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ON STRONG NORMALITY

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Dedicated to the memory of Professor Pierre Liardet

ABSTRACT. We introduce the concept of strong normality by defining strong normal numbers and provide various properties of these numbers, including the fact that almost all real numbers are strongly normal.

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Given a fixed integer $q \geq 2$, an irrational number α is said to be a *normal number* in base q (or a q-normal number) if any preassigned sequence of k digits (in base q) appears in the q-ary expansion of α at the expected frequency, namely $1/q^k$.

Normal numbers have been studied since Borel [1] in 1909. Hence the vast literature concerning normal numbers (see for instance Champernowne [3], Copeland and Erdős [4], Davenport and Erdős [5], and the recent book of Bugeaud [2]). In a series of papers (see [6] through [11]), the first two authors obtained new results concerning normal numbers, including various ways of constructing new families of normal numbers.

In this paper, we will identify a very special family of normal numbers—that we will call *strongly normal numbers*—which are connected with arithmetical functions that have a local normal distribution, such as the function $\omega(n)$ which counts the number of distinct prime factors of n.

Let us first recall some definitions.

A sequence $(x_n)_{n\in\mathbb{N}}$ of real numbers is said to be *uniformly distributed modulo 1* (or *mod 1*) if for every interval $[\alpha,\beta) \subseteq [0,1)$,

$$\lim_{N\to\infty} \frac{1}{N}\#\big\{n \leq N : \{x_n\} \in [a,b)\big\} = b - a\,.$$

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In other words, a sequence of real numbers is said to be uniformly distributed mod 1 if every subinterval of the unit interval gets its fair share of the fractional parts of the elements of this sequence.

Recall also that, given a set of N real numbers $x_1, \ldots, x_N$, the *discrepancy* of this set is defined as the quantity

$$D(x_1, \ldots, x_N) := \sup_{[a,b) \subseteq [0,1)} \left| \frac{1}{N} \sum_{\substack{n \le N \\ \{x_n\} \in [a,b)}} 1 - (b-a) \right| .$$

It is known that a sequence $(x_n)_{n \in \mathbb{N}}$ of real numbers is uniformly distributed mod 1 if $D(x_1, \ldots, x_N) \to 0$ as $N \to \infty$ (see Theorem 1.1 in the book of Kuipers and Niederreiter [15]).

Also, given an integer $q \ge 2$, it can be shown (see Theorem 8.1 in the book of Kuipers and Niederreiter [15]) that a real number α is normal in base q if and only if the sequence $(\{q^n \alpha\})_{n \in \mathbb{N}}$ is uniformly distributed mod 1.

We are now ready to introduce the concept of *strong normality*. For each positive integer N, let

$$M = M_N := \lfloor \delta_N \sqrt{N} \rfloor, \text{ where } \delta_N \to 0 \text{ and } \delta_N \log N \to \infty \text{ as } N \to \infty . \qquad (1)$$

We shall say that an infinite sequence of real numbers $(x_n)_{n \ge 1}$ is *strongly uniformly distributed* mod 1 if

$$D(x_{N+1}, \ldots, x_{N+M}) \to 0 \quad \text{as} \quad N \to \infty$$

for every choice of δ_N satisfying (1).

Remark 1. Observe that if a sequence of real numbers $(x_n)_{n \in \mathbb{N}}$ is strongly uniformly distributed mod 1, then it must be uniformly distributed mod 1 as well. The proof goes as follows. Assume that $(x_n)_{n \in \mathbb{N}}$ is strongly uniformly distributed mod 1 and define the sequence $(\epsilon_k)_{k \in \mathbb{N}}$ by

$$\epsilon_k = \begin{cases} 1 & \text{if} \quad k \le e\,, \\ 1/\log k & \text{if} \quad k > e\,. \end{cases}$$

Also, for each integer $k \ge 1$, let $U_k = \lfloor k^2 \epsilon_k \rfloor$ and $V_k = U_{k+1} - U_k - 1$. Moreover, setting $N = U_k$ and $M = M_N = V_k$, one can verify that (1) is satisfied as $k \to \infty$. To see this, observe that

$$\begin{aligned} V_k &= (k+1)^2 \epsilon_{k+1} - k^2 \epsilon_k + O(1) = 2k\epsilon_{k+1} + k^2(\epsilon_{k+1} - \epsilon_k) + O(1) \\ &= 2k\epsilon_{k+1} + O\left(\frac{k}{\log^2 k} \right) = \big(1 + o(1)\big) 2k\epsilon_k \qquad \text{as } k \to \infty . \end{aligned} \qquad (2)$$

Now, for each $k \in \mathbb{N}$, define δ_{U_k} implicitly by $V_k = \lfloor \delta_{U_k} \sqrt{U_k} \rfloor$. Using this in (2), it follows that

$$2k\epsilon_k \big(1 + o(1)\big) = \delta_{U_k} k \sqrt{\epsilon_k} \big(1 + o(1)\big) \qquad (k \to \infty)\,,$$

from which we obtain that

$$\delta_{U_k} = \big(1 + o(1)\big) 2\sqrt{\epsilon_k} \qquad (k \to \infty)\,.$$

Hence, it follows that, as $k \to \infty$,

$$\delta_N = \delta_{U_k} \to 0 \text{ and } \delta_N \log N = \big(1+o(1)\big) 2\sqrt{\epsilon_k} \log U_k = \big(1+o(1)\big) 4\sqrt{\log k} \to \infty\,,$$

implying that condition (1) is satisfied and also, using the fact that $(x_n)_{n \in \mathbb{N}}$ is strongly uniformly distributed mod 1, that

$$D(x_{U_k}, \ldots, x_{U_{k+1}-1}) = D(x_N, \ldots, x_{N+M}) \to 0 \qquad (k \to \infty)\,. \tag{3}$$

We shall now use this result to prove that

$$D(x_1, \ldots, x_N) \to 0 \qquad (N \to \infty)\,. \tag{4}$$

To do so, for each $N \in \mathbb{N}$, let t_N be the unique integer k for which $U_k \le N < U_{k+1}$, from which it follows that

$$\frac{N - U_{t_N}}{N} \le \frac{U_{t_N+1} - U_{t_N}}{N} \to 0 \qquad (N \to \infty)\,. \tag{5}$$

With this set up, we have

$$N D(x_1, \ldots, x_N) \le \sum_{\ell=1}^{t_N - 1} (U_{\ell+1} - U_\ell) D(x_{U_\ell}, \ldots, x_{U_{\ell+1}-1}) + (N - U_{t_N})\,. \tag{6}$$

Applying (3) successively with $k = \ell$ for $\ell = 1, \ldots, t_N - 1$, it follows, in light of (5), that the right hand side of (6) is $o(N)$ as $N \to \infty$. From this, (4) follows immediately, thus proving our claim.

Remark 2. It follows from the above that if α is a strongly normal number, then it must also be a normal number. Indeed, by definition, the sequence $(\{\alpha q^n\})_{n \in \mathbb{N}}$ is strongly uniformly distributed mod 1 and therefore, in light of Remark 1, then it must be uniformly distributed mod 1, which in turn (as we saw above) is equivalent to the statement that α is a normal number.

Given a fixed integer $q \ge 2$, we say that an irrational number α is a *strongly normal number* in base q (or a *strongly q-normal number*) if the sequence $(x_n)_{n \in \mathbb{N}}$, defined by $x_n = \{q^n \alpha\}$, is strongly uniformly distributed mod 1. First, observe that there exist normal numbers which are not strongly normal. For instance, consider the Champernowne number

$$\theta := 0.1\,10\,11\,100\,101\,110\,111\,1000\,1001\,1010\,1011\,1100\,1101\,1110\,1111\,\ldots$$

that is the number made up of the concatenation of the positive integers written in base 2. It is known since Champernowne [3] that θ is normal. However, one can show that θ is not a strongly normal number. Indeed, given a positive integer n, let $S_n = \lfloor 2^n/(\sqrt{n}\log n)\rfloor$ and consider the sequence

$$2^{2n}+1, 2^{2n}+2, 2^{2n}+3, \ldots, 2^{2n}+S_n\,, \tag{7}$$

writing each of the above S_n integers in binary. Each of the resulting binary integers contains $2n+1$ digits, implying that the total number of digits appearing in the sequence (7) is equal to $(2n+1)S_n$.

Now, letting $\lambda(m)$ stand for the number of digits in the integer m, the total number N of digits of the concatenated integers preceding the number $2^{2n}+1$ is, as n becomes large,

$$N = \sum_{m\le 2^{2n}} \lambda(m) = 2n+1+\sum_{m\le 2^{2n}} \left\lceil \frac{\log m}{\log 2}\right\rceil = \big(1+o(1)\big)2n\cdot 2^{2n}. \tag{8}$$

We can write the first digits of the Champernowne number as

$$\begin{aligned}\theta &= 0.\epsilon_1\epsilon_2\ldots\epsilon_N\overline{2^{2n}+1}\,\overline{2^{2n}+2}\,\ldots\,\overline{2^{2n}+S_n}\ldots\\ &= 0.\epsilon_1\epsilon_2\ldots\epsilon_N\,\rho\;\ldots,\end{aligned}$$

say, where in fact, $\rho = \overline{2^{2n}+1}\,\overline{2^{2n}+2}\,\ldots\,\overline{2^{2n}+S_n} = \epsilon_{N+1}\ldots\epsilon_{N+\lambda(\rho)}$. (Here, $\overline{n_1}\,\overline{n_2}\ldots\overline{n_r}$ stands for the concatenation of all the digits appearing successively in the integers $n_1, n_2, \ldots, n_r$.) We will first show that the proportion of zeros in the word ρ is too large. For this we shall first count the number of 1's in ρ. Setting $\beta(m)$ as the number of 1's in the integer m, the total number of 1's in ρ is equal to

$$\sum_{m\le S_n}\beta(m) = \frac{1}{2}\frac{S_n\log S_n}{\log 2}+O(S_n)\,,$$

from which we can deduce that the total number of zeros in ρ is

$$\sum_{m=1}^{S_n} n + \sum_{m=1}^{S_n}\big(n-\beta(m)\big) = 2nS_n - \frac{1}{2}\frac{S_n\log S_n}{\log 2}+O(S_n)\,. \tag{9}$$

Since $\lambda(\rho) = (2n+1)S_n$ and recalling that $S_n = \lfloor 2^n/(\sqrt{n}\log n)\rfloor$, it follows from (9) that the proportion of zeros in ρ is equal to, as $n\to\infty$,

$$\begin{aligned}\frac{1}{\lambda(\rho)}\times\text{the number of zeros in }\rho &= \frac{2n}{2n+1}-\frac{1}{2}\frac{\log S_n}{(2n+1)\log 2}+o(1)\\ &= 1+o(1)-\frac{1}{2}\frac{n\log 2-\frac{1}{2}\log n}{(2n+1)\log 2}+o(1)\\ &= 1-\frac{1}{4}+o(1) = \frac{3}{4}+o(1)\,.\end{aligned}$$

Then, since

$$\left| \sum_{\substack{N+1 \le \nu \le N+M \\ \{2^\nu \theta\} < \frac{1}{2}}} 1 - \frac{1}{2}(2n+1)S_n \right| \ge \frac{1}{4}(2n+1)S_n \,,$$

it follows that, setting $x_n := \{2^n\theta\}$ and choosing

$$M = M_N = (2n+1)S_n \approx \sqrt{N}/\log\log N$$

(where we used (8)), thereby complying with condition (1), the discrepancy of the sequence of numbers $x_{N+1}, \ldots, x_{N+M}$ is

$$\begin{aligned} &D(x_{N+1}, \ldots, x_{N+M}) \\ &= \sup_{[a,b) \subseteq [0,1)} \frac{1}{(2n+1)S_n} \left| \sum_{\substack{N+1 \le \nu \le N+M \\ \{2^\nu \theta\} \in [a,b)}} 1 - (b-a)\big((2n+1)S_n\big) \right| \\ &\ge \frac{\frac{1}{4}(2n+1)S_n}{(2n+1)S_n} = \frac{1}{4} \end{aligned}$$

and therefore does not tend to 0, thereby implying that θ is not strongly normal.

Remark 3. Observe that instead of choosing $M_N = \lfloor \delta_N \sqrt{N} \rfloor$ as we did in (1), we could have set $M_N = \lfloor \delta_N N^\gamma \rfloor$, where γ is fixed real number belonging to the interval $(0,1)$, and then introduce the corresponding concept of a γ-strongly uniformly distributed sequence mod 1, with corresponding γ-strong normal numbers. In this case, one could easily show that if $0 < \gamma_1 < \gamma_2 < 1$, then any γ_1-strong normal number is also be a γ_2-strong normal number.

Remark 4. A further discussion on appropriate choices of M_N in the definition of strong normality is exposed in Section 9.

Identifying which real numbers are normal is not an easy task. For instance, no one has been able to prove that any of the classical constants π, e, $\sqrt{2}$ and $\log 2$ is normal, even though numerical evidence indicates that all of them are. Even constructing normal numbers is not an easy task. Hence, one might believe that constructing strongly normal numbers will even be more difficult. So, here we first show how one can construct large families of strongly normal numbers. On the other hand, it has been shown by Borel [1] that almost all real numbers are normal. Although the set of strongly normal numbers is "much smaller" than the whole set of normal numbers, in this paper, we will prove that almost all numbers are strongly normal. After studying the multidimensional case, we examine the relation between arithmetic functions with local normal distribution and strong normality.

1. A simple criteria for strong normality

Our first two propositions provide a simple criteria for strong uniform distribution mod 1 and for strong normality. They are direct consequences of the definition of strong normality.

Proposition 1. *Let $\mathcal{D}$ be the set of all continuous functions $f : [0,1] \to [0,1)$ such that $\int_0^1 f(x)\,dx = 0$. Then, the sequence $(x_n)_{n\geq 1}$ is strongly uniformly distributed mod 1 if and only if, for all $f \in \mathcal{D}$, letting $M = M_N$ be as in* (1),

$$\frac{1}{M}\sum_{j=1}^{M} f(\{x_{N+j}\}) \to 0 \qquad \text{as } N \to \infty\,.$$

Given a positive real number $\alpha < 1$ whose q-ary expansion is written as $\alpha = 0.\epsilon_1\epsilon_2\ldots$, where each $\epsilon_j \in \mathcal{A}_q := \{0,1,\ldots,q-1\}$. For an arbitrary word $\beta = \delta_1\ldots\delta_k \in \mathcal{A}_q^k$, let $R_{N,M}(\beta)$ stand for the number of times that the word β appears as a subword of the word $\epsilon_{N+1}\ldots\epsilon_{N+M}$.

Proposition 2. *A positive real number $\alpha < 1$ is strongly q-normal if and only if, given an arbitrary word $\beta = \delta_1\ldots\delta_k \in \mathcal{A}_q^k$ and $M = M_N$ as in* (1),

$$\lim_{N\to\infty}\frac{R_{N,M}(\beta)}{M} = \frac{1}{q^k}\,.$$

2. The construction of strongly normal numbers

We first show how one can go about constructing strongly normal numbers. One way is as follows. First, we start with a normal number in base $q \geq 2$, say $\alpha = 0.\epsilon_1\epsilon_2\ldots$, and then for each positive integer T, we consider the corresponding word $\alpha_T = \epsilon_1\epsilon_2\ldots\epsilon_T$. One can show that, if the sequences of integers $T_1 < T_2 < \cdots$ and $m_1 < m_2 < \cdots$ are chosen appropriately, and if, for short, we write γ^m for the concatenation of m times the word γ, that is $\gamma^m = \underbrace{\gamma\ldots\gamma}_{m \text{ times}}$, then the number

$$\beta = 0.\alpha_{T_1}^{m_1}\alpha_{T_2}^{m_2}\ldots$$

is a strongly normal number in base q.

We first show that the choice $T_\ell = \ell$ and $m_\ell = \ell$ is an appropriate one and in fact we state this as a proposition.

Proposition 3. *Let α be a q-normal number. Then, using the above notation, the number*

$$\beta = 0.\alpha_1^1\alpha_2^2\alpha_3^3\ldots$$

is a strongly normal number in base q.

Proof. Given a word $\gamma = c_1 \ldots c_r \in \mathcal{A}_q^r$ and an arbitrary word $\delta_1 \ldots \delta_h \in \mathcal{A}_q^h$, let $E_\gamma(\delta_1 \ldots \delta_h)$ be the number of occurrences of γ as a subword in $\delta_1 \ldots \delta_h$ and let

$$\Delta_\gamma(\delta_1 \ldots \delta_h) := \left| E_\gamma(\delta_1 \ldots \delta_h) - \frac{h}{q^r} \right| .$$

On the other hand, let $\kappa_1, \kappa_2, \ldots$ stand for the q-ary digits of β, so that $\beta = 0.\kappa_1\kappa_2\ldots$, and let $M = M_N$ be as in (1). Finally set

$$\mu := \kappa_{N+1}\kappa_{N+2}\ldots\kappa_{N+M}.$$

We will count how many times the word γ occurs as a subword of μ.

Denoting by $\lambda(\gamma)$ the length of the word γ, observe that

$$\lambda(\alpha_1^1\alpha_2^2\ldots\alpha_R^R) = \sum_{\ell=1}^{R} \ell^2 = \frac{R(R+1)(2R+1)}{6} \qquad (R \in \mathbb{N}) .$$

Setting

$$K_R := \frac{R(R+1)(2R+1)}{6} \qquad (R \in \mathbb{N}) ,$$

it is easily seen that no more than one K_ν is located in the interval $[N+1, N+M]$. Indeed, let us show that

$$\text{if} \quad N < K_\nu \le N + M, \quad \text{then} \quad K_{\nu+1} > N + M. \tag{10}$$

Indeed, it is clear that

$$K_{\nu+1} = K_\nu + (\nu+1)^2 > N + (\nu+1)^2 \qquad (\nu \ge 1) \tag{11}$$

and that, since $\nu^3 > K_\nu > N$, it follows that $\nu > N^{1/3}$, so that $(\nu+1)^2 > N^{2/3}$, which combined with (11) implies that

$$K_{\nu+1} > N + N^{2/3} > N + \lfloor \delta_N \sqrt{N} \rfloor = N + M,$$

thus proving our claim (10).

Now, assume that N is large and let R be the largest integer such that $K_R \le N + M$. We then have two distinct possibilities:

Case I : $N \le K_R$;

Case II : $K_R < N$.

If Case II holds, then

$$E_\gamma(\mu) = \frac{M}{R+1} E_\gamma(\alpha_{R+1}) + O(R) + O\left(\frac{M}{R}\right),$$

from which it follows that

$$\Delta_\gamma(\mu) \leq \frac{M}{R+1} \Delta_\gamma(\alpha_{R+1}) + O\left(R + \frac{M}{R}\right). \tag{12}$$

Because α is normal, we have that $\dfrac{\Delta_\gamma(\alpha_R)}{R} \to 0$ as $R \to \infty$, while on the other hand, $\left(R + \dfrac{M}{R}\right) \cdot \dfrac{1}{M} \to 0$ as $M \to \infty$. Using this in (12), it follows that

$$\frac{1}{M} \left| E_\gamma(\mu) - \frac{M}{q^r} \right| \to 0 \qquad \text{as } M \to \infty\,,$$

so that in light of Proposition 2, the number β is strongly normal in base q.

Since Case I can be handled in a similar way, the proposition is proved. □

Remark 5. Other choices of T_ℓ and m_ℓ can also lead to the construction of strongly normal numbers. For instance, let $R > 0$ be a fixed integer and, for each real number $x > 0$, define

$$x_1 := \log_+ x = \max(1, \log x), \qquad x_{\ell+1} = \log_+ x_\ell \qquad (\ell = 1, 2, \dots)\,.$$

Given a real number

$$\alpha = 0.\epsilon_1\epsilon_2 \ldots \in \mathcal{A}_q^{\mathbb{N}}\,,$$

set

$$F(\alpha; \beta) = \#\{(\gamma_1, \gamma_2) : \alpha = \gamma_1\beta\gamma_2\},$$

that is the number of occurrences of the word β in the digits of the word α. One can construct a real number α such that, for every integer $k \geq 1$,

$$\max_{\beta \in \mathcal{A}_q^k} \left| \frac{1}{M_N} F(\epsilon_{N+1} \ldots \epsilon_{N+M_N}; \beta) - \frac{1}{q^k} \right| \to 0 \quad \text{as } N \to \infty\,. \tag{13}$$

Indeed, for each integer $\ell \geq 1$, let us choose $T_\ell = \ell$ and $m_\ell = 2^{2^{\cdot^{\cdot^{\cdot^{2^\ell}}}}}$, that is

$$\ell = \underbrace{\log_2 \log_2 \ldots \log_2}_{R+1 \text{ times}} m_\ell\,.$$

Now, starting with a q-ary normal number $\gamma = 0.\epsilon_1\epsilon_2 \ldots$, and, for each positive integer T, set $\gamma_T = 0.\epsilon_1\epsilon_2 \ldots \epsilon_T$. Then, one can show that the number

$$\beta = 0.\gamma_1^{m_1}\gamma_2^{m_2} \ldots$$

does indeed satisfy condition (13) and is therefore a strongly q-normal number.

3. Preliminary lemmas

The classical *Borel-Cantelli lemma* can be stated as follows.

Lemma 1. *Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $A_1, A_2, \ldots$ be a list of the elements of $\mathcal{F}$. Let $E = \{x : x$ belongs to infinitely many A_j's$\}$. Assuming that*

$$\sum_{j=1}^{\infty} P(A_j) < \infty\,,$$

then $P(E) = 0$.

Given a probability space $(\Omega, \mathcal{F}, P)$, we say that $A_1, A_2, \ldots$ is a list of *completely independent elements* of $\mathcal{F}$ if, given any finite increasing sequence of integers, say $i_1 < i_2 < \cdots < i_k$, we have

$$P(A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_k}) = P(A_{i_1})P(A_{i_2}) \cdots P(A_{i_k})\,.$$

The second Borel-Cantelli lemma can be considered as the converse of the classical Borel-Cantelli lemma. It can be stated as follows.

Lemma 2. *Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $A_1, A_2, \ldots$ be a list of completely independent elements of $\mathcal{F}$. Letting E be as in Lemma 1 and assuming that*

$$\sum_{j=1}^{\infty} P(A_j) = \infty\,,$$

then $P(E) = 1$.

A real number is *simply normal* in base q if in its base q expansion, every digit $0, 1, \ldots, q-1$ occurs with the same frequency $1/q$. The following lemma offers a simple way of establishing if a given real number is a normal number.

Lemma 3. *Let $q \geq 2$ be an integer. If a real number α is simply normal in base q^r for each $r \in \mathbb{N}$, then α is normal in base q.*

Proof. A proof of this result can be found in the book of Kuipers and Niederreiter [15]. □

In the spirit of Proposition 2, we will say that a real number $\alpha < 1$ is a *simply strong normal number in base q* if for every digit $d \in \mathcal{A}_q$,

$$\lim_{N \to \infty} \frac{R_{N,M}(d)}{M} = \frac{1}{q}\,.$$

Lemma 4. *Let $q \geq 2$ be an integer. If a real number α is a simply strong normal in base q^r for each $r \in \mathbb{N}$, then α is strongly normal in base q.*

Proof. This result can be proved along the same lines as one would use to prove Lemma 3. □

Lemma 5. *For each integer $k \geq 1$, let*

$$\pi_k(x) := \#\{n \leq x : \omega(n) = k\}\,.$$

Then, the relation

$$\pi_k(x) = \big(1 + o(1)\big)\frac{x}{\log x}\frac{(\log\log x)^{k-1}}{(k-1)!} \qquad (x \to \infty)$$

holds uniformly for

$$|k - \log\log x| \leq \frac{1}{\delta_x}\sqrt{\log\log x}\,, \tag{14}$$

where δ_x is some function of x chosen appropriately and which tends to 0 as $x \to \infty$.

Proof. This follows from Theorem 10.4 stated in the book of De Koninck and Luca [12]. □

Lemma 6. *Letting δ_x be as in the statement of Lemma 5,*

$$\max_{\substack{k \ \text{satisfying (14)} \\ \ell \in [0, \lceil \delta_x^{3/2}\sqrt{\log\log x}\,\rceil]}} \left|\frac{\pi_{k+\ell}(x)}{\pi_k(x)} - 1\right| \to 0 \qquad \text{as } x \to \infty\,.$$

Proof. Given k satisfying (14), let θ_k be defined implicitly by $k = \log\log x + \theta_k$, and let $\ell \in [0, \lceil \delta_x^{3/2}\sqrt{\log\log x}\,\rceil]$. Then, in light of Lemma 5, we have, as $x \to \infty$,

$$\begin{aligned}
\frac{\pi_{k+\ell}(x)}{\pi_k(x)} &= \big(1+o(1)\big)\frac{(\log\log x)^\ell}{k^\ell \prod_{\nu=0}^{\ell-1}\left(1+\frac{\nu}{k}\right)} \\
&= \big(1+o(1)\big)\left(\frac{\log\log x}{k}\right)^\ell \exp\left\{-\frac{\ell(\ell-1)}{2k} + O\left(\frac{\ell^3}{k^2}\right)\right\} \\
&= \big(1+o(1)\big)\left(\frac{1}{1+\theta_k/\log\log x}\right)^\ell \big(1+o(1)\big) \\
&= \big(1+o(1)\big)\exp\left\{-\frac{\ell\theta_k}{\log\log x} + O\left(\frac{\ell\theta_k^2}{(\log\log x)^2}\right)\right\} \\
&= 1 + o(1),
\end{aligned}$$

thereby completing the proof of Lemma 6. □

For any particular set of primes $\mathcal{P}$, we introduce the expressions

$$\Omega_{\mathcal{P}}(n) := \sum_{\substack{p^a \,\|\, n \\ p\in\mathcal{P}}} a \quad \text{and} \quad E(x) := \sum_{\substack{p\le x \\ p\in\mathcal{P}}} \frac{1}{p}\,. \tag{15}$$

The following two results, which we also state as lemmas, are due respectively to Halász [13] and Kátai [14].

Lemma 7. (Halász) *Let $0<\delta\le 1$ and let $\mathcal{P}$ be a set of primes with corresponding functions $\Omega_{\mathcal{P}}(n)$ and $E(x)$ given in (15). Then, the estimate*

$$\sum_{\substack{n\le x \\ \Omega_{\mathcal{P}}(n)=k}} 1 = \frac{xE(x)^k}{k!} e^{-E(x)} \left\{ 1 + O\left(\frac{|k-E(x)|}{E(x)}\right) + O\left(\frac{1}{\sqrt{E(x)}}\right)\right\}$$

holds uniformly for all integers k and real numbers $x\ge 3$ satisfying

$$E(x) \ge \frac{8}{\delta^3} \quad \textit{and} \quad \delta \le \frac{k}{E(x)} \le 2-\delta\,.$$

Lemma 8. (Kátai) *For $1\le h\le x$, let*

$$\begin{aligned} A_k(x,h) &:= \sum_{\substack{x\le n\le x+h \\ \omega(n)=k}} 1\,, \\ \delta_k(x,h) &:= \frac{A_k(x,h)}{h} - \frac{\pi_k(x)}{x}\,, \\ E(x,h) &:= \sum_{k=1}^{\infty} \delta_k^2(x,h)\,. \end{aligned}$$

Letting $\varepsilon>0$ be an arbitrarily small number and $x^{7/12+\varepsilon}\le h\le x$, then

$$E(x,h) \ll \frac{1}{\log^2 x\cdot\sqrt{\log\log x}}\,.$$

4. Main results

Theorem 1. *The Lebesgue measure of the set of all those real numbers $\alpha\in[0,1]$ which are not strongly q-normal is equal to 0.*

Let r be a fixed positive integer and set $E:=[0,1)^r$. Consider an r dimensional sequence $(\underline{x}_n)_{n\in\mathbb{N}} := \big(x_1^{(n)},\dots,x_r^{(n)}\big)_{n\in\mathbb{N}}$ in $\mathbb{R}^r$. This sequence is said to be *uniformly distributed mod E* if, for all intervals $[a_j, b_j) \subseteq [0,1)$, $j = 1, \dots, r$, we have

$$\lim_{N\to\infty} \frac{1}{N} \# \left\{ n \le N : \{x_j^{(n)}\} \in [a_j, b_j) \text{ for } j = 1, \dots, r \right\} = \prod_{j=1}^{r} (b_j - a_j) .$$

Accordingly, the *discrepancy* of the finite sequence $\underline{x}_1, \dots, \underline{x}_N$ in $\mathbb{R}^r$ is defined as

$$D(\underline{x}_1, \dots, \underline{x}_N) = \sup_{\substack{[a_j,b_j)\subseteq[0,1) \\ j=1,\dots,r}} \left| \frac{1}{N} \sum_{\substack{\{x_j^{(n)}\}\in[a_j,b_j) \\ j=1,\dots,r}} 1 - \prod_{j=1}^{r} (b_j - a_j) \right| .$$

Then, we shall say that an infinite sequence $(\underline{x}_n)_{n\in\mathbb{N}}$ is *strongly uniformly distributed mod E* if

$$D(\underline{x}_N, \dots, \underline{x}_{N+M}) \to 0 \qquad \text{as } N \to \infty$$

for every choice of δ_N satisfying (1).

In what follows, we let $q_1, \dots, q_r$ be fixed integers ≥ 2.

Theorem 2. *The Lebesgue measure of the set of all those r-tuples $(\alpha_1, \dots, \alpha_r) \in [0,1)^r$ for which the sequence $(\underline{x}_n)_{n\in\mathbb{N}}$, where $\underline{x}_n := (\{\alpha_1 q_1^n\}, \dots, \{\alpha_r q_r^n\})$, is not strongly uniformly distributed in $[0,1)^r$ is equal to 0.*

Theorem 3. *Assume that for each $i = 1, 2, \dots, r$, the number α_i is strongly q_i-normal. Let $E = [0,1)^r$ and assume that f is a continuous periodic function mod E and that it satisfies $\int_0^1 \cdots \int_0^1 f(x_1, \dots, x_r)\, dx_1 \cdots dx_r = 0$. Further set*

$$y_n = f\left(\alpha_1 q_1^{\omega(n)}, \dots, \alpha_r q_r^{\omega(n)}\right) \qquad (n = 1, 2, \dots) .$$

Then,

$$\frac{1}{x} \sum_{n\le x} y_n \to 0 \qquad \text{as } x \to \infty . \tag{16}$$

Moreover, further defining $\underline{z}_n := \left(\{\alpha_1 q_1^{\omega(n)}\}, \dots, \{\alpha_r q_r^{\omega(n)}\}\right)$ for $n = 1, 2, \dots$, we have that $(\underline{z}_n)_{n\in\mathbb{N}}$ is uniformly distributed in E.

The following result is a direct consequence of Theorem 3 and is related to the result stated in Lemma 7.

Theorem 4. *Let g be any one of the arithmetic functions*

$$\omega(n) := \sum_{p|n} 1, \qquad \Omega(n) := \sum_{p^a \| n} a, \qquad \Omega_{\mathcal{P}}(n) := \sum_{\substack{p^a \| n \\ p \in \mathcal{P}}} a$$

and let

$$\underline{x}_n := \left(\{\alpha_1 q_1^{g(n)}\}, \ldots, \{\alpha_r q_r^{g(n)}\} \right).$$

Then, for almost all $(\alpha_1, \ldots, \alpha_r) \in [0,1)^r$, the sequence $(\underline{x}_n)_{n \geq 1}$ is uniformly distributed in $[0,1)^r$.

The following result is a consequence of Lemma 8 and we shall omit its proof since it is essentially along the same lines as that of Theorem 3.

Theorem 5. *For each integer $i = 1, \ldots, r$, assume that α_i is strongly q_i-normal and set*

$$\underline{x}_n := \left(\{\alpha_1 q_1^{\omega(n)}\}, \ldots, \{\alpha_r q_r^{\omega(n)}\} \right).$$

Then, with $M = \lfloor N^{7/12+\varepsilon} \rfloor$,

$$D(\underline{x}_{N+1}, \ldots, \underline{x}_{N+M}) \to 0 \qquad \textit{as } N \to \infty.$$

5. Proof of Theorem 1

Theorem 1 will follow immediately from the following lemma.

Lemma 9. *Let $(\Omega, \mathcal{A}, P)$ be a probability space, where $\Omega = [0,1)$, let $\mathcal{A}$ be the ring of Borel sets and let P be the Lebesgue measure. Let $q \geq 2$ be a fixed integer and set $\mathcal{A}_q := \{0, 1, \ldots, q-1\}$. Let $\epsilon_n \in \mathcal{A}_q$, $n = 1, 2, \ldots$, be independent random variables such that $P(\epsilon_n = a) = 1/q$ for each $a \in \mathcal{A}_q$. For each $\omega \in \Omega$, let*

$$\alpha(\omega) := 0.\epsilon_1(\omega)\epsilon_2(\omega)\ldots$$

For an arbitrary $\delta > 0$, let

$$E_\delta := \left\{ \omega \in \Omega : \limsup_{N \to \infty} \max_{d \in \mathcal{A}_q} \left| \frac{1}{M} \sum_{\substack{n=N+1 \\ \epsilon_n = d}}^{N+M} 1 - \frac{1}{q} \right| > \delta \right\},$$

where M satisfies (1)*. Then,*

$$P(E_\delta) = 0 \quad \textit{for every } \delta > 0. \tag{17}$$

Moreover, setting

$$E^* := \left\{ \omega \in \Omega : \limsup_{N\to\infty} \max_{d\in\mathcal{A}_q} \left| \frac{1}{M} \sum_{\substack{n=N+1 \\ \epsilon_n=d}}^{N+M} 1 - \frac{1}{q} \right| \neq 0 \right\},$$

we have $P(E^*) = 0$.

Proof of Lemma 9. Let $U \in \mathbb{N}$ and given any $d \in \mathcal{A}_q$, let

$$\alpha_d(\epsilon_1, \dots, \epsilon_U) = \sum_{\substack{i\in\{1,\dots,U\} \\ \epsilon_i=d}} 1 .$$

leftline It is clear that

$$P\big(\alpha_d(\epsilon_1, \dots, \epsilon_U) = j\big) = \frac{1}{q^U} \binom{U}{j} (q-1)^{U-j}.$$

For each $0 < \delta < 1/q$, set

$$S = S(\delta) := \left\{ \omega \in \Omega : \max_{d\in\mathcal{A}_q} \left| \alpha_d(\epsilon_1, \dots, \epsilon_U) - \frac{U}{q} \right| > \delta U \right\}. \tag{18}$$

If $\omega \in S$, then clearly the inequality

$$\alpha_d(\epsilon_1, \dots, \epsilon_U) < \frac{U}{q} - \delta \frac{U}{q}$$

holds for at least one $d \in \mathcal{A}_q$, in which case we have

$$\begin{aligned} P(S) &\le \frac{q}{q^U} \sum_{0\le j\le(1-\delta)U/q} \binom{U}{j} \cdot (q-1)^{U-j} \\ &= q \left(1 - \frac{1}{q}\right)^U \sum_{0\le j\le V} \binom{U}{j} \frac{1}{(q-1)^j}, \quad \text{where } V = \lfloor (1-\delta)U/q \rfloor . \end{aligned} \tag{19}$$

Now let

$$t_j = \binom{U}{j} \frac{1}{(q-1)^j} \qquad (j = 0, 1, \dots, V).$$

Then, for each integer $j \ge 1$, we have

$$\frac{t_{j-1}}{t_j} = (q-1) \frac{j}{U-j+1} < \frac{(q-1)(1-\delta)U/q}{U+1-(1-\delta)U/q} < \frac{(q-1)(1-\delta)}{q-(1-\delta)} < 1-\delta,$$

so that $t_{j-1} < (1-\delta)t_j$, thus implying that

$$\sum_{0\le j\le V} t_j \le t_V \left(1 + (1-\delta) + (1-\delta)^2 + \cdots \right) = \frac{t_V}{\delta}.$$

Using the Stirling formula in the form

$$\log n! = n\log(n/e) + \frac{1}{2}\log(2\pi n) + \theta_n \qquad \text{with } \theta_n \to 0$$

and setting $V=\kappa U$, where $\kappa = \dfrac{\lfloor \frac{1-\delta}{q} U\rfloor}{U} = \dfrac{1-\delta}{q} + O\left(\dfrac{1}{U}\right)$, we then have

$$\begin{aligned}
\log t_V &= U\log U - \kappa U\log(\kappa U) - (1-\kappa)U\log\big((1-\kappa)U\big) - \kappa U\log(q-1)\\
&\quad +\frac{1}{2}\log\frac{1}{\kappa(1-\kappa)} - \frac{1}{2}\log(2\pi) + O(\theta_V)\\
&= \big(-\kappa\log\kappa - (1-\kappa)\log(1-\kappa) - \kappa\log(q-1)\big)U\\
&\quad +\frac{1}{2}\log\frac{1}{\kappa(1-\kappa)} - \frac{1}{2}\log(2\pi) - \frac{1}{2}\log U + O(\theta_V)\,.
\end{aligned}$$

Letting $h(\kappa) = \kappa\log\dfrac{1}{(q-1)\kappa} + (1-\kappa)\log\dfrac{1}{1-\kappa}$, it follows that

$$\log t_V = Uh(\kappa) + \frac{1}{2}\log\frac{1}{\kappa(1-\kappa)} - \frac{1}{2}\log(2\pi) - \frac{1}{2}\log U + O(\theta_V)\,.$$

Observe that

$$h(1/q) = \log\frac{q}{q-1} \quad \text{and} \quad h(\kappa) < \big(1-c(\delta)\big)\log\frac{q}{q-1}\,,$$

where $c(\delta)>0$ provided $\delta>0$.

Using this in (19), we obtain that

$$\begin{aligned}
P(S) &\leq q\exp\left\{U\log(1-1/q) + \log V + U\big(1-c(\delta)\big)\log\frac{q}{q-1}\right\}\\
&< \exp\{-c_1(\delta)U\}\,, \qquad (20)
\end{aligned}$$

where $c_1(\delta)>0$ is some constant depending only on δ and q.

For each integer $r\geq 1$, let $N_r = q^r$ and consider the interval $\mathcal{L}_r = [N_r, N_{r+1}-1]$. Let us cover a given interval $\mathcal{L}_r$ by the union of $K_r := 1 + \left\lfloor \frac{(q-1)q^r}{r^2}\right\rfloor$ consecutive intervals $\mathcal{T}_1^{(r)}, \mathcal{T}_2^{(r)}, \ldots, \mathcal{T}_{K_r+1}^{(r)}$, each of length $U_r := r^2$. Now, we define the sets $S_i^{(r)}$, for $i=1,\ldots,K_r+1$, as we did for the set S in (18), but this time with the independent variables

$$\epsilon_{N_r+(i-1)U_r+\ell} \qquad (\ell = 1,2,\ldots,U_r)\,.$$

For these new independent variables, if we proceed as we did to obtain (20), then we have

$$P\big(S_i^{(r)}\big) \leq q^r\exp\left\{-c_1(\delta)r^2\right\} \quad (i=1,\ldots,K_r+1)\,,$$

so that

$$P\left(\bigcup_{i=1}^{K_r+1} S_i^{(r)}\right) \ll K_r q^r \exp\left\{-c_1(\delta)r^2\right\} \le \frac{q^{2r+1}}{r^2}\exp\left\{-c_1(\delta)r^2\right\}$$
$$= \exp\left\{-c_1(\delta)r^2 + (2r+1)\log q - 2\log r\right\} < \frac{1}{r^3}\,,$$

provided r is sufficiently large.

Since the series $\sum 1/r^3$ converges, we may apply Lemma 1 and conclude that the set

$$E_\delta := \#\left\{\omega : \omega \in \bigcup_{i=1}^{K_r+1} S_i^{(r)} \text{ for infinitely many } r\right\}$$

is such that $P(E_\delta) = 0$, thus establishing (17). From this result, then it follows also that $P(E^*) = 0$. □

6. Proof of Theorem 2

The proof is quite straightforward. Indeed, first set $Q := q_1q_2\cdots q_r$ and let $\alpha = 0.a_1a_2\ldots$ be a strongly Q-normal number, where each a_j satisfies

$$a_j \in \mathcal{A}_Q\,, \qquad a_j \equiv b_j^{(\ell)} \pmod{q_\ell} \quad \text{with } b_j^{(\ell)} \in \mathcal{A}_{q_\ell} \qquad (\ell = 1,\ldots,r)\,.$$

Writing

$$\alpha_\ell = 0.b_1^{(\ell)}b_2^{(\ell)}\ldots \qquad (\ell = 1,\ldots,r)\,,$$

then each α_ℓ is a strongly q_ℓ-normal number. This means that the sequence $(x_n)_{n\in\mathbb{N}}$ defined by

$$x_n := \Big(\{\alpha_1 q_1^n\},\ldots,\{\alpha_r q_r^n\}\Big) \qquad (n = 1,2\ldots)$$

is strongly uniformly distributed mod $[0,1)^r$, thus completing the proof of Theorem 2.

7. Proof of Theorem 3

Let x be a large number and let us set $S := \lfloor \delta_x^{3/2}\sqrt{\log\log x}\rfloor$, where δ_x is as in Lemma 5. Moreover, for each positive integer $m \le 1/\delta_x^{5/2}$, let us consider the interval

$$U_m := \big[\lfloor\log\log x\rfloor + mS, \lfloor\log\log x\rfloor + (m+1)S - 1\big] = [t_m, t_{m+1}-1]\,,$$

say.

For each integer $k \geq 1$, let $u_k := f(\alpha_1 q_1^k, \ldots, \alpha_r q_r^k)$. Observe that

$$\begin{aligned}
\sum_{n \leq x} y_n &= \sum_{k \geq 1} u_k \pi_k(x) \\
&= \sum_{|k - \log\log x| \leq \sqrt{\log\log x}/\delta_x} u_k \pi_k(x) + \sum_{|k - \log\log x| > \sqrt{\log\log x}/\delta_x} u_k \pi_k(x) \\
&= S_1(x) + S_2(x)\,, \qquad (21)
\end{aligned}$$

say.

It follows from the Turán-Kubilius inequality that

$$\frac{1}{x} S_2(x) \to 0 \qquad \text{as } x \to \infty\,. \qquad (22)$$

For the evaluation of $S_1(x)$, we proceed as follows. Let x be a large number. Then, for each positive integer $m \leq 1/\delta_x^{5/2}$, let us consider the interval

$$U_m := \big[\lfloor \log\log x \rfloor + mS, \lfloor \log\log x \rfloor + (m+1)S - 1\big] = [t_m, t_{m+1} - 1]\,,$$

say. We then have

$$S_1(x) = \sum_{|m| \leq 1/\delta_x^{5/2}} \sum_{k \in U_m} u_k \pi_k(x) = \sum_{|m| \leq 1/\delta_x^{5/2}} S^{(m)}(x)\,, \qquad (23)$$

say.

Using Lemma 6, it follows that, as x becomes large,

$$\left| S^{(m)}(x) - \pi_{t_m}(x) \sum_{k \in U_m} u_k \right| \leq o(1) \pi_{t_m}(x) \sum_{k \in U_m} 1\,. \qquad (24)$$

Since

$$\pi_{t_m}(x) \sum_{k \in U_m} 1 = \big(1 + o(1)\big) \sum_{k \in U_m} \pi_k(x)\,,$$

it follows that

$$\pi_{t_m}(x) \sum_{k \in U_m} u_k = \big(1 + o(1)\big) \left\{ \sum_{k \in U_m} \pi_k(x) \right\} \frac{1}{\sum_{k \in U_m} 1} \sum_{k \in U_m} u_k\,. \qquad (25)$$

Now the fact that each α_i is strongly q_i-normal for $i = 1, \ldots, r$ implies that

$$\frac{1}{\sum_{k \in U_m} 1} \sum_{k \in U_m} u_k = o(1) \qquad \text{for each } m \leq 1/\delta_x^{5/2}.$$

Combining this with (24) and (25), it follows that

$$\left| S^{(m)}(x) \right| \leq o(1) \sum_{k \in U_m} \pi_k(x)\,,$$

which substituted in (23) yields

$$S_1(x) \leq o(1) \sum_{|m| \leq 1/\delta_x^{5/2}} \sum_{k \in U_m} \pi_k(x) = o(x) \,. \tag{26}$$

Using (22) and (26) in (21) completes the proof of (16). The second part of Theorem 3 then immediately follows from (16).

8. Final remarks

When we introduced the notion of *strongly normal number in base* q, we chose for simplicity to consider intervals $[N+1, N+M]$ with $M = \lfloor \delta_N \sqrt{N} \rfloor$. However, it is interesting to observe that we could have chosen much smaller intervals, namely with $M = \lfloor \log^2 N \rfloor$, and nevertheless still preserve the property that almost all real numbers are strongly normal. Indeed, following the proof used in Lemma 9, as we consider an arbitrary sequence of digits $\epsilon_{N+1}\epsilon_{N+2}\ldots\epsilon_{N+M}$, with $M = \lfloor \log^2 N \rfloor$, and examine the occurrence of an arbitrary digit $d \in \mathcal{A}_q$ in this sequence, we could define r as the unique integer such that $q^r \leq n < q^{r+1}$, in which case we would have

$$r^2 \leq \left(\frac{\log n}{\log q} \right)^2 < (r+1)^2.$$

In the end, we would see that

$$\left| \frac{1}{\log^2 n} \sum_{\substack{\nu=n+1 \\ \epsilon_\nu = d}}^{n+\lfloor \log^2 n \rfloor} 1 - \frac{1}{q} \right| > \delta$$

holds only for finitely many n's and that this is true for each $\delta > 0$. We can conclude from this that, for almost all α,

$$\lim_{n\to\infty} \max_{d \in \mathcal{A}_q} \left| \frac{1}{\log^2 n} \sum_{\substack{\nu=n+1 \\ \epsilon_\nu = d}}^{n+\lfloor \log^2 n \rfloor} 1 - \frac{1}{q} \right| = 0 \,,$$

thus also establishing that we could have defined the notion of *strongly normal numbers* with $M = \lfloor \log^2 N \rfloor$ instead of with $M = \lfloor \delta_N \sqrt{N} \rfloor$.

Now, could we have chosen M even smaller, say $M = \lfloor \log N \rfloor$? Not really! Indeed, assume that $(\epsilon_n)_{n \geq 1}$ are independent random variables such that

$P(\epsilon_n = a) = 1/q$ for each $a \in \mathcal{A}_q$. For $N \in \mathbb{N}$, let $H = H_N = \left\lfloor \dfrac{q^{N+1} - q^N}{N} \right\rfloor$ and the set

$$B_\ell^{(N)} := \{\omega : \epsilon_{q^N+\ell N+\nu} = 0,\ \nu = 0,1,\ldots,N-1\} \qquad (\ell = 0,1,\ldots,H-1)\,.$$

The events $B_\ell^{(N)}$ $(\ell = 0,1,\ldots,H-1)$ are independent and $P\left(B_\ell^{(N)}\right) = 1/q^N$. Hence, with $D_N = \bigcup_{\ell=0}^{H-1} B_\ell^{(N)}$, we have

$$P(D_N) = \frac{H}{q^N} \geq \frac{1}{2^N}\,.$$

On the other hand, $D_1, D_2, \ldots$ are independent and $\sum_{N=1}^{\infty} P(D_N) = \infty$. Hence, by the second Borel-Cantelli lemma (see Lemma 2), we may conclude that for almost all events ω, there exists an infinite sequence of N's, say $n_1, n_2, \ldots$ such that

$$\epsilon_{n_\nu+1} = 0, \quad \epsilon_{n_\nu+2} = 0, \ldots, \epsilon_{n_\nu+m_\nu} = 0, \quad \text{where} \quad m_\nu \geq c\frac{\log n_\nu}{\log q}\,.$$

We have thus shown that one could encounter a normal number α with sequences of digits covering intervals of the form $[N+1, N+M]$, with $M \approx \log N$, made up only of zeros.

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REFERENCES

[1] BOREL, E.: *Les probabilités dénombrables et leurs applications arithmétiques*, Rend. Circ. Mat. Palermo **27** (1909), 247–271.

[2] BUGEAUD, Y.: *Distribution modulo one and Diophantine approximation*. Cambridge Tracts in Math. Vol. 193. Cambridge University Press, Cambridge, 2012.

[3] CHAMPERNOWNE, D. G.: *The construction of decimals normal in the scale of ten*, J. London Math. Soc. **8** (1933), 254–260.

[4] COPELAND, A. H.—ERDŐS, P.: *Note on normal numbers*, Bull. Amer. Math. Soc. **52** (1946), 857–860.

[5] DAVENPORT, H.—ERDŐS, P.: *Note on normal decimals*, Canad. J. Math. **4** (1952), 58–63.

[6] DE KONINCK, J. M.—KÁTAI, I.: *Construction of normal numbers by classified prime divisors of integers*, Funct. Approx. Comment. Math. **45** (2011), no. 2, 231–253.

[7] DE KONINCK, J. M.—KÁTAI, I.: *On a problem on normal numbers raised by Igor Shparlinski*, Bull. Aust. Math. Soc. **84** (2011), 337–349.

[8] DE KONINCK, J. M.—KÁTAI, I.: *Normal numbers created from primes and polynomials*, Unif. Distrib. Theory **7** (2012), no. 2, 1–20.

[9] DE KONINCK, J. M.—KÁTAI, I.: *Some new methods for constructing normal numbers*, Ann. Sci. Math. Québec **36** (2012), no. 2, 349–359.

[10] DE KONINCK, J. M.—KÁTAI, I.: *Construction of normal numbers by classified prime divisors of integers II*, Funct. Approx. Comment. Math. **49** (2013), no. 1, 7–27.

[11] DE KONINCK, J. M.—KÁTAI, I.: *Normal numbers generated using the smallest prime factor function*, Ann. Sci. Math. Québec **38** (2014), no. 2, 133–144.

[12] DE KONINCK, J. M.—LUCA, F.: *Analytic Number Theory: Exploring the Anatomy of Integers*, Grad. Stud. Math. Vol. 134, Amer. Math. Soc., Providence, RI. 2012.

[13] HALÁSZ, G.: *On the distribution of additive and the mean values of multiplicative arithmetic functions*, Studia Sci. Math. Hungar. **6** (1971), 211–233.

[14] KÁTAI, I.: *A remark on a paper of K. Ramachandra*, in: Number Theory (Ootacamund, 1984), Lecture Notes in Math. Vol. 1122, Springer, Berlin, 1985, pp. 147–152.

[15] KUIPERS, L.—NIEDERREITER, H.: *Uniform Distribution of Sequences*, John Wiley & Sons, New York, 1974.

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