 \le i \le s)\}$. Denote $\mathbf{b}^{\mathbf{m}} = \prod_{i=1}^s b_i^{m_i}$. A $(\mathbf{t}, \mathbf{m}, \mathbf{e}, s)$-net in multiple bases $(b_1, \ldots, b_s)$ is a point set of $\mathbf{b}^{\mathbf{m}}$ points such that $A_{\mathbf{b}^{\mathbf{m}}}(E) = \prod_{i=1}^s b_i^{t_i}$ for every elementary interval $E = E_{\mathbf{b}}(e_1 j_1, \ldots, e_s j_s)$ in multiple bases $(b_1, \ldots, b_s)$ with $\mu(E) = \prod_{i=1}^s b_i^{t_i - m_i}$ and $\mathbf{j}$ satisfying $(e_1 j_1, \ldots, e_s j_s) = \mathbf{m} - \mathbf{t}$.

**Definition 2.** A $(\mathbf{t}, \mathbf{e}, s)$-sequence in multiple bases $(b_1, \ldots, b_s)$ is an infinite sequence, $X = (X_n)_{n \ge 0}$, of points in $[0,1]^s$ such that for all integers $\ell \ge 0$ and all $\mathbf{m}$ with $m_i \ge t_i$ $(1 \le i \le s)$ satisfying $\mathbf{m} - \mathbf{t} \in M(\mathbf{e})$, the point set $\{[X_{\ell \mathbf{b}^{\mathbf{m}}}]_{\mathbf{b},\mathbf{m}}, \ldots, [X_{(\ell+1)\mathbf{b}^{\mathbf{m}}-1}]_{\mathbf{b},\mathbf{m}}\}$ is a $(\mathbf{t}, \mathbf{m}, \mathbf{e}, s)$-net, where $[X_n]_{\mathbf{b},\mathbf{m}}$ means the $m_i$-digit truncation in base $b_i$ of the $i$th coordinate of a point $X_n$ for $1 \le i \le s$.

It is obvious that a $(t, \mathbf{e}, s)$-sequence in a single base $b$ [20] is identical to a $(\mathbf{t}, \mathbf{e}, s)$-sequence in multiple bases $(b_1, \ldots, b_s)$, where $b_1 = \cdots = b_s = b$ and $\mathbf{t} = (t_1, \ldots, t_s)$ is any tuple satisfying $t = t_1 + \cdots + t_s$.

**Remark 1.** The $s$-dimensional Halton sequence [5] is a $(\mathbf{t}, \mathbf{e}, s)$-sequence in multiple bases $(b_1, \ldots, b_s)$ such that $\mathbf{t} = (0, \ldots, 0)$, $\mathbf{e} = (1, \ldots, 1)$, and $b_i$ is the $i$th smallest prime for $i = 1, \ldots, s$.

**Remark 2.** The $s$-dimensional Niederreiter sequence in base $b$ [11] is a $(\mathbf{t}, \mathbf{e}, s)$--sequence in multiple bases $(b_1, \ldots, b_s)$ such that $\mathbf{t} = (0, \ldots, 0)$, $\mathbf{e} = (e_1, \ldots, e_s)$, and $b_1 = \cdots = b_s = b$, where $e_i$, $1 \le i \le s$, is equal to the degree of the $i$th polynomial from the list of all monic irreducible polynomials over GF($b$) sorted in nondecreasing order of degree.

**Remark 3.** The $s$-dimensional Sobol' sequence [17] is a $(\mathbf{t}, \mathbf{e}, s)$-sequence in multiple bases $(b_1, \ldots, b_s)$ such that $\mathbf{t} = (0, \ldots, 0)$, $\mathbf{e} = (e_1, \ldots, e_s)$, and $b_1 = \cdots = b_s = 2$, where $e_1 = 1$, and $e_i$, $2 \le i \le s$, is equal to the degree of the $(i-1)$-th polynomial from the list of all primitive polynomials over GF(2) sorted in nondecreasing order of degree.

**Remark 4.** The $s$-dimensional generalized Faure sequence in base $b$ [18] is a $(\mathbf{t}, \mathbf{e}, s)$-sequence in multiple bases $(b_1, \ldots, b_s)$ such that $\mathbf{t} = (0, \ldots, 0)$, $\mathbf{e} = (1, \ldots, 1)$, and $b_1 = \cdots = b_s = b$, where $b$ is the smallest prime with $b \ge s$.

**Remark 5.** Each of the $s$-dimensional Xing-Niederreiter sequence in base $b$ and genus $g$ [29] and the $s$-dimensional Hofer-Niederreiter sequence in base $b$ and genus $g$ [7] is a $(\mathbf{t}, \mathbf{e}, s)$-sequence in multiple bases $(b_1, \ldots, b_s)$ such that $\mathbf{t} = (t_1, \ldots, t_s)$ is any tuple satisfying $g = t_1 + \cdots + t_s$, $\mathbf{e} = (e_1, \ldots, e_s)$, and $b_1 = \cdots = b_s = b$, where $e_i$ is equal to the degree of the $i$th place from the list of all places sorted in nondecreasing order of degree.

Next, we will give a theorem on the upper bound for the discrepancy of $(\mathbf{t}, \mathbf{e}, s)$-sequences in multiple bases $(b_1, \ldots, b_s)$. Before that, we need the following lemmas.

**Lemma 1.** *Let $(b_1, \ldots, b_s)$ be an integer vector with $b_i \ge 2$ $(1 \le i \le s)$, and $\mathbf{e} = (e_1, \ldots, e_s)$ and $\mathbf{j} = (j_1, \ldots, j_s)$ be integer vectors with $e_i \ge 1$ and $j_i \ge 0$ for $i = 1, \ldots, s$. Let $I$ be an interval given by*

$$I = \prod_{i=1}^{s} \left[ \frac{a_i}{b_i^{e_i j_i}}, \frac{c_i}{b_i^{e_i j_i}} \right), \quad \left( 0 \le a_i < c_i \le b_i^{e_i j_i} \right),$$

*where $a_i$ and $c_i$ are integers for $i = 1, \ldots, s$. Denote $B = \prod_{i=1}^{s} b_i^{t_i}$ and $\mathbf{b}^{\mathbf{ej}} = \prod_{i=1}^{s} b_i^{e_i j_i}$. Then, for the first $N$ points of the truncated version of a $(\mathbf{t}, \mathbf{e}, s)$-sequence in multiple bases $(b_1, \ldots, b_s)$, we have*

$$|A_N(I) - N\mu(I)| \le B \prod_{i=1}^{s} (c_i - a_i)$$

*for every positive integer $N$, and $A_N(I) \le B \prod_{i=1}^{s} (c_i - a_i)$ if $N \le B\mathbf{b}^{\mathbf{ej}}$, where the truncation size is taken to be large enough depending on $\mathbf{j}$.*

P r o o f. From Definition 2, for the truncated version of a $(\mathbf{t}, \mathbf{e}, s)$-sequence in multiple bases $(b_1, \dots, b_s)$, we see that for any integer $\ell \geq 0$,

$$A_{\ell B \mathbf{b}^{\mathbf{ej}}}(I) = \ell B \prod_{i=1}^{s} (c_i - a_i).$$

The last statement follows by taking $\ell = 1$ and the fact that $A_N(I)$ is nondecreasing in $N$. For an arbitrary $N$, there is $\ell$ such that $\ell B \mathbf{b}^{\mathbf{ej}} \leq N < (\ell+1) B \mathbf{b}^{\mathbf{ej}}$. Thus, we have

$$A_N(I) - N\mu(I) \leq A_{(\ell+1)B\mathbf{b}^{\mathbf{ej}}}(I) - \ell B \mathbf{b}^{\mathbf{ej}} \prod_{i=1}^{s} \frac{c_i - a_i}{b_i^{e_i j_i}} = B \prod_{i=1}^{s} (c_i - a_i),$$

and

$$N\mu(I) - A_N(I) \leq (\ell+1) B \mathbf{b}^{\mathbf{ej}} \prod_{i=1}^{s} \frac{c_i - a_i}{b_i^{e_i j_i}} - A_{\ell B \mathbf{b}^{\mathbf{ej}}}(I) = B \prod_{i=1}^{s} (c_i - a_i).$$

This completes the proof. □

**Lemma 2.** *Let $b \geq 4$ be an integer. Then, we have $x^b \geq b^x$ for $2 \leq x \leq b$.*

P r o o f. Since $\log x$ is a concave function, if $b \geq 4$ we have

$$\frac{x}{b} \leq \frac{\log x}{\log b}$$

for $2 \leq x \leq b$. Thus, the proof is complete. □

**Lemma 3.** *Let $b_1, \dots, b_s$ be integers with $2 \leq b_1 \leq \cdots \leq b_s$. Then for any integer $N$ with $1 < N \leq b_s$ we have*

$$\prod_{i=1}^{s} \big( (2b_i - 1) \log_{b_i} N \big) > N.$$

P r o o f. By Lemma 2 and simple calculations for $b_s = 2$ and 3, we have

$$N^{2b_s - 1} > b_s^N$$

for $b_s \geq 2$ and $2 \leq N \leq b_s$. Therefore, $(2b_s - 1) \log_{b_s} N > N \log_{b_s} b_s = N$. For $i = 1, \dots, s-1$, we observe $(2b_i - 1) \log_{b_i} N \geq (2b_i - 1) \log_{b_i} 2 > \log_{b_i} b_i = 1$. Thus, the proof is complete. □

We now prove the main theorem in this section, whose proof is based on an excellent exposition by Matoušek [10, pp. 41-43].

**Theorem 1.** *Let $(b_1, \dots, b_s)$ be an integer vector with $b_i \geq 2$ $(1 \leq i \leq s)$. The star discrepancy for the first $N$ points, denoted by $P_N$, of a $(\mathbf{t}, \mathbf{e}, s)$-sequence in multiple bases $(b_1, \dots, b_s)$ satisfies*

$$ND_s^*(P_N) \leq B \prod_{i=1}^{s} \left( \frac{2b_i^{e_i} - 1}{e_i \log b_i} \right) (\log N)^s \tag{2}$$

*for all $N > 1$, where $B = \prod_{i=1}^{s} b_i^{t_i}$.*

P r o o f. Take any $\mathbf{z} = (z^{(1)}, \dots, z^{(s)}) \in [0,1]^s$. Each $z^{(i)}$ is expanded in base $b_i^{e_i}$ as $\sum_{j=1}^{\infty} a_j^{(i)} b_i^{-e_i j}$, with $0 \leq a_j^{(i)} \leq b_i^{e_i} - 1$. Let $n_i = \lceil \log_{b_i^{e_i}} N \rceil$ and define $z_0^{(i)} = 0$ and $z_{n_i+1}^{(i)} = z^{(i)}$. Consider the numbers $z_k^{(i)} = \sum_{j=1}^{k} a_j^{(i)} b_i^{-e_i j}$ for $k = 1, \dots, n_i$. Then, we obtain the splitting of $J = \prod_{i=1}^{s} [0, z^{(i)})$ as

$$I(\mathbf{j}) = \prod_{i=1}^{s} \left[ z_{j_i}^{(i)}, z_{j_i+1}^{(i)} \right),$$

where $\mathbf{j} = (j_1, \dots, j_s)$ with $0 \leq j_i \leq n_i$ $(1 \leq i \leq s)$. For the truncated version of $(\mathbf{t}, \mathbf{e}, s)$-sequences in multiple bases, Lemma 1 gives

$$\left| A_N\big(I(\mathbf{j})\big) - N\mu\big(I(\mathbf{j})\big) \right| \leq B \prod_{i=1}^{s} (b_i^{e_i} - 1)$$

for all $\mathbf{j}$ with $0 \leq j_i < n_i$ $(1 \leq i \leq s)$. Denote the remaining intervals by

$$R_h = \bigcup_{\mathbf{j} \in \mathcal{J}_h} I(\mathbf{j}),$$

for $h = 1, \dots, s$, where $\mathcal{J}_h = \{(j_1, \dots, j_s) |\ 0 \leq j_i \leq n_i \ (1 \leq i \neq h \leq s)$ and $j_h = n_h\}$. Since $\mu(R_h) \leq b_h^{-e_h n_h} \leq 1/N$ and $R_h$ contains at most $B$ points, we have

$$|A_N(R_h) - N\mu(R_h)| \leq B$$

for $h = 1, \dots, s$. Therefore,

$$|A_N(J) - N\mu(J)| \leq B \prod_{i=1}^{s} (b_i^{e_i} - 1) \times \prod_{i=1}^{s} \lceil \log_{b_i^{e_i}} N \rceil + Bs \tag{3}$$

for any $J$. Since the right-hand side is independent of the interval $J$, it is an upper bound of $ND_s^*(P_N) = \sup_J |A_N(J) - N\mu(J)|$.

Assuming that $N \geq \max_{1 \leq i \leq s} b_i^{e_i}$, we can further bound the right-hand side of (3) from above as follows:

$$
\begin{aligned}
ND_s^*(P_N) &\le B\left(\prod_{i=1}^{s}\Big(\big(b_i^{e_i}-1\big)\big(\log_{b_i^{e_i}} N+1\big)\Big)+s\right)\\
&\le B\left(\prod_{i=1}^{s}\Big(2\big(b_i^{e_i}-1\big)\log_{b_i^{e_i}} N\Big)+s\right)\\
&\le B\prod_{i=1}^{s}\Big(\big(2b_i^{e_i}-2\big)\log_{b_i^{e_i}} N+1\Big)\\
&\le B\prod_{i=1}^{s}\Big(\big(2b_i^{e_i}-1\big)\log_{b_i^{e_i}} N\Big).
\end{aligned}
$$

In fact, this upper bound holds for all $N > 1$, because, from Lemma 3, it is bigger than $N\big(\ge ND_s^*(P_N)\big)$ for all $N$ with $1 < N \le \max_{1\le i\le s} b_i^{e_i}$.

Hence, the discrepancy bound (2) for the truncated version is obtained. As shown in [20], the discrepancy bound for the untruncated version remains the same as the truncated version. Thus, the proof is complete. □

If the discrepancy is analyzed based on Atanassov's signed splitting method [1], we obtain

$$ND_s^*(P_N) \le \frac{B}{s!}\prod_{i=1}^{s}\left(\frac{b_i^{e_i}-1}{2e_i\log b_i}\right)(\log N)^s + O((\log N)^{s-1})$$

for all $N > 1$ [21]. (The proof is a simple generalization of the one given in [22] for $(t, \mathbf{e}, s)$-sequences in a single base $b$.) Although the leading constant becomes smaller than the one in Theorem 1, it looks difficult to apply this upper bound to the tractability analysis to obtain better results.

Based on the above theorem, we can derive the following corollaries for the four types of low-discrepancy sequences.

**Corollary 1.** *Let $P_N$ be the first $N$ points of the $s$-dimensional Halton sequence. Let $P_N^u$ be the projection of $P_N$ on the lower dimensional space $[0,1]^{|u|}$. Then for any nonempty subset $u \subseteq \{1,\dots,s\}$, the star discrepancy of $P_N^u$ satisfies*

$$D_{|u|}^*(P_N^u) \le \frac{1}{N}\prod_{i\in u}(C\, i\log N)$$

*for all $N > 1$, where $C = 6$.*

P r o o f. From Remark 1 and Theorem 1, we have

$$ND_{|u|}^*(P_N^u) \le \prod_{i\in u}\left(\frac{2b_i-1}{\log b_i}\log N\right) \tag{4}$$

for all $N > 1$, where $b_i$ is the $i$th smallest prime for $i = 1,\dots,s$.

Since $b_i/\log b_i < 3\,i$ for $i = 1, 2, \ldots$ (see, e.g., Dusart [4]), the right-hand side of (4) is further bounded from above by

$$\prod_{i\in u} (6\,i \log N)\,.$$

Thus, the proof is complete. □

**Corollary 2.** *Let $P_N$ be the first $N$ points of the $s$-dimensional Niederreiter sequence in base $b$. Let $P_N^u$ be the projection of $P_N$ on the lower dimensional space $[0,1]^{|u|}$. Then for any nonempty subset $u \subseteq \{1, \ldots, s\}$, the star discrepancy of $P_N^u$ satisfies*

$$D_{|u|}^*(P_N^u) \le \frac{1}{N} \prod_{i\in u} (C\ i \log N)$$

*for all $N > 1$, where $C = 4b^2/\log b$.*

P r o o f. From Remark 2 and Theorem 1, we have

$$N D_{|u|}^*(P_N^u) \le \prod_{i\in u} \left( \frac{2b^{e_i} - 1}{e_i \log b} \log N \right) \tag{5}$$

for all $N > 1$, where $e_i$, $1 \le i \le s$, is equal to the degree of the $i$th polynomial from the list of all monic irreducible polynomials over GF($b$) sorted in nondecreasing order of degree.

According to Niederreiter [11], we have

$$e_i \le \log_b i + \log_b \log_b (i + b) + 2.$$

Thus, the right-hand side of (5) is bounded from above by

$$\prod_{i\in u} \left( \frac{2b^2 i \log_b (i + b)}{e_i \log b} \log N \right).$$

Since $\log_b(i + b) \le 2e_i$ for $i = 1, 2, \ldots$, we can further bound it by

$$\prod_{i\in u} \left( \frac{4b^2 i}{\log b} \log N \right).$$

Thus, the proof is complete. □

**Corollary 3.** *Let $P_N$ be the first $N$ points of the $s$-dimensional Sobol' sequence. Let $P_N^u$ be the projection of $P_N$ on the lower dimensional space $[0,1]^{|u|}$. Then for any nonempty subset $u \subseteq \{1, \ldots, s\}$, the star discrepancy of $P_N^u$ satisfies*

$$D_{|u|}^*(P_N^u) \le \frac{1}{N} \prod_{i\in u} \big(C\ i \log_2 \log_2 (i + 3) \log N\big)$$

*for all $N > 1$, where $C > 1$ is a constant.*

P r o o f. From Remark 3 and Theorem 1, we have

$$ND^*_{|u|}(P^u_N) \le \prod_{i\in u}\left(\frac{2^{e_i+1}-1}{e_i \log 2}\log N\right) \tag{6}$$

for all $N > 1$, where $e_1 = 1$, and $e_i$, $2 \le i \le s$, is equal to the degree of the $(i-1)$th polynomial from the list of all primitive polynomials over GF(2) sorted in nondecreasing order of degree.

According to Sobol'[17], we have

$$e_i \le \log_2 i + \log_2 \log_2(i+1) + \log_2 \log_2 \log_2(i+3) + c$$

for $i = 1, 2, \ldots$, where $c$ is a positive constant independent of $i$.

Thus, the right-hand side of (6) is bounded from above by

$$\prod_{i\in u}\left(\frac{2^{c+1} i \log_2(i+1) \log_2 \log_2(i+3)}{e_i \log 2}\log N\right).$$

Since $\log_2(i+1) < 2e_i$ for $i = 1, 2, \ldots$, we can further bound it by

$$\prod_{i\in u}\left(\frac{2^{c+2}}{\log 2}\, i \log_2 \log_2(i+3) \log N\right).$$

Thus, the proof is complete. □

**Corollary 4.** *Let $P_N$ be the first $N$ points of the $s$-dimensional generalized Faure sequence in base $b$. Let $P^u_N$ be the projection of $P_N$ on the lower dimensional space $[0,1]^{|u|}$. Then for any nonempty subset $u \subseteq \{1, \ldots, s\}$, the star discrepancy of $P^u_N$ satisfies*

$$D^*_{|u|}(P^u_N) \le \frac{1}{N}\left(C\frac{s \log N}{\log s}\right)^{|u|}$$

*for all $N > 1$ and $s > 1$, where $C = 4$.*

P r o o f. From Remark 4 and Theorem 1, we have

$$ND^*_{|u|}(P^u_N) \le \left(\frac{2b-1}{\log b}\log N\right)^{|u|} \tag{7}$$

for all $N > 1$, where $b$ is the smallest prime with $b \ge s$.

According to the Bertrand postulate, $s \le b \le 2s$, the right-hand side of (7) is bounded from above by

$$\left(\frac{4s \log N}{\log s}\right)^{|u|}.$$

Thus, the proof is complete. □

**Corollary 5.** *Let $P_N$ be the first $N$ points of the $s$-dimensional Xing-Niederreiter sequence in base $b$ and genus $g$ or of the $s$-dimensional Hofer-Niederreiter sequence in base $b$ and genus $g$. Let $P_N^u$ be the projection of $P_N$ on the lower dimensional space $[0,1]^{|u|}$. Then for any nonempty subset $u \subseteq \{1,\dots,s\}$, the star discrepancy of $P_N^u$ satisfies*

$$D^*_{|u|}(P_N^u) \le \frac{b^g}{N} \prod_{i \in u} (C\ i \log N)$$

*for all $N > 1$, where $C > 1$ is a constant.*

P r o o f. From Remark 5 and Theorem 1, we have

$$N D^*_{|u|}(P_N^u) \le b^g \prod_{i \in u} \left( \frac{2b^{e_i} - 1}{e_i \log b} \log N \right) \tag{8}$$

for all $N > 1$, where $e_i$, $1 \le i \le s$, is equal to the degree of the $i$th place from the list of all places sorted in nondecreasing order of degree. According to the prime number theorem for global function fields [9]:

$$\#(e) = \frac{b}{b-1} \cdot \frac{b^e}{e} + o\left(\frac{b^e}{e}\right) \quad \text{as} \quad e \to \infty, \tag{9}$$

where $\#(e), e = 1, 2, \dots$, is the number of places of degree $\le e$, we can further bound (8) by

$$b^g \prod_{i \in u} (C\ i \log N)\,,$$

because $\#(e_i - 1) < i \le \#(e_i)$ for $i = 1, 2, \dots$ Thus, the proof is complete. Notice that the theorem (9) holds independently of the genus $g$. □

## 3. Tractability analysis of several low-discrepancy sequences

In this section, we apply the results obtained in the preceding section to the tractability analysis for finite-order weights, of the Halton sequence, of the Niederreiter sequence, of the Sobol' sequence, of the generalized Faure sequence, of the Xing-Niederreiter sequence, and of thel Hofer-Niederreiter sequence. First, we define the worst case integration error by

$$e(P_N; H) = \sup_{||f||_H \le 1} \left| \int_{[0,1]^s} f(\mathbf{x}) d\mathbf{x} - \frac{1}{N} \sum_{i=0}^{N-1} f(X_i) \right|,$$

where $P_N = \{X_0, X_1, \dots, X_{N-1}\}$ is a set of $N$ points in $[0,1]^s$ and $H$ denotes a normed space of integrands $f(\mathbf{x})$. We denote the initial error, which indicates the error when $P_N$ is empty, by $e(0;H)$. In this section, we consider two types of spaces. One is the anchored Sobolev space with an arbitrary anchor $\mathbf{a} = (a_1, \dots, a_s) \in [0,1]^s$, and the other is the unanchored Sobolev space. For more detailed background and theoretical properties of these spaces, consult the blue book by Novak and Woźniakowski [13]. As in [13], we say that the weights $\gamma = \{\gamma_{s,u}\}$ are finite-order if there exists an integer $\omega$ such that $\gamma_{s,u} = 0$ for all $s$ and for all $u$ with $|u| > \omega$, and say that the order is $\omega^*$ if $\omega^*$ is the smallest integer with this property, where we define $\gamma_{s,\emptyset} = 1$. Hereafter, we denote the anchored Sobolev space by $H(K_{s,\gamma,A})$, and the unanchored Sobolev space by $H(K_{s,\gamma,B})$, and let $H(K_{s,\gamma})$ be $H(K_{s,\gamma,A})$ or $H(K_{s,\gamma,B})$.

### 3.1. Strong tractability of the Halton sequence

First, we give the following lemma.

**Lemma 4.** *Let $P_N$ be the point set consisting of the first $N$ points of the $s$-dimensional Halton sequence. Then*

$$e^2\big(P_N; H(K_{s,\gamma})\big) \le \frac{1}{N^2} \sum_{\emptyset \ne u \subseteq \{1,\dots,s\}} \gamma_{s,u} \prod_{i \in u} (C_1\, i \log_2 N)^2$$

*for all $N > 1$, where $C_1 = 12 \log 2$.*

Proof. According to Theorem 16.27 and its proof in [13], we have

$$e^2\big(P_N; H(K_{s,\gamma})\big) \le \sum_{\emptyset \ne u \subseteq \{1,\dots,s\}} \gamma_{s,u} \big(2^{|u|} D^*_{|u|}(P_N^u)\big)^2 \tag{10}$$

for both anchored and unanchored Sobolev spaces. Applying Corollary 1 completes the proof. □

Then, we obtain the main theorem on the tractability using the Halton sequence. Although the proof is almost the same as that of Theorem 16.28 of [13], it is added for the reader's convenience and for completeness. We define $m_i = a_i^2 - a_i + 1/3$ for $i = 1, \dots, s$.

**Theorem 2.** *Let $P_N$ be the point set of the first $N$ points of the $s$-dimensional Halton sequence, where $N > 1$.*

(A) *Consider the anchored Sobolev space $H(K_{s,\gamma,A})$ with an arbitrary anchor* $\mathbf{a}$.

- *For arbitrary finite-order weights $\{\gamma_{s,u}\}$ of order $\omega$, we have*

$$\frac{e\big(P_N; H(K_{s,\gamma,A})\big)}{e\big(0; H(K_{s,\gamma,A})\big)} \le \frac{C_2\ s^\omega \log_2^\omega N}{N},$$

*where $C_2$ is a positive constant independent of $N$ and $s$.*

- *If the finite-order weights $\{\gamma_{s,u}\}$ of order $\omega$ satisfy*

$$M := \sup_{s\in\mathbb{N}} \left( \frac{\sum_{u\subseteq\{1,\dots,s\},|u|\le\omega} \gamma_{s,u} \prod_{i\in u} i^2}{\sum_{u\subseteq\{1,\dots,s\},|u|\le\omega} \gamma_{s,u} \prod_{i\in u} m_i} \right) < \infty, \tag{11}$$

*then for arbitrary $\delta > 0$ there exists a positive constant $C_\delta$ independent of $N$ and $s$ such that*

$$\frac{e\big(P_N; H(K_{s,\gamma,A})\big)}{e\big(0; H(K_{s,\gamma,A})\big)} \le C_\delta N^{-1+\delta}.$$

(B) *Consider the unanchored Sobolev space $H(K_{s,\gamma,B})$.*

- *For arbitrary bounded finite-order weights $\{\gamma_{s,u}\}$ of order $\omega$, we have*

$$e\big(P_N; H(K_{s,\gamma,B})\big) \le \frac{C_3\ s^{3\omega/2} \log_2^\omega N}{N},$$

*where $C_3$ is a positive constant independent of $N$ and $s$.*

- *If the finite-order weights $\{\gamma_{s,u}\}$ of order $\omega$ satisfy*

$$M' := \sup_{s\in\mathbb{N}} \left( \sum_{u\subseteq\{1,\dots,s\},|u|\le\omega} \gamma_{s,u} \prod_{i\in u} i^2 \right) < \infty, \tag{12}$$

*then for arbitrary $\delta > 0$ there exists a positive constant $C'_\delta$ independent of $N$ and $s$ such that*

$$e\big(P_N; H(K_{s,\gamma,B})\big) \le C'_\delta N^{-1+\delta}.$$

P r o o f. According to Lemma 4, we have the worst case error as

$$\begin{aligned} e^2\big(P_N; H(K_{s,\gamma,A})\big) &\le \frac{1}{N^2} \sum_{u\subseteq\{1,\dots,s\},0<|u|\le\omega} \gamma_{s,u} \prod_{i\in u} (C_1 i \log_2 N)^2 \\ &= \frac{1}{N^2} \sum_{\ell=1}^{\omega} \left( (C_1 \log_2 N)^{2\ell} \sum_{u\subseteq\{1,\dots,s\},|u|=\ell} \gamma_{s,u} \prod_{i\in u} i^2 \right). \end{aligned}$$

For the anchored case (A), the initial error is given as

$$e^2\big(0; H(K_{d,\gamma,A})\big) = \sum_{u\subseteq\{1,\dots,s\},|u|\le\omega} \gamma_{s,u} \prod_{i\in u} m_i.$$

First, we consider arbitrary finite-order weights of order $\omega$. Since $1/12 \le m_i \le 1/3$ for all $i = 1, \ldots, s$, and from Lemma 4, we have

$$\begin{aligned}
\frac{e^2\bigl(P_N; H(K_{s,\gamma,A})\bigr)}{e^2\bigl(0; H(K_{s,\gamma,A})\bigr)} &\le \frac{1}{N^2} \frac{\sum_{u\subseteq\{1,\ldots,s\},0<|u|\le\omega} \gamma_{s,u} \prod_{i\in u} (C_1 i \log_2 N)^2}{\sum_{u\subseteq\{1,\ldots,s\},0\le|u|\le\omega} \gamma_{s,u} \prod_{i\in u} m_i}, \\
&\le \frac{12^\omega}{N^2} \frac{\sum_{u\subseteq\{1,\ldots,s\},0<|u|\le\omega} \gamma_{s,u} \prod_{i\in u} (C_1 i \log_2 N)^2}{\sum_{u\subseteq\{1,\ldots,s\},0\le|u|\le\omega} \gamma_{s,u}}, \\
&\le \frac{12^\omega}{N^2} \max_{u:|u|\le\omega} \prod_{i\in u} (C_1 i \log_2 N)^2, \\
&\le \frac{12^\omega (C_1 s \log_2 N)^{2\omega}}{N^2} = \frac{C_2^2 s^{2\omega} \log_2^{2\omega} N}{N^2},
\end{aligned}$$

where $C_2 = 2^\omega \sqrt{3^\omega} C_1^\omega$.

Now consider finite-order weights of order $\omega$ satisfying (11). For arbitrary $\delta > 0$ define

$$B_\delta = \max_{1\le\ell\le\omega} \left( \left( \frac{C_1}{2\delta \log 2} \right)^{2\ell} (2\ell)! \right).$$

Then, we have

$$\begin{aligned}
\frac{e^2\bigl(P_N; H(K_{s,\gamma,A})\bigr)}{e^2\bigl(0; H(K_{s,\gamma,A})\bigr)} &\le \frac{1}{N^2} \sum_{\ell=1}^{\omega} \left( (C_1 \log_2 N)^{2\ell} \frac{\sum_{u\subseteq\{1,\ldots,s\},|u|=\ell} \gamma_{s,u} \prod_{i\in u} i^2}{\sum_{u\subseteq\{1,\ldots,s\},0\le|u|\le\omega} \gamma_{s,u} \prod_{i\in u} m_i} \right) \\
&\le \frac{M}{N^2} \sum_{\ell=1}^{\omega} \left( \frac{C_1 \log N}{\log 2} \right)^{2\ell} \le \frac{B_\delta M}{N^2} \sum_{\ell=1}^{\omega} \frac{(2\delta \log N)^{2\ell}}{(2\ell)!} \\
&\le \frac{B_\delta M}{N^2} \exp(2\delta \log N) = C_\delta^2 N^{-2+2\delta},
\end{aligned}$$

where $C_\delta = \sqrt{B_\delta M}$. This completes the proof for the case (A).

For the unanchored case (B), in which the initial error $e\bigl(0; H(K_{d,\gamma,B})\bigr) = 1$, let $\Gamma^*$ satisfy $\gamma_{s,u} \le \Gamma^*$ for all $s$ and for all $u \subseteq \{1, \ldots, s\}$. Then, we have

$$\begin{aligned}
e^2\bigl(P_N; H(K_{s,\gamma,B})\bigr) &\le \frac{\Gamma^*}{N^2} \sum_{\ell=1}^{\omega} \left( (C_1 \log_2 N)^{2\ell} \sum_{u\subseteq\{1,\ldots,s\},|u|=\ell} \prod_{i\in u} i^2 \right) \\
&\le \frac{\Gamma^*}{N^2} \sum_{\ell=1}^{\omega} \left( s^{2\ell} (C_1 \log_2 N)^{2\ell} \sum_{u\subseteq\{1,\ldots,s\},|u|=\ell} 1 \right)
\end{aligned}$$

$$\le \frac{\Gamma^* s^{2\omega} (C_1 \log_2 N)^{2\omega}}{N^2} \sum_{\ell=1}^{\omega} \binom{s}{\ell}$$
$$\le \frac{\Gamma^* s^{3\omega} (C_1 \log_2 N)^{2\omega}}{N^2} = \frac{C_3^2 s^{3\omega} \log_2^{2\omega} N}{N^2},$$

where $C_3 = \sqrt{\Gamma^*} C_1^{\omega}$. Note that $\sum_{\ell=1}^{\omega} \binom{s}{\ell} \le \sum_{\ell=1}^{\omega} \frac{s^\ell}{\ell} \le s^\omega$ for $1 \le \omega \le s$.

For the second part of (B), under the condition (12) we have

$$e^2\big(P_N; H(K_{s,\gamma,B})\big) \le \frac{1}{N^2} \sum_{\ell=1}^{\omega} \left( (C_1 \log_2 N)^{2\ell} \sum_{u \subseteq \{1,\dots,s\}, |u|=\ell} \gamma_{s,u} \prod_{i \in u} i^2 \right)$$
$$\le \frac{M'}{N^2} \sum_{\ell=1}^{\omega} \left( \frac{C_1 \log N}{\log 2} \right)^{2\ell} \le C_\delta'^2 N^{-2+2\delta},$$

where $C'_\delta = \sqrt{B_\delta M'}$. Thus, the proof for the case (B) is complete. □

### 3.2. Strong tractability of the Niederreiter sequence

We prove the next lemma.

**Lemma 5.** *Let $P_N$ be the point set consisting of the first $N$ points of the $s$-dimensional Niederreiter sequence in base $b$. Then*

$$e^2\big(P_N; H(K_{s,\gamma})\big) \le \frac{1}{N^2} \sum_{\emptyset \ne u \subseteq \{1,\dots,s\}} \gamma_{s,u} \prod_{i \in u} (C_1\ i \log_2 N)^2$$

*for all $N > 1$, where $C_1 = \frac{8b^2}{\log_2 b} > 1$ is a constant independent of $N$ and $s$.*

Proof. As in the proof of Lemma 4, applying Corollary 2 to (10) completes the proof. □

Now we obtain the main theorem on the tractability using the Niederreiter sequence.

**Theorem 3.** *Let $P_N$ be the point set of the first $N$ points of the $s$-dimensional Niederreiter sequence in base $b$, where $N > 1$.*

(A) *Consider the anchored Sobolev space $H(K_{s,\gamma,A})$ with an arbitrary anchor* **a**.

- *For arbitrary finite-order weights $\{\gamma_{s,u}\}$ of order $\omega$, we have*

$$\frac{e\big(P_N; H(K_{s,\gamma,A})\big)}{e\big(0; H(K_{s,\gamma,A})\big)} \le \frac{C_2\ s^\omega \log_2^\omega N}{N},$$

*where $C_2$ is a positive constant independent of $N$ and $s$.*

- *If the finite-order weights* $\{\gamma_{s,u}\}$ *of order* $\omega$ *satisfy*

$$\sup_{s\in\mathrm{N}}\left(\frac{\sum_{u\subseteq\{1,\dots,s\},|u|\le\omega}\gamma_{s,u}\prod_{i\in u}i^2}{\sum_{u\subseteq\{1,\dots,s\},|u|\le\omega}\gamma_{s,u}\prod_{i\in u}m_i}\right)<\infty,$$

*then for arbitrary* $\delta>0$ *there exists a positive constant* $C_\delta$ *independent of* $N$ *and* $s$ *such that*

$$\frac{e\big(P_N;H(K_{s,\gamma,A})\big)}{e\big(0;H(K_{s,\gamma,A})\big)}\le C_\delta N^{-1+\delta}.$$

(B) *Consider the unanchored Sobolev space* $H(K_{s,\gamma,B})$.

- *For arbitrary bounded finite-order weights* $\{\gamma_{s,u}\}$ *of order* $\omega$*, we have*

$$e\big(P_N;H(K_{s,\gamma,B})\big)\le\frac{C_3\ s^{3\omega/2}\log_2^{\omega}N}{N},$$

*where* $C_3$ *is a positive constant independent of* $N$ *and* $s$.

- *If the finite-order weights* $\{\gamma_{s,u}\}$ *of order* $\omega$ *satisfy*

$$\sup_{s\in\mathrm{N}}\left(\sum_{u\subseteq\{1,\dots,s\},|u|\le\omega}\gamma_{s,u}\prod_{i\in u}i^2\right)<\infty,$$

*then for arbitrary* $\delta>0$ *there exists a positive constant* $C'_\delta$ *independent of* $N$ *and* $s$ *such that*

$$e\big(P_N;H(K_{s,\gamma,B})\big)\le C'_\delta N^{-1+\delta}.$$

P r o o f. Based on Lemma 5, the proof is done in the same way as that of Theorem 2. □

### 3.3. Strong tractability of the Sobol' sequence

We show the next lemma.

**Lemma 6.** *Let* $P_N$ *be the point set consisting of the first* $N$ *points of the* $s$*-dimensional Sobol' sequence. Then*

$$e^2\big(P_N;H(K_{s,\gamma})\big)\le\frac{1}{N^2}\sum_{\emptyset\ne u\subseteq\{1,\dots,s\}}\gamma_{s,u}\prod_{i\in u}\big(C_1\ i\log_2\log_2(i+3)\log_2 N\big)^2$$

*for all* $N>1$*, where* $C_1>1$ *is a constant independent of* $N$ *and* $s$.

P r o o f. As in the proof of Lemma 4, applying Corollary 3 to (10) completes the proof. □

Now we obtain the main theorem on the tractability using the Sobol' sequence.

**Theorem 4.** *Let $P_N$ be the point set of the first $N$ points of the $s$-dimensional Sobol' sequence, where $N > 1$.*

(A) *Consider the anchored Sobolev space $H(K_{s,\gamma,A})$ with an arbitrary anchor* $\mathbf{a}$.

- *For arbitrary finite-order weights $\{\gamma_{s,u}\}$ of order $\omega$, we have*

$$\frac{e\big(P_N; H(K_{s,\gamma,A})\big)}{e\big(0; H(K_{s,\gamma,A})\big)} \le \frac{C_2 \left(s \log_2 \log_2(s+3)\right)^{\omega} \log_2^{\omega} N}{N},$$

  *where $C_2$ is a positive constant independent of $N$ and $s$.*

- *If the finite-order weights $\{\gamma_{s,u}\}$ of order $\omega$ satisfy*

$$\sup_{s\in\mathbb{N}} \left( \frac{\sum_{u\subseteq\{1,\dots,s\},|u|\le\omega} \gamma_{s,u} \prod_{i\in u} \big(i \log_2 \log_2(i+3)\big)^2}{\sum_{u\subseteq\{1,\dots,s\},|u|\le\omega} \gamma_{s,u} \prod_{i\in u} m_i} \right) < \infty,$$

  *then for arbitrary $\delta > 0$ there exists a positive constant $C_\delta$ independent of $N$ and $s$ such that*

$$\frac{e\big(P_N; H(K_{s,\gamma,A})\big)}{e\big(0; H(K_{s,\gamma,A})\big)} \le C_\delta N^{-1+\delta}.$$

(B) *Consider the unanchored Sobolev space $H(K_{s,\gamma,B})$.*

- *For arbitrary bounded finite-order weights $\{\gamma_{s,u}\}$ of order $\omega$, we have*

$$e\big(P_N; H(K_{s,\gamma,B})\big) \le \frac{C_3 \left(s^{3/2} \log_2 \log_2(s+3)\right)^{\omega} \log_2^{\omega} N}{N},$$

  *where $C_3$ is a positive constant independent of $N$ and $s$.*

- *If the finite-order weights $\{\gamma_{s,u}\}$ of order $\omega$ satisfy*

$$\sup_{s\in\mathbb{N}} \left( \sum_{u\subseteq\{1,\dots,s\},|u|\le\omega} \gamma_{s,u} \prod_{i\in u} \big(i \log_2 \log_2(i+3)\big)^2 \right) < \infty,$$

  *then for arbitrary $\delta > 0$ there exists a positive constant $C'_\delta$ independent of $N$ and $s$ such that*

$$e\big(P_N; H(K_{s,\gamma,B})\big) \le C'_\delta N^{-1+\delta}.$$

P r o o f. Based on Lemma 6, the proof is done in the same way as that of Theorem 2. □

### 3.4. Strong tractability of the generalized Faure sequence

We present the next lemma.

**Lemma 7.** *Let $P_N$ be the point set consisting of the first $N$ points of the $s$-dimensional generalized Faure sequence in base $b$. Then*

$$e^2\big(P_N; H(K_{s,\gamma})\big) \le \frac{1}{N^2} \sum_{\emptyset \ne u \subseteq \{1,\dots,s\}} \gamma_{s,u} \left(C_1 \frac{s \log_2 N}{\log_2 s}\right)^{2|u|}$$

*for all $N > 1$ and $s > 1$, where $C_1 = 8$.*

Proof. As in the proof of Lemma 4, applying Corollary 4 to (10) completes the proof. □

We now prove the main theorem on the tractability using the generalized Faure sequence.

**Theorem 5.** *Let $P_N$ be the point set of the first $N$ points of the $s$-dimensional generalized Faure sequence in base $b$, where $N > 1$ and $s > 1$.*

(A) *Consider the anchored Sobolev space $H(K_{s,\gamma,A})$ with an arbitrary anchor* $\mathbf{a}$.

- *For arbitrary finite-order weights $\{\gamma_{s,u}\}$ of order $\omega$, we have*

$$\frac{e\big(P_N; H(K_{s,\gamma,A})\big)}{e\big(0; H(K_{s,\gamma,A})\big)} \le \frac{C_2\, s^\omega \log_2^\omega N}{(\log_2^\omega s) N},$$

  *where $C_2$ is a positive constant independent of $N$ and $s$.*

- *If the finite-order weights $\{\gamma_{s,u}\}$ of order $\omega$ satisfy*

$$\sup_{s \in \mathbb{N}} \left( \frac{\sum_{u \subseteq \{1,\dots,s\}, |u| \le \omega} \gamma_{s,u} (s/\log_2 s)^{2|u|}}{\sum_{u \subseteq \{1,\dots,s\}, |u| \le \omega} \gamma_{s,u} \prod_{i \in u} m_i} \right) < \infty, \tag{13}$$

  *then for arbitrary $\delta > 0$ there exists a positive constant $C_\delta$ independent of $N$ and $s$ such that*

$$\frac{e\big(P_N; H(K_{s,\gamma,A})\big)}{e\big(0; H(K_{s,\gamma,A})\big)} \le C_\delta N^{-1+\delta}.$$

(B) *Consider the unanchored Sobolev space $H(K_{s,\gamma,B})$.*

- *For arbitrary bounded finite-order weights $\{\gamma_{s,u}\}$ of order $\omega$, we have*

$$e\big(P_N; H(K_{s,\gamma,B})\big) \le \frac{C_3\, s^{3\omega/2} \log_2^\omega N}{(\log_2^\omega s) N},$$

  *where $C_3$ is a positive constant independent of $N$ and $s$.*

- *If the finite-order weights* $\{\gamma_{s,u}\}$ *of order* $\omega$ *satisfy*

$$\sup_{s\in\mathrm{N}}\left(\sum_{u\subseteq\{1,\dots,s\},|u|\le\omega}\gamma_{s,u}(s/\log_2 s)^{2|u|}\right)<\infty, \tag{14}$$

*then for arbitrary* $\delta>0$ *there exists a positive constant* $C'_\delta$ *independent of* $N$ *and* $s$ *such that*

$$e\big(P_N;H(K_{s,\gamma,B})\big)\le C'_\delta N^{-1+\delta}.$$

P r o o f. Based on Lemma 7, the proof is done in the same way as that of Theorem 2. □

### 3.5. Strong tractability of the Xing-Niederreiter sequence and the Hofer-Niederreiter sequence

We show the next lemma.

**Lemma 8.** *Let* $P_N$ *be the point set consisting of the first* $N$ *points of the* $s$*-dimensional Xing-Niederreiter sequence in base* $b$ *and genus* $g$ *or of the* $s$*-dimensional Hofer-Niederreiter sequence in base* $b$ *and genus* $g$*. Then*

$$e^2\big(P_N;H(K_{s,\gamma})\big)\le\frac{b^{2g}}{N^2}\sum_{\emptyset\ne u\subseteq\{1,\dots,s\}}\gamma_{s,u}\prod_{i\in u}(C_1\ i\log_2 N)^2$$

*for all* $N>1$*, where* $C_1>1$ *is a constant independent of* $N$ *and* $s$*.*

P r o o f. As in the proof of Lemma 4, applying Corollary 5 to (10) completes the proof. □

Now we obtain the main theorem on the tractability using the Xing-Niederreiter sequence or the Hofer-Niederreiter sequence.

**Theorem 6.** *Let* $P_N$ *be the point set of the first* $N$ *points of the* $s$*-dimensional Xing-Niederreiter sequence in base* $b$ *and genus* $g$ *or of the* $s$*-dimensional Hofer-Niederreiter sequence in base* $b$ *and genus* $g$*, where* $N>1$*.*

(A) *Consider the anchored Sobolev space* $H(K_{s,\gamma,A})$ *with an arbitrary anchor* $\mathbf{a}$*.*

- *For arbitrary finite-order weights* $\{\gamma_{s,u}\}$ *of order* $\omega$*, we have*

$$\frac{e\big(P_N;H(K_{s,\gamma,A})\big)}{e\big(0;H(K_{s,\gamma,A})\big)}\le\frac{C_2\ s^\omega\log_2^\omega N}{N},$$

*where* $C_2$ *is a positive constant independent of* $N$ *and* $s$*.*

- *If the finite-order weights* $\{\gamma_{s,u}\}$ *of order* $\omega$ *satisfy*

$$\sup_{s\in\mathbb{N}}\left(\frac{\sum_{u\subseteq\{1,\dots,s\},|u|\le\omega}\gamma_{s,u}\prod_{i\in u}i^2}{\sum_{u\subseteq\{1,\dots,s\},|u|\le\omega}\gamma_{s,u}\prod_{i\in u}m_i}\right)<\infty, \tag{15}$$

*then for arbitrary* $\delta>0$ *there exists a positive constant* $C_\delta$ *independent of* $N$ *and* $s$ *such that*

$$\frac{e\big(P_N;H(K_{s,\gamma,A})\big)}{e\big(0;H(K_{s,\gamma,A})\big)}\le C_\delta N^{-1+\delta}.$$

(B) *Consider the unanchored Sobolev space* $H(K_{s,\gamma,B})$.

- *For arbitrary bounded finite-order weights* $\{\gamma_{s,u}\}$ *of order* $\omega$*, we have*

$$e\big(P_N;H(K_{s,\gamma,B})\big)\le\frac{C_3\ s^{3\omega/2}\log_2^{\omega}N}{N},$$

*where* $C_3$ *is a positive constant independent of* $N$ *and* $s$.

- *If the finite-order weights* $\{\gamma_{s,u}\}$ *of order* $\omega$ *satisfy*

$$\sup_{s\in\mathbb{N}}\left(\sum_{u\subseteq\{1,\dots,s\},|u|\le\omega}\gamma_{s,u}\prod_{i\in u}i^2\right)<\infty, \tag{16}$$

*then for arbitrary* $\delta>0$ *there exists a positive constant* $C'_\delta$ *independent of* $N$ *and* $s$ *such that*

$$e\big(P_N;H(K_{s,\gamma,B})\big)\le C'_\delta N^{-1+\delta}.$$

P r o o f. Based on Lemma 8, the proof is done in the same way as that of Theorem 2. □

## 4. Discussion

According to [7], the $s$-dimensional Niederreiter-Xing sequence in base $b$ and genus $g$ [12], in which only rational places are used and thereby global function fields of different genera $g$ are employed for different dimensions $s$, is a $(t,\mathbf{e},s)$--sequence in a single base $b$ such that $t=g$ and $e_1=\cdots=e_s=1$. It is an open problem to determine the tractability using points from the Niederreiter--Xing sequence. Although we know that $t=g=O(s)$, this property makes the problem intractable, because an upper bound for the worst case error must contain a multiplicative factor $b^{O(s)}$, since the $t$-value of any lower dimensional projection $P_N^{|u|}$ remains the same, i.e., $t=g$.

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