

ON A GOLAY-SHAPIRO-LIKE SEQUENCE

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Dedicated to the memory of Pierre Liardet

ABSTRACT. A recent paper by P. Lafrance, N. Rampersad, and R. Yee studies the sequence of occurrences of 10 as a scattered subsequence in the binary expansion of integers. They prove in particular that the summatory function of this sequence has the “root N ” property, analogously to the summatory function of the Golay-Shapiro sequence. We prove here that the root N property does not hold if we twist the sequence by powers of a complex number of modulus one, hence showing a fundamental difference with the Golay-Shapiro sequence.

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1. Introduction

In the paper [7] the authors study a sequence $(i_n)_{n \geq 0}$ involving the number $inv_2(n)$ of inversions in the binary expansion of the integer n , i.e., the number of occurrences of 10 as a scattered subsequence of the binary representation of the integer n . More precisely, defining $i_n := (-1)^{inv_2(n)}$ they prove in particular the following result.

THEOREM 1 (Theorem 2 in [7]). *There exists a bounded, continuous, nowhere differentiable, 1-periodic function G such that*

$$S(N) := \sum_{0 \leq n \leq N} i_n = \sqrt{N}G(\log_4 N).$$

This shows that the behavior of the summatory function of sequence $(i_n)_{n \geq 0}$ is quite similar to the behavior of the summatory function of the Golay-Shapiro sequence (see [4, 3]). Recall that the ± 1 Golay-Shapiro sequence $(a_n)_{n \geq 0}$ is

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defined by $a_n = (-1)^{w_n}$, where w_n counts the number of possibly overlapping 11's in the binary expansion of the integer n . This sequence can also be defined by $a_0 = 1$, and for all $n \geq 0$, the recurrence relations $a_{2n} = a_n$ and $a_{2n+1} = (-1)^n a_n$ (see [2]). It is then natural to ask, as the authors of [7] do, whether the sequence $(i_n)_{n \geq 0}$ satisfies the fundamental “root N ” property of the Golay-Shapiro sequence, namely

$$\sup_{\theta \in \mathbb{R}} \left| \sum_{0 \leq n \leq N} a_n e^{2i\pi n\theta} \right| = O(\sqrt{N}).$$

This question is furthermore justified not only by the fact that $(a_n)_{n \geq 0}$ admits a “digital” representation as $(i_n)_{n \geq 0}$ does (namely $a_n = (-1)^{u_n}$, where u_n is the number of possibly overlapping 11's in the binary expansion of n), but also by the fact that many other “digital” sequences have the root N property (see [1]). The purpose of this paper is to prove that the sequence $(i_n)_{n \geq 0}$ does not satisfy the root N property.

REMARK 1. The Golay-Shapiro sequence is also called the Rudin-Shapiro or the Shapiro-Rudin sequence. Since Rudin [8, p. 855] acknowledges Shapiro's priority of [9], and since [9] and [6] appeared the same year, the sequence should indeed be called the “Golay-Shapiro sequence”. Note that the fact that this sequence appears in a somewhat disguised form in the paper of Golay [6] can be found in the article of Brillhart and Morton [5] where they write that Odlyzko pointed out to them [6, bottom of p. 469].

2. Preliminary results

First we recall a property of sequence $(i_n)_{n \geq 0}$ given in [7].

PROPOSITION 1 (Proposition 1 of [7]). *The sequence $(i_n)_{n \geq 0}$ satisfies $i_0 = 1$ and the following recurrence relations: for all $n \geq 0$,*

$$i_{2n+1} = i_n, \quad i_{4n} = i_n, \quad i_{4n+2} = -i_{2n}.$$

This proposition implies the following result on the summatory function of $(i_n z^n)_{n \geq 0}$.

PROPOSITION 2. *Let z be a complex number. Define the sum $T(N, z)$ by*

$$T(N, z) := \sum_{0 \leq n \leq 2^N - 1} z^n \begin{pmatrix} i_n \\ i_{2n} \end{pmatrix}.$$

Then we have

$$T(N+1, z) = \begin{pmatrix} z & 1 \\ 1 & -z \end{pmatrix} T(N, z^2).$$

Proof. Separating even and odd indices in $T(N+1, z)$ and using Proposition 1 yields

$$\begin{aligned} T(N+1, z) &= \sum_{0 \leq 2n \leq 2^{N+1}-1} z^{2n} \begin{pmatrix} i_{2n} \\ i_{4n} \end{pmatrix} + \sum_{0 \leq 2n+1 \leq 2^{N+1}-1} z^{2n+1} \begin{pmatrix} i_{2n+1} \\ i_{4n+2} \end{pmatrix} \\ &= \sum_{0 \leq n \leq 2^N-1} z^{2n} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} i_n \\ i_{2n} \end{pmatrix} + \sum_{0 \leq n \leq 2^N-1} z^{2n+1} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} i_n \\ i_{2n} \end{pmatrix} \\ &= \begin{pmatrix} z & 1 \\ 1 & -z \end{pmatrix} T(N, z^2). \end{aligned} \quad \square$$

It happens that a single common transformation gives a simpler form for all matrices

$$\begin{pmatrix} z & 1 \\ 1 & -z \end{pmatrix}.$$

PROPOSITION 3. Let i be a square root of -1 and P be the matrix defined by

$$P := \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}.$$

Then

$$P^{-1} = \frac{1}{2} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} \quad \text{and} \quad P^{-1} \begin{pmatrix} z & 1 \\ 1 & -z \end{pmatrix} P = \begin{pmatrix} 0 & z-i \\ z+i & 0 \end{pmatrix}.$$

Proof. Straightforward. \square

PROPOSITION 4. Let $M(z) = \begin{pmatrix} z & 1 \\ 1 & -z \end{pmatrix}$. Then we have

$$\sum_{0 \leq n \leq 2^{2N}-1} i_n z^n = \frac{1}{2} (c_N(1-i) + d_N(1+i)),$$

where

$$c_N = \prod_{0 \leq k \leq N-1} (z^{4^k} - i) (z^{2 \cdot 4^k} + i) \quad \text{and} \quad d_N = \prod_{0 \leq k \leq N-1} (z^{4^k} + i) (z^{2 \cdot 4^k} - i).$$

Proof. Using Proposition 2 we have

$$T(N+1, z) = M(z) T(N, z^2) = M(z) M(z^2) T(N-1, z^4) = \dots$$

hence, starting from $T(N, z)$ yields

$$T(N, z) = M(z)M(z^2) \cdots M(z^{2^{N-1}})T(0, z^{2^N}).$$

Replacing N by $2N$, defining $\widetilde{M}(z) := \begin{pmatrix} 0 & z-i \\ z+i & 0 \end{pmatrix}$, and using Proposition 3 gives

$$\begin{aligned} T(2N, z) &= \left(P\widetilde{M}(z)P^{-1}\right) \cdots \left(P\widetilde{M}(z^{2^{2N-1}})P^{-1}\right)T(0, z^{2^{2N}}) \\ &= P\widetilde{M}(z) \cdots \widetilde{M}(z^{2^{2N-1}})P^{-1}T(0, z^{2^{2N}}) \\ &= P \begin{pmatrix} 0 & z-i \\ z+i & 0 \end{pmatrix} \cdots \begin{pmatrix} 0 & z^{2^{2N-1}}-i \\ z^{2^{2N-1}}+i & 0 \end{pmatrix} P^{-1}T(0, z^{2^{2N}}). \end{aligned}$$

Grouping the matrices in the last equality pairwise and noting that

$$\begin{pmatrix} 0 & z-i \\ z+i & 0 \end{pmatrix} \begin{pmatrix} 0 & z^2-i \\ z^2+i & 0 \end{pmatrix} = \begin{pmatrix} (z-i)(z^2+i) & 0 \\ 0 & (z+i)(z^2-i) \end{pmatrix}$$

we obtain

$$T(2N, z) = P \begin{pmatrix} c_N & 0 \\ 0 & d_N \end{pmatrix} P^{-1}T(0, z^{2^{2N}}) \quad (1)$$

with

$$c_N = \prod_{0 \leq k \leq N-1} (z^{4^k} - i)(z^{2 \cdot 4^k} + i)$$

and

$$d_N = \prod_{0 \leq k \leq N-1} (z^{4^k} + i)(z^{2 \cdot 4^k} - i) \quad (2)$$

Since for any complex number Z we have

$$T(0, Z) = Z^0 \begin{pmatrix} i_0 \\ i_0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

Equality (1) can be rewritten as

$$T(2N, z) = \frac{1}{2} \begin{pmatrix} c_N(1-i) + d_N(1+i) \\ c_N(1+i) + d_N(1-i) \end{pmatrix}. \quad (3)$$

So that we have

$$\sum_{0 \leq n \leq 2^{2N}-1} i_n z^n = \frac{1}{2} (c_N(1-i) + d_N(1+i)). \quad \square$$

3. The main result

Now we state and prove our main theorem.

THEOREM 2. *Define $j := e^{2i\pi/3}$. Let z be a complex number such that there exists an integer $r > 0$ with $z^{4^r} = j$. Then there exists a positive constant C (depending on z) such that, for all N large enough, the following inequality holds:*

$$\left| \sum_{0 \leq n \leq 2^{2N}-1} i_n z^n \right| \geq C(2 + \sqrt{3})^N.$$

In particular, for such a complex number z , $\left| \sum_{0 \leq k \leq n-1} i_k z^k \right|$ is not $O(\sqrt{n})$.

Proof. If $z^{4^r} = j$, then, for all $d \geq r$, we clearly have

$$z^{4^d} = \left(z^{4^r} \right)^{4^{r-d}} = j$$

(note that $j^3 = 1$ hence $j^4 = j$). Thus, since

$$(j - i)(j^2 + i) = 2 - \sqrt{3} \quad \text{and} \quad (j + i)(j^2 - i) = 2 + \sqrt{3},$$

we have

$$\begin{aligned} c_N &= \prod_{0 \leq k \leq N-1} \left(z^{4^k} - i \right) \left(z^{2 \cdot 4^k} + i \right) \\ &= \prod_{0 \leq k \leq r-1} \left(z^{4^k} - i \right) \left(z^{2 \cdot 4^k} + i \right) \prod_{r \leq k \leq N-1} (j - i)(j^2 + i) \\ &= A_z (2 - \sqrt{3})^N \end{aligned}$$

for some constant

$$A_z = \prod_{0 \leq k \leq r-1} \left(z^{4^k} - i \right) \left(z^{2 \cdot 4^k} + i \right) [(j - i)(j^2 + i)]^{-1} \neq 0.$$

Similarly $d_n = B_z (2 + \sqrt{3})^N$ for some nonzero constant B_z .

Finally, this gives

$$\begin{aligned} \left| \sum_{0 \leq n \leq 2^{2N}-1} i_n z^n \right| &= \frac{|d_N(1 + i) + (1 - i)c_N|}{2} \\ &\geq \frac{|(2 + \sqrt{3})^N |B_z| - |A_z|(2 - \sqrt{3})^N|}{\sqrt{2}} \\ &\geq C_z (2 + \sqrt{3})^N \end{aligned}$$

for some positive constant C_z and N large enough. □

REMARK 2. Actually Theorem 2 above shows that there exists a dense set of real numbers θ such that

$$\left| \sum_{0 \leq n \leq N-1} i_n e^{2i\pi n\theta} \right| \text{ is not } O(\sqrt{N}).$$

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