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# DISTRIBUTION OF LEADING DIGITS OF NUMBERS

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*Dedicated to the memory of Professor Pierre Liardet*

ABSTRACT. Applying the theory of distribution functions of sequences we find the relative densities of the first digits also for sequences $x_n$ not satisfying Benford's law. Especially for sequence $x_n = n^r$, $n = 1, 2, \dots$ and $x_n = p_n^r$, $n = 1, 2, \dots$, where $p_n$ is the increasing sequence of all primes and $r > 0$ is an arbitrary real. We also add rate of convergence to such densities.

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## 1. Introduction

In this paper we give a complete solution of the first digit problem in base $b \geq 2$ for the sequence $n^r$, $n = 1, 2, \dots$, and for the sequence $p_n^r$, $n = 1, 2, \dots$, where $p_n$ is the $n$th prime and $r > 0$ is an arbitrary real. They do not satisfy Benford's law. For example for $n^r$ we use the following main steps:

- Denote $F_N(x) = \frac{1}{N}\#\{n \leq N; \log_b(n^r) \bmod 1 \in [0, x)\}$.
- Then $F_N\Big(\log_b\big(\frac{D+1}{b^{s-1}}\big)\Big) - F_N\Big(\log_b\big(\frac{D}{b^{s-1}}\big)\Big)$ = the density of the number of $n \leq N$ for which the leading block of $s$ digits of $n^r$ is equal to $D$, $D = d_1 d_2 \cdots d_s$.
- Let $N_i$ be an increasing sequence of positive integer such that $\log_b N_i^r \bmod 1 \to w$ for some $w \in [0, 1]$.

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- Then $$\lim_{i\to\infty}\left(F_{N_i}\left(\log_b\left(\frac{D+1}{b^{s-1}}\right)\right)-F_{N_i}\left(\log_b\left(\frac{D}{b^{s-1}}\right)\right)\right)$$
$$=g_w\left(\log_b\left(\frac{D+1}{b^{s-1}}\right)\right)-g_w\left(\log_b\left(\frac{D}{b^{s-1}}\right)\right),$$
- where $g_w(x)=\frac{1}{b^{\frac{w}{r}}}\cdot\frac{b^{\frac{x}{r}}-1}{b^{\frac{1}{r}}-1}+\frac{\min(b^{\frac{x}{r}},b^{\frac{w}{r}})-1}{b^{\frac{w}{r}}}$.

In Theorem 5 we give a rate of such limit. The same limit holds also for the sequence $p_n^r$, $n=1,2,\dots$, but the rate of convergence described in Theorem 6 is more complicated.

In the following we present some clarifications.

### 1.1. Benford's law

For a real number $x$, let $[x]$ be the integral part of $x$, and let $x \bmod 1=\{x\}=x-[x]$. Let $b\ge 2$ be an integer considered as a base for the development of a real number $x>0$ and $M_b(x)$ be the mantissa of $x$ defined by $x=M_b(x)\times b^{n(x)}$ such that $1\le M_b(x)<b$ holds, where $n(x)$ is a uniquely determined integer. Let $D=d_1d_2\cdots d_s$ be a positive integer expressed in the base $b$, that is

$$D=d_1\times b^{s-1}+d_2\times b^{s-2}+\cdots+d_{s-1}\times b+d_s\,,$$

where $d_1\neq 0$ and at the same time $D=d_1d_2\cdots d_s$ is considered as an $s-$consecutive block of digits in the base $b$. Note that for $x$ of the type

$$x=0.00\cdots 0d_1d_2\cdots d_s\cdots\,,$$

where $d_1>0$, we have $M_b(x)=d_1.d_2\cdots d_s\cdots$ and the first zero digits is omitted. Thus arbitrary $x>0$ has the first $s$-digits, starting a non-zero digit, equal to $d_1d_2\cdots d_s$ if and only if [1]

$$d_1.d_2\cdots d_s\le M_b(x)<d_1.d_2\cdots(d_s+1). \tag{1}$$

Since $\log_b M_b(x)=\log_b x \bmod 1$ the inequality (1) is equivalent to

$$\log_b\left(\frac{D}{b^{s-1}}\right)\le\log_b x \bmod 1<\log_b\left(\frac{D+1}{b^{s-1}}\right). \tag{2}$$

Underline once again

$$\frac{D}{b^{s-1}}=d_1.d_2\cdots d_s,\quad \frac{D+1}{b^{s-1}}=d_1.d_2\cdots(d_s+1).$$

[1]If $d_1=d_2=\cdots=d_s=b-1$, then we have $d_1.d_2\cdots(d_s+1)=b$.

**Definition 1** (P. Diaconis (1977)). A sequence $x_n$, $n = 1, 2, \dots$, of positive real numbers satisfies *Benford's law* (abbreviated to B.L.) [2] in base $b$, if for every $s = 1, 2, \dots$ and every $s$-digits integer $D = d_1 d_2 \cdots d_s$ we have the density

$$\lim_{N\to\infty} \frac{\#\{n \le N;\ \text{leading block of } s \text{ digits (beginning with} \ne 0) \text{ of } x_n = D\}}{N}$$
$$= \log_b\left(\frac{D+1}{b^{s-1}}\right) - \log_b\left(\frac{D}{b^{s-1}}\right). \tag{3}$$

From (2) and from definition (3) it follows immediately:

**Theorem 1.** *A sequence $x_n$, $n = 1, 2, \dots$, of positive real numbers satisfies B.L. in base $b$ if and only if the sequence $\log_b x_n \bmod 1$ is uniformly distributed in $[0, 1)$ (for definition see page 26).*

## 1.2. Historical comments

The first digit problem: An infinite sequence $x_n \ge 1$ of real numbers satisfies *Benford's law*, if the frequency (the asymptotic density) of occurrences of a given first digit $a$, when $x_n$ is expressed in the decimal form is given by $\log_{10}\left(1 + \frac{1}{a}\right)$ for every $a = 1, 2, \dots, 9$ (0 as a possible first digit is not admitted). Since $x_n$ has the first digit $a$ if and only if $\log_{10} x_n \bmod 1 \in [\log_{10} a, \log_{10}(a+1))$ , Benford's law for $x_n$ follows from the u.d. of $\log_{10} x_n \bmod 1$. F. Benford (1938) compared the empirical frequency of occurrences of $a$ with $\log_{10}\big((a+1)/a\big)$ in twenty different domains such as the areas of 335 rivers; the size of 3259 U.S. populations; the street address of first 342 persons listed in American Men of Sciences, etc. which led him to the conclusion that "the logarithmic law applies particularly to those outlaw numbers that are without known relationships …" For the asymptotic density of the second-place digit $b$ he found $\sum_{a=1}^{9} \log_{10}\left(1 + \frac{1}{10a+b}\right)$.

F. Benford rediscovered Newcomb's observation from (1881).

Many authors think that if the sequence $x_n$ does not satisfy B.L., then the relative density of indices $n$ for which the $b$-expansion of $x_n$ start with leading digits $D = d_1 d_2 \cdots d_s$

$$\frac{1}{N}\#\left\{n \le N;\ \log_b\left(\frac{D}{b^{s-1}}\right) \le \{\log_b x_n\} < \log_b\left(\frac{D+1}{b^{s-1}}\right)\right\}$$

*does not follow any distribution in the sense of natural density*, see S. Eliahou, B. Massé and D. Schneider [5]. These authors as an alternate result shown that the sequence $\log_{10} n^r \bmod 1$, $n = 1, 2, \dots, [e^r]$ and the sequence $\log_{10} p_n^r \bmod 1$, $n = 1, 2, \dots, [e^r]$, where $p_n$ are all prime numbers, have the discrepancy $O(r^{-1})$.

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[2] precisely generalized or strong

Thus, for $r \to \infty$, these sequences tends to uniform distribution and thus $n^r$ and $p_n^r$ tends to B.L.

### 1.3. General scheme of solution of the First digit problem

**Definition 2.** Let $y_n$, $n = 1, 2, \dots$, be a sequence of real numbers and define the step distribution function of $y_n \bmod 1$

$$F_N(x) = \frac{\#\{n \le N; y_n \bmod 1 \in [0, x)\}}{N} \tag{4}$$

for $x \in [0, 1]$. The limit $g(x)$ of a subsequence $F_{N_k}(x)$ of $F_N(x)$

$$\lim_{k\to\infty} F_{N_k}(x) = g(x) \tag{5}$$

for every $x \in [0, 1]$, [3] is called distribution function (abbreviating d.f.) of $y_n$, where $N_1 < N_2 < \cdots$ is related sequence of indices. Let $G(y_n \bmod 1)$ be the set of all possible limits (5). If $G(y_n \bmod 1) = \{g(x)\}$, $g(x) = x$, then the sequence $y_n$, $n = 1, 2, \dots$ is called uniformly distributed mod 1 (abbreviating u.d. mod1.)

From Definition 2 and (2) immediately follows the basic theorem.

**Theorem 2.** *Let $g(x) \in G(\log_b x_n \bmod 1)$ and $\lim_{i\to\infty} F_{N_i}(x) = g(x)$. Then for $D = d_1 d_2 \cdots d_s$*

$$\lim_{N_i\to\infty} \frac{\#\{n \le N_i;\ \text{first } s \text{ digits (starting a non-zero digit) of } x_n = D\}}{N_i}$$
$$= g\left(\log_b\left(\frac{D+1}{b^{s-1}}\right)\right) - g\left(\log_b\left(\frac{D}{b^{s-1}}\right)\right). \tag{6}$$

This is the general scheme of solution of the First digit problem for the sequence $x_n$, $n = 1, 2, \dots$, for which $\log_b x_n \bmod 1$ need not be u.d. sequence. This approach is also presented in [2].

Proof. Using $F_N(x) = \frac{1}{N}\#\{n \le N; x_n \bmod 1 \in [0, x)\}$ we have

$$F_N\left(\log_b\left(\frac{D+1}{b^{s-1}}\right)\right) - F_N\left(\log_b\left(\frac{D}{b^{s-1}}\right)\right)$$
$$= \frac{1}{N}\#\{n \le N;\ \text{first } s \text{ digits (starting a non-zero digit) of } x_n = D\}.$$

□

[3] Similar theory of d.f.s can be found by using weak limits, i.e. the limit (5) for every continuity point $x$ of $g(x)$. But in the following we use only continuous $g(x)$.

## 2. Distribution functions of sequences involving logarithm

D.f.'s of $\log_b x_n \bmod 1$ which we need in (6) can be computed by following theorems:

**Theorem 3** ([9]). *Let the real-valued function $f(x)$ be strictly increasing for $x \geq 1$ and let $f^{-1}(x)$ be its inverse function and*

$$F_N(x) = \frac{1}{N}\#\{n \leq N; f(n) \bmod 1 \in [0,x)\}.$$

*Assume that*

(i) $\lim_{x\to\infty} f'(x) = 0$,

(ii) $\lim_{k\to\infty} f^{-1}(k+1) - f^{-1}(k) = \infty$,

(iii) $\lim_{k\to\infty} \frac{f^{-1}(k+w(k))}{f^{-1}(k)} = \psi(w)$ *for every sequence* $w(k) \in [0,1]$ *for which* $\lim_{k\to\infty} w(k) = w$, *where this limit defines the function* $\psi : [0,1] \to [1,\psi(1)]$,

(iv) $\psi(1) > 1$.

*Then for the sequence $f(n) \bmod 1$, $n = 1,2,\dots$, we have*

$$G(f(n) \bmod 1) = \left\{ g_w(x) = \frac{1}{\psi(w)}\frac{\psi(x)-1}{\psi(1)-1} + \frac{\min(\psi(x),\psi(w))-1}{\psi(w)};\ w \in [0,1] \right\}. \quad (7)$$

*Now, if $f(N_i) \bmod 1$ is a subsequence $f(n) \bmod 1$ such that $f(N_i) \bmod 1 \to w$, then $F_{N_i}(x) \to g_w(x)$ for every $x \in [0,1]$.*

Similar theorem is valid also for $f(p_n)$, where $p_n$ are primes.

**Theorem 4** ([7]). *Let $F_N(x) = \frac{1}{N}\#\{n \leq N; f(p_n) \bmod 1 \in [0,x)\}$ for $x \in [0,1]$, where $p_n$ is the increasing sequence of all primes. Assume* (i)-(iv) *from Theorem 3. Then the sequence $f(p_n) \bmod 1$, $n = 1,2,\dots$, has*

$$G\big(f(p_n) \bmod 1\big) = G\big(f(n) \bmod 1\big).$$

*Now, if $f(p_{N_i}) \bmod 1$ is a subsequence $f(p_n) \bmod 1$ such that $f(p_{N_i}) \bmod 1 \to w$, then $F_{N_i}(x) \to g_w(x)$ for every $x \in [0,1]$.*

### 2.1. Natural numbers

Applying Theorem 3 to the sequence $f(n) = \log_b n^r$, $n = 1,2,\dots$ we have

$$f^{-1}(x) = b^{\frac{x}{r}}, \lim_{k\to\infty} \frac{f^{-1}(k+w)}{f^{-1}(k)} = \frac{b^{\frac{k+w}{r}}}{b^{\frac{k}{r}}} = b^{\frac{w}{r}} = \psi(w),$$

$$G(\log_b n^r \bmod 1) = \left\{ g_w(x) = \frac{1}{b^{\frac{w}{r}}}\frac{b^{\frac{x}{r}}-1}{b^{\frac{1}{r}}-1} + \frac{\min(b^{\frac{x}{r}}, b^{\frac{w}{r}})-1}{b^{\frac{w}{r}}};\quad w \in [0,1] \right\}.$$

If
$$\lim_{i\to\infty}\{f(N_i)\} = \lim_{i\to\infty}\{\log_b(N_i^r)\} = w,$$
then we have
$$\lim_{i\to\infty}\frac{\#\{n\le N_i;\ \text{first } s \text{ digits of } n^r \text{ are } d_1d_2\dots d_s\}}{N_i}$$
$$= g_w\big(\log_b d_1.d_2\dots(d_s+1)\big) - g_w\big(\log_b d_1.d_2\dots d_s\big). \qquad (8)$$
Some examples of $w$ :
**1.** Assume that $N_i = b^i$ and $r$ is a positive integer, then
$$\lim_{i\to\infty}\{\log_b(b^{ir})\} = 0 = w$$
and thus for (8) we have
$$=\frac{b^{(\log_b d_1.d_2\dots(d_s+1))/r}-1}{b^{1/r}-1} - \frac{b^{(\log_b d_1.d_2\dots d_s)/r}-1}{b^{1/r}-1}$$
$$=\frac{(d_1.d_2\dots(d_s+1))^{(1/r)} - (d_1.d_2\dots d_s))^{(1/r)}}{b^{1/r}-1}$$
$$=\frac{1}{b^{s-1}}\frac{1}{b-1} \text{ if } r=1.$$

**2.** Put $N_i = [b^{\frac{i+w}{r}}]$, where $i$ is a positive integer. The $\lim_{i\to\infty}\{\log_b(N_i^r)\} = w$.

P r o o f. We have $N_i = [b^{\frac{i+w}{r}}] = b^{\frac{i+w'}{r}}$, where $w' \le w$ and $\{\log_b(N_i^r)\} = w'$. Further $|b^{\frac{i+w'}{r}} - b^{\frac{i+w}{r}}| = |w'-w|b^{\frac{i+x}{r}}(\log b)\frac{1}{r}$. Since $|b^{\frac{i+w'}{r}} - b^{\frac{i+w}{r}}| < 1$, then $|w'-w|\to 0$. Thus $\lim_{i\to\infty}\{\log_b(N_i^r)\} = w$. □

### 2.2. Primes

Applying Theorem 4 for the sequence
$$f(p_n) = \log_b p_n^r, \quad n = 1,2,\dots,$$
where $p_n$ is the $n$th prime and $r>0$, we have
$$G(\log_b p_n^r \bmod 1) = \Big\{g_w(x) = \frac{1}{b^{\frac{w}{r}}}\frac{b^{\frac{x}{r}}-1}{b^{\frac{1}{r}}-1} + \frac{\min(b^{\frac{x}{r}}, b^{\frac{w}{r}})-1}{b^{\frac{w}{r}}};\ w\in[0,1]\Big\}.$$
If $\{f(p_{N_i})\} = \{\log_b(p_{N_i}^r)\}\to w$, then
$$\lim_{i\to\infty}\frac{\#\{n\le N_i;\ \text{first } s \text{ digits of } p_n^r = d_1d_2\dots d_s\}}{N_i}$$
$$= g_w\big(\log_b d_1.d_2\dots(d_s+1)\big) - g_w\big(\log_b d_1.d_2\dots d_s\big).$$

Some examples of $w$ and $N_i$:

**1.** If $0 < w < 1$ and $N_i = \pi(b^{\frac{i+w}{r}})$, then $\lim_{i\to\infty}\{\log_b p^r_{N_i}\} = w$.

If $N_i = \pi(b^{\frac{i}{r}}) + 1$, then $\lim_{i\to\infty}\{\log_b p^r_{N_i}\} = 0$.

If $N_i = \pi(b^{\frac{i}{r}})$, then $\lim_{i\to\infty}\{\log_b p^r_{N_i}\} = 1$.

P r o o f. Let $0 < w < 1$ and $N_i = \pi(b^{\frac{i+w}{r}})$. We have

$$p_{N_i} \le b^{\frac{w+i}{r}} < p_{N_i+1},$$

and so

$$\log_b p^r_{N_i} \le w + i < \log_b p^r_{N_i+1}.$$

Now we have

$$\begin{aligned}\log_b p^r_{N_i+1} - \log_b p^r_{N_i} &= r\log_b \frac{p_{N_i+1}}{p_{N_i}}\\ &= r\log_b\Big(1 + \frac{p_{N_i+1} - p_{N_i}}{p_{N_i}}\Big)\\ &= r\log_b\left(1 + \frac{1}{p^{1-\theta}_{N_i}}\right) \to 0 \quad (i\to\infty),\end{aligned} \tag{9}$$

where the last equality is obtained by using the result

$$p_{n+1} - p_n = O(p_n^\theta) \quad \text{with} \quad \theta = 0.525 \quad \text{(see [1])}.$$

Therefore there exists $i_0$ such that for $i \ge i_0$, $[\log_b p^r_{N_i}] = i$ and

$$\begin{aligned}0 &\le w - \{\log_b p^r_{N_i}\} = i + w - (i + \{\log_b p^r_{N_i}\}) = i + w - \log_b p^r_{N_i}\\ &< \log_b p^r_{N_i+1} - \log_b p^r_{N_i} \to 0 \quad (i\to\infty).\end{aligned}$$

Hence, $\lim_{i\to\infty}\{\log p^r_{N_i}\} = w$. Similarly the other two cases are proved. $\square$

### 2.3. Summary

From the above it follows that the sequences

$$\log_b n^r \bmod 1, \quad n = 1, 2, \dots, \quad \text{and} \quad \log_b p^r_n \bmod 1, \quad n = 1, 2, \dots,$$

have the same distribution functions $g_w(x) = \frac{1}{b^{\frac{w}{r}}}\frac{b^{\frac{x}{r}}-1}{b^{\frac{1}{r}}-1} + \frac{\min(b^{\frac{x}{r}}, b^{\frac{w}{r}})-1}{b^{\frac{w}{r}}}$; $w \in [0,1]$.

Since $\lim_{r\to\infty}\frac{b^{\frac{x}{r}}-1}{b^{\frac{1}{r}}-1} = x$, then $\lim_{r\to\infty} g_w(x) = x$. Thus, as $r \to \infty$ the sequences $n^r$ and $p^r_n$ tends to B.L. This is qualitative proof of results in [5].

## 3. Rate of convergence

### 3.1. The first digit problem for the sequence $n^r, n = 1, 2, \ldots$

**Theorem 5.** *Let $N, b$ be positive integers, $b > 1$, $r > 0$, $w_0 \in [0,1]$. Denote*

$$F_N(x) = \frac{\#\{n \le N; \log_b(n^r) \bmod 1 \in [0,x)\}}{N},$$
$$g_{w_0}(x) = \frac{1}{b^{\frac{w_0}{r}}} \cdot \frac{b^{\frac{x}{r}} - 1}{b^{\frac{1}{r}} - 1} + \frac{\min(b^{\frac{x}{r}}, b^{\frac{w_0}{r}}) - 1}{b^{\frac{w_0}{r}}},$$
$$w = \{r \log_b N\}.$$

*Then for every $x \in [0,1]$ we have*

$$|F_N(x) - g_{w_0}(x)| \le \frac{|w - w_0|}{r} \cdot \log b \cdot b^{\frac{1}{r}} \cdot (b^{\frac{1}{r}} + 1) + \frac{3}{N} + \frac{r \log_b N}{N}.$$

**Corollary 1.** *Let $r > 0$, $w = \{r \log_b N\}$. Then for $D = d_1 d_2 \cdots d_s$ the ration $\frac{1}{N}\#\{1 \le n \le N;$ the leading block of $s$ digits of $n^r$ is equal to $D\}$ it can be approximated by*

$$g_w\left(\log_b\left(\frac{D+1}{b^{s-1}}\right)\right) - g_w\left(\log_b\left(\frac{D}{b^{s-1}}\right)\right)$$

*with the error term*

$$2\left(\frac{3}{N} + \frac{r \log_b N}{N}\right) = O_{b,r}\left(\frac{\log N}{N}\right).$$

Proof of Corrolary 1. We have

$$\frac{1}{N}\#\{1 \le n \le N; \text{the leading block of } s \text{ digits of } n^r \text{ is equal to } D\}$$
$$= F_N\left(\log_b\left(\frac{D+1}{b^{s-1}}\right)\right) - F_N\left(\log_b\left(\frac{D}{b^{s-1}}\right)\right).$$

Applying Theorem 5 with $w_0 = w$, we have $|F_N(x) - g_w(x)| \le \frac{3}{N} + \frac{r \log_b N}{N}$. $\square$

**Example 1.** Let $r > 0$ and $i$ be a positive integer. By Corollary 1 for $D = d_1 d_2 \cdots d_s$ the fraction

$$\frac{1}{b^i}\#\{1 \le n \le b^i; \text{the leading block of } s \text{ digits of } n^r \text{ is equal to } D\}$$

it can be approximated by

$$\frac{(\frac{D+1}{b^{s-1}})^{\frac{1}{r}} - (\frac{D}{b^{s-1}})^{\frac{1}{r}}}{b^{\frac{1}{r}} - 1} \quad \text{with the error term} \quad \frac{2ri+6}{b^i}.$$

Now we prove Theorem 5.

P r o o f. Firstly we repeat a proof of (7). For a positive integer $N$ define

- $K_N = [f(N)]$, abbreviating $K_N = K$,
- $w_N = \{f(N)\}$, abbreviating $w_N = w$,
- $A_N([x,y)) = \#\{n \le N; f(n) \in [x,y)\}$,
- $F_N(x) = \frac{\#\{n \le N; f(n) \bmod 1 \in [0,x)\}}{N}$.

Clearly $f^{-1}(K+w) = N$ and for every $x \in [0,1]$ and $F_N(x)$ in (4) we have

$$F_N(x) = \frac{\sum_{k=0}^{K-1} A_N([k,k+x))}{N} + \\ + \frac{A_N([K,K+x) \cap [K,K+w))}{N} + \frac{O(A_N([1,f^{-1}(0)))}{N}$$

From monotonicity of $f(x)$ it follows $A_N([x,y)) = f^{-1}(y) - f^{-1}(x) + \theta$, where $|\theta| \le 1$. Thus

$$F_N(x) = \frac{\sum_{k=0}^{K-1} (f^{-1}(k+x) - f^{-1}(k))}{N} \\ + \frac{\min(f^{-1}(K+x), f^{-1}(K+w)) - f^{-1}(K)}{N} + \frac{O(K)}{N} + \frac{O(f^{-1}(0))}{N}, \quad (10)$$

where

$$O(K) \le K+1 \text{ and } O(f^{-1}(0)) \le f^{-1}(0).$$

The assumption (ii) implies $1/(f^{-1}(k+1) - f^{-1}(k)) \to 0$ which together with Cauchy-Stolz (other name is Stolz-Cesàro, see [10, p. 4–7]) lemma implies that

$$K/f^{-1}(K) \to 0 \quad \text{and thus} \quad K/N \to 0 \quad \text{as} \quad N \to \infty.$$

Furthermore we can express the first term of $F_N(x)$ in (10) as

$$\frac{\sum_{k=0}^{K-1} (f^{-1}(k+x) - f^{-1}(k))}{\sum_{k=0}^{K-1} (f^{-1}(k+1) - f^{-1}(k))} \cdot \frac{f^{-1}(K) - f^{-1}(0)}{f^{-1}(K+w)} \quad (11)$$

and the second term as

$$\frac{\min\left(\frac{f^{-1}(K+x)}{f^{-1}(K)}, \frac{f^{-1}(K+w)}{f^{-1}(K)}\right) - 1}{\frac{f^{-1}(K+w)}{f^{-1}(K)}}. \quad (12)$$

Using the assumption (ii) and (iii) the Cauchy-Stolz lemma implies

$$\lim_{K\to\infty} \frac{\sum_{k=0}^{K-1} (f^{-1}(k+x) - f^{-1}(k))}{\sum_{k=0}^{K-1} (f^{-1}(k+1) - f^{-1}(k))} = \lim_{k\to\infty} \frac{f^{-1}(k+x) - f^{-1}(k)}{f^{-1}(k+1) - f^{-1}(k)} = \frac{\psi(x)-1}{\psi(1)-1}. \quad (13)$$

Now, for increasing subsequence $N_i$ of indices $N$, denote

$$K_i = [f(N_i)] \quad \text{and} \quad w_i = \{f(N_i)\}.$$

If $w_i \to w_0$, then by (iii)

$$f^{-1}(K_i)/f^{-1}(K_i + w_i) \to 1/\psi(w_0), \quad \text{and} \quad f^{-1}(K_i + x)/f^{-1}(K_i) \to \psi(x).$$

Thus (11), (12), (13) imply (7)

$$F_{N_i}(x) \to g_{w_0}(x) = \frac{1}{\psi(w_0)} \cdot \frac{\psi(x)-1}{\psi(1)-1} + \frac{\min(\psi(x), \psi(w_0))-1}{\psi(w_0)} \quad \text{for all} \quad x \in [0,1).$$

In the following we prove a quantitative form of (7). Put

- $K = [f(N)]$,
- $w = \{f(N)\}$,
- $N = f^{-1}(K + w)$. Then

$$\begin{aligned}
&F_N(x) - g_{w_0}(x) \\
&= \left( \frac{\sum_{k=0}^{K-1}(f^{-1}(k+x) - f^{-1}(k))}{\sum_{k=0}^{K-1}(f^{-1}(k+1) - f^{-1}(k))} \cdot \frac{f^{-1}(K) - f^{-1}(0)}{f^{-1}(K+w)} - \frac{\psi(x)-1}{\psi(1)-1} \cdot \frac{1}{\psi(w_0)} \right) \\
&\quad + \left( \frac{\min\left(\frac{f^{-1}(K+x)}{f^{-1}(K)}, \frac{f^{-1}(K+w)}{f^{-1}(K)}\right) - 1}{\frac{f^{-1}(K+w)}{f^{-1}(K)}} - \frac{\min(\psi(x), \psi(w_0)) - 1}{\psi(w_0)} \right) \\
&\quad + \left( \frac{O(K)}{N} + \frac{O(f^{-1}(0))}{N} \right)
\end{aligned}$$

$$= (\mathrm{I}) \left( \frac{\sum_{k=0}^{K-1}(f^{-1}(k+x) - f^{-1}(k))}{\sum_{k=0}^{K-1}(f^{-1}(k+1) - f^{-1}(k))} - \frac{\psi(x)-1}{\psi(1)-1} \right) \left( \frac{f^{-1}(K) - f^{-1}(0)}{f^{-1}(K+w)} \right) \tag{14}$$

$$+ (\mathrm{II}) \left( \frac{f^{-1}(K) - f^{-1}(0)}{f^{-1}(K+w)} - \frac{1}{\psi(w_0)} \right) \left( \frac{\psi(x)-1}{\psi(1)-1} \right) \tag{15}$$

$$+ (\mathrm{III}) \left( \min\left( \frac{f^{-1}(K+x)}{f^{-1}(K)}, \frac{f^{-1}(K+w)}{f^{-1}(K)} \right) - \min\big(\psi(x), \psi(w_0)\big) \right) \left( \frac{1}{\frac{f^{-1}(K+w)}{f^{-1}(K_i)}} \right) \tag{16}$$

$$+ (\mathrm{IV}) \left( \psi(w_0) - \frac{f^{-1}(K+w)}{f^{-1}(K)} \right) \left( \frac{\min(\psi(x), \psi(w_0)) - 1}{\frac{f^{-1}(K+w)}{f^{-1}(K)} \cdot \psi(w_0)} \right) \tag{17}$$

$$+ (\mathrm{V}) \left( \frac{O(K)}{N} + \frac{O(f^{-1}(0))}{N} \right). \tag{18}$$

Every second term in (I)=(14), (II)=(15), (III)=(16), (IV)=(17) is bounded precisely

$$\left(\frac{f^{-1}(K)-f^{-1}(0)}{f^{-1}(K+w)}\right)\leq 1,\quad \left(\frac{\psi(x)-1}{\psi(1)-1}\right)\leq 1,\quad \left(\frac{1}{\frac{f^{-1}(K+w)}{f^{-1}(K)}}\right)\leq 1,$$

$$\left(\frac{\min(\psi(x),\psi(w_0))-1}{\frac{f^{-1}(K+w)}{f^{-1}(K)}\cdot\psi(w_0)}\right)\leq\psi(1)-1.$$

Now put $f(x)=\log_b(x^r)$. This function satisfies Theorem 3 and

$$f^{-1}(x)=b^{\frac{x}{r}},$$

$$\lim_{k\to\infty}\frac{f^{-1}(k+x)}{f^{-1}(k)}=\frac{b^{\frac{k+x}{r}}}{b^{\frac{k}{r}}}=b^{\frac{x}{r}}=\psi(x),$$

$$g_{w_0}(x)=\frac{1}{b^{\frac{w_0}{r}}}\cdot\frac{b^{\frac{x}{r}}-1}{b^{\frac{1}{r}}-1}+\frac{\min(b^{\frac{x}{r}},b^{\frac{w_0}{r}})-1}{b^{\frac{w_0}{r}}},$$

$$f^{-1}(0)=b^{\frac{0}{r}}=1,\quad \psi(1)=b^{\frac{1}{r}},$$

$$K=[r\log_b N],\quad \omega=\{r\log_b N\},\quad N=b^{\frac{K+w}{r}}.$$

Then for the first terms in (I)=(14), (II)=(15), (III)=(16), (IV)=(17) we have:

$$\text{(I)}\left|\frac{\sum_{k=0}^{K-1}(f^{-1}(k+x)-f^{-1}(k))}{\sum_{k=0}^{K-1}(f^{-1}(k+1)-f^{-1}(k))}-\frac{\psi(x)-1}{\psi(1)-1}\right|$$

$$=\left|\frac{\sum_{k=0}^{K-1}(b^{\frac{k+x}{r}}-b^{\frac{k}{r}})}{\sum_{k=0}^{K-1}(b^{\frac{k+1}{r}}-b^{\frac{k}{r}})}-\frac{b^{\frac{x}{r}}-1}{b^{\frac{1}{r}}-1}\right|=0,$$

$$\text{(II)}\left|\frac{f^{-1}(K)-f^{-1}(0)}{f^{-1}(K+w)}-\frac{1}{\psi(w_0)}\right|=\left|\frac{b^{\frac{K}{r}}-1}{b^{\frac{K+w}{r}}}-\frac{1}{b^{\frac{w_0}{r}}}\right|\leq\left|\frac{1}{b^{\frac{w}{r}}}-\frac{1}{b^{\frac{w_0}{r}}}\right|+\frac{1}{N},$$

$$\text{(III)}\left|\min\left(\frac{f^{-1}(K+x)}{f^{-1}(K)},\frac{f^{-1}(K+w)}{f^{-1}(K)}\right)-\min\bigl(\psi(x),\psi(w_0)\bigr)\right|$$

$$=|\min\bigl(\psi(x),\psi(w)\bigr)-\min\bigl(\psi(x),\psi(w_0)\bigr)|$$

$$\leq|\psi(w)-\psi(w_0)|=|b^{\frac{w}{r}}-b^{\frac{w_0}{r}}|(\text{ for a proof see in appendix}),\tag{19}$$

$$\text{(IV)}\left|\psi(w_0)-\frac{f^{-1}(K+w)}{f^{-1}(K)}\right|\bigl(\psi(1)-1\bigr)=|b^{\frac{w}{r}}-b^{\frac{w_0}{r}}|(b^{\frac{1}{r}}-1),$$

$$\text{(V)}\left(\frac{O(K)}{N}+\frac{O(f^{-1}(0))}{N}\right)\leq\frac{K+1+f^{-1}(0)}{N}\leq\frac{r\log_b N+2}{N}.$$

In the end of proof we use

$$|b^{\frac{w}{r}} - b^{\frac{w_0}{r}}| \le \frac{|w-w_0|}{r} \cdot \log b \cdot b^{\frac{1}{r}}. \qquad \square$$

### 3.2. First digit problem for $p_n^r$, $n = 1, 2, \dots$, where $p_n$ is the $n$th prime

**Theorem 6.** *Let $N$ be an arbitrary integer, $r > 0$, $w_0 \in [0,1]$, Denote*

$$F_N(x) = \frac{\#\{n \le N; \log_b({p_n}^r) \bmod 1 \in [0,x)\}}{N},$$

$$g_{w_0}(x) = \frac{1}{b^{\frac{w_0}{r}}} \cdot \frac{b^{\frac{x}{r}} - 1}{b^{\frac{1}{r}} - 1} + \frac{\min(b^{\frac{x}{r}}, b^{\frac{w_0}{r}}) - 1}{b^{\frac{w_0}{r}}},$$

$$K = [r \log_b p_N], \quad w = \{r \log_b p_N\}.$$

*Let $k_1$ be an integer such that*

$$\max\left(\frac{\log 59}{\log b} r, \frac{5rb^{\frac{1}{r}} + \frac{r}{2} + b^{\frac{1}{r}} \log b}{(b^{\frac{1}{r}} - 1)\log b}\right) < k_1 < K.$$

*Then for every $x \in [0,1]$ we have*

$$\begin{aligned}|F_N(x) - g_{w_0}(x)| \le & \frac{b^{\frac{1}{r}}(2\log b + 11r)}{(\log b)(b^{\frac{1}{r}} - 1)k_1 - (5r + \log b)b^{\frac{1}{r}} - \frac{r}{2}} \\ & + \frac{K}{k_1} \frac{1}{b^{\frac{K-k_1}{r}}} \frac{\log b + \frac{3}{2k_1}}{\log b + \frac{2}{K}} + 3(b^{\frac{1}{r}})^2 \frac{\log b + \frac{11}{2} r}{K \log b + \frac{r}{4}} \\ & + 3(b^{\frac{1}{r}})^2 \frac{|w - w_0|}{r} \log b + \frac{r \log_b p_N + 1}{N}.\end{aligned}$$

**Corollary 2.** *Let $r > 0$, $i$ be a positive integer, and let $N = \pi(b^{\frac{i}{r}})$. Then for $D = d_1 d_2 \cdots d_s$ the fraction*

$\frac{1}{N}\#\{1 \le n \le N;$ *the leading block of $s$ digits of $p_n^r$ is equal to $D$*$\}$

*it can be approximated by*

$$\frac{\left(\frac{D+1}{b^{s-1}}\right)^{\frac{1}{r}} - \left(\frac{D}{b^{s-1}}\right)^{\frac{1}{r}}}{b^{\frac{1}{r}} - 1}$$

*with the error term*

$$\begin{aligned}&\left|F_N\left(\frac{D+1}{b^{s-1}}\right) - g_1\left(\frac{D+1}{b^{s-1}}\right)\right| + \left|F_N\left(\frac{D}{b^{s-1}}\right) - g_1\left(\frac{D}{b^{s-1}}\right)\right| \\ &= O_{b,r}\left(\frac{1}{\log p_N}\right)\end{aligned}$$

*given by Theorem 6.*

P r o o f o f C o r o l l a r y 2. For $N = \pi(b^{\frac{i}{r}})$ we have $r \log_b p_N \le i < r \log_b p_{N+1}$. Since $r \log_b p_{N+1} - r \log_b p_N \to 0$ by (9), we obtain $[r \log_b p_N] = i - 1$ for sufficiently large $i$. Therefore

$$\begin{aligned}|w-1| &= |\log_b p_N^r - (i-1) - 1| = i - \log_b p_N^r \le r(\log_b p_{N+1} - \log_b p_N)\\ &\le \frac{r}{\log b}\cdot\frac{1}{p_N}(p_{N+1}-p_N) \le \frac{r}{\log b}\cdot\frac{C}{p_N^{1-\theta}},\end{aligned}$$

where $C$ is a positive constant, $\theta = 0.525$ (see [1]). We apply Theorem 6 with $N = \pi(b^{\frac{i}{r}})$, $w_0 = 1$, $K = [r \log_b p_N]$, $k_1 = [\frac{r}{2} \log_b p_N]$, $w = \{r \log_b p_N\}$. □

P r o o f o f T h e o r e m 6. Firstly we repeat the proof of Theorem 4. Let $0 \le x \le 1$. We have

$$\begin{aligned}&\{f(p_n)\} < x \Longleftrightarrow 0 \le f(p_n) - k < x \Longleftrightarrow\\ &k \le f(p_n) < k + x \Longleftrightarrow f^{-1}(k) \le p_n < f^{-1}(k+x).\end{aligned}$$

Let $\pi(t) = \#\{p \le t; p - \text{prime}\}$. Using $\pi(t) = \#\{p < t; p - \text{prime}\} + O(1)$, then we have

$$\begin{aligned}F_N(x) =& \frac{\sum_{k=0}^{K-1}(\pi(f^{-1}(k+x)) - \pi(f^{-1}(k)))}{N} +\\ &+ \frac{\min(\pi(f^{-1}(K+x)), \pi(f^{-1}(K+w))) - \pi(f^{-1}(K))}{N}\\ &+ \frac{O(K)}{N} + \frac{O(\pi(f^{-1}(0)))}{N},\end{aligned} \tag{20}$$

where

$$O(K) \le K+1, \qquad O\Big(\pi\big(f^{-1}(0)\big)\Big) \le \pi\big(f^{-1}(0)\big).$$

Let $K = [f(p_N)]$, $w = \{f(p_N)\}$. Then $p_N = f^{-1}(K+w)$. Then we have

$$N = \pi\big(f^{-1}(K+w)\big).$$

We can express the first term of $F_N(x)$ in (20) as

$$\frac{\sum_{k=0}^{K-1}(\pi(f^{-1}(k+x)) - \pi(f^{-1}(k)))}{\sum_{k=0}^{K-1}(\pi(f^{-1}(k+1)) - \pi(f^{-1}(k)))} \cdot \frac{\pi(f^{-1}(K)) - \pi(f^{-1}(0))}{\pi(f^{-1}(K+w))}$$

and the second term as

$$\frac{\min\Big(\frac{\pi(f^{-1}(K+x))}{\pi(f^{-1}(K))}, \frac{\pi(f^{-1}(K+w))}{\pi(f^{-1}(K))}\Big) - 1}{\frac{\pi(f^{-1}(K+w))}{\pi(f^{-1}(K))}}. \tag{21}$$

The Cauchy-Stolz lemma implies

$$\lim_{K\to\infty}\frac{\sum_{k=0}^{K-1}(\pi(f^{-1}(k+x))-\pi(f^{-1}(k)))}{\sum_{k=0}^{K-1}(\pi(f^{-1}(k+1))-\pi(f^{-1}(k)))}=\lim_{k\to\infty}\frac{\pi(f^{-1}(k+x))-\pi(f^{-1}(k))}{\pi(f^{-1}(k+1))-\pi(f^{-1}(k))}. \tag{22}$$

The Prime number theorem in the form

$$\pi(t)=\frac{t}{\log t}+O\left(\frac{t}{\log^2 t}\right)$$

gives

$$\frac{\pi(f^{-1}(k+x))}{\pi(f^{-1}(k)))}=\frac{f^{-1}(k+x)}{f^{-1}(k)}\cdot\frac{\log f^{-1}(k)}{\log f^{-1}(k+x)}\cdot\frac{1+O\left(\frac{1}{\log f^{-1}(k+x)}\right)}{1+O\left(\frac{1}{\log f^{-1}(k)}\right)}.$$

We need

$$\lim_{k\to\infty}\frac{\log f^{-1}(k)}{\log f^{-1}(k+x)}=1. \tag{23}$$

The limit (23) is proved in [7] by using the following steps:

$$\begin{aligned}\frac{\log f^{-1}(k+x)}{\log f^{-1}(k)}&=\frac{1}{\log f^{-1}(k)}\big(\log f^{-1}(k+x)-\log f^{-1}(k)\big)+1\\&=\frac{1}{\log f^{-1}(k)}\log\left(\frac{f^{-1}(k+x)}{f^{-1}(k)}\right)+1\to 1\end{aligned}$$

since by assumption $\frac{f^{-1}(k+x)}{f^{-1}(k)}\to\psi(x)\ge 1$. Thus

$$\lim_{k\to\infty}\frac{\pi(f^{-1}(k+x))}{\pi(f^{-1}(k))}=\lim_{k\to\infty}\frac{f^{-1}(k+x)}{f^{-1}(k)}=\psi(x) \tag{24}$$

and (22) has the form $\frac{\psi(x)-1}{\psi(1)-1}$. Applying (24) on (21) we have

$$\lim_{K\to\infty}\frac{\min\left(\frac{\pi(f^{-1}(K+x))}{\pi(f^{-1}(K))},\frac{\pi(f^{-1}(K+w))}{\pi(f^{-1}(K))}\right)-1}{\frac{\pi(f^{-1}(K+w))}{\pi(f^{-1}(K))}}=\frac{\min(\psi(x),\psi(w))-1}{\psi(w)}.$$

Here for given $N$,

$p_N$ is $N$th prime, $K=[f(p_N)]$, $w=\{f(p_N)\}$ and $w_0\in[0,1]$ is arbitrary.

Similarly as for the sequence $f(n)$ mod 1 we have (18), (14), (15), (16), (17) for the sequence $f(p_n)$ mod 1 and by applying $\pi(x)$ we have the following

$$
\begin{aligned}
&F_N(x) - g_{w_0}(x) \\
&= \left( \frac{\sum_{k=0}^{K-1}(\pi(f^{-1}(k+x)) - \pi(f^{-1}(k)))}{\sum_{k=0}^{K-1}(\pi(f^{-1}(k+1)) - \pi(f^{-1}(k)))} \cdot \frac{\pi(f^{-1}(K)) - \pi(f^{-1}(0))}{\pi(f^{-1}(K+w))} \right. \\
&\qquad\qquad \left. - \frac{\psi(x)-1}{\psi(1)-1} \cdot \frac{1}{\psi(w_0)} \right) \\
&\quad + \left( \frac{\min\left(\frac{\pi(f^{-1}(K+x))}{\pi(f^{-1}(K))}, \frac{\pi(f^{-1}(K+w))}{\pi(f^{-1}(K))}\right) - 1}{\frac{\pi(f^{-1}(K+w))}{\pi(f^{-1}(K))}} - \frac{\min(\psi(x), \psi(w_0)) - 1}{\psi(w_0)} \right) \\
&\qquad\qquad + \left( \frac{O(K)}{N} + \frac{O(\pi(f^{-1}(0)))}{N} \right)
\end{aligned}
$$

$$
= (\mathrm{I}) \left( \frac{\sum_{\mathrm{k}=0}^{\mathrm{K}-1}(\pi(\mathrm{f}^{-1}(\mathrm{k}+\mathrm{x})) - \pi(\mathrm{f}^{-1}(\mathrm{k})))}{\sum_{\mathrm{k}=0}^{\mathrm{K}-1}(\pi(\mathrm{f}^{-1}(\mathrm{k}+1)) - \pi(\mathrm{f}^{-1}(\mathrm{k})))} - \frac{\psi(\mathrm{x})-1}{\psi(1)-1} \right) \left( \frac{\pi(\mathrm{f}^{-1}(\mathrm{K})) - \pi(\mathrm{f}^{-1}(0))}{\pi(\mathrm{f}^{-1}(\mathrm{K}+\mathrm{w}))} \right) \tag{25}
$$

$$
+ (\mathrm{II}) \left( \frac{\pi(\mathrm{f}^{-1}(\mathrm{K})) - \pi(\mathrm{f}^{-1}(0))}{\pi(\mathrm{f}^{-1}(\mathrm{K}+\mathrm{w}))} - \frac{1}{\psi(\mathrm{w}_0)} \right) \left( \frac{\psi(\mathrm{x}) - 1}{\psi(1) - 1} \right) \tag{26}
$$

$$
+ (\mathrm{III}) \left( \min\left( \frac{\pi(\mathrm{f}^{-1}(\mathrm{K}+\mathrm{x}))}{\pi(\mathrm{f}^{-1}(\mathrm{K}))}, \frac{\pi(\mathrm{f}^{-1}(\mathrm{K}+\mathrm{w}))}{\pi(\mathrm{f}^{-1}(\mathrm{K}))} \right) - \min\big(\psi(\mathrm{x}), \psi(\mathrm{w}_0)\big) \right) \left( \frac{1}{\frac{\pi(\mathrm{f}^{-1}(\mathrm{K}+\mathrm{w}))}{\pi(\mathrm{f}^{-1}(\mathrm{K}))}} \right) \tag{27}
$$

$$
+ (\mathrm{IV}) \left( \psi(\mathrm{w}_0) - \frac{\pi(\mathrm{f}^{-1}(\mathrm{K}+\mathrm{w}))}{\pi(\mathrm{f}^{-1}(\mathrm{K}))} \right) \left( \frac{\min(\psi(\mathrm{x}), \psi(\mathrm{w}_0)) - 1}{\frac{\pi(\mathrm{f}^{-1}(\mathrm{K}+\mathrm{w}))}{\pi(\mathrm{f}^{-1}(\mathrm{K}))} \cdot \psi(\mathrm{w}_0)} \right) \tag{28}
$$

$$
+ (\mathrm{V}) \left( \frac{\mathrm{O}(\mathrm{K})}{\mathrm{N}} + \frac{\mathrm{O}(\pi(\mathrm{f}^{-1}(0)))}{\mathrm{N}} \right). \tag{29}
$$

Every second term in (I)-(V)=(25), (26), (27) and (28) is bounded, precisely

$$
\begin{gathered}
\left( \frac{\pi(f^{-1}(K)) - \pi(f^{-1}(0))}{\pi(f^{-1}(K+w))} \right) \le 1, \\
\left( \frac{\psi(x) - 1}{\psi(1) - 1} \right) \le 1, \qquad \left( \frac{1}{\frac{\pi(f^{-1}(K+w))}{\pi(f^{-1}(K))}} \right) \le 1, \\
\left( \frac{\min(\psi(x), \psi(w_0)) - 1}{\frac{\pi(f^{-1}(K+w))}{\pi(f^{-1}(K))} \cdot \psi(w_0)} \right) \le \psi(1) - 1.
\end{gathered}
$$

Thus for the first terms (again writing as (I)-(V)) we have

$$|F_N(x) - g_{w_0}(x)|$$

$$\leq \text{(I)} \left| \frac{\sum_{\mathrm{k}=0}^{\mathrm{K}-1}(\pi(\mathrm{f}^{-1}(\mathrm{k}+\mathrm{x})) - \pi(\mathrm{f}^{-1}(\mathrm{k})))}{\sum_{\mathrm{k}=0}^{\mathrm{K}-1}(\pi(\mathrm{f}^{-1}(\mathrm{k}+1)) - \pi(\mathrm{f}^{-1}(\mathrm{k})))} - \frac{\psi(\mathrm{x}) - 1}{\psi(1) - 1} \right| \tag{30}$$

$$+ \text{(II)} \left| \frac{\pi(\mathrm{f}^{-1}(\mathrm{K})) - \pi(\mathrm{f}^{-1}(0))}{\pi(\mathrm{f}^{-1}(\mathrm{K}+\mathrm{w}))} - \frac{1}{\psi(\mathrm{w}_0)} \right| \tag{31}$$

$$+ \text{(III)} \left| \min\left( \frac{\pi(\mathrm{f}^{-1}(\mathrm{K}+\mathrm{x}))}{\pi(\mathrm{f}^{-1}(\mathrm{K}))}, \frac{\pi(\mathrm{f}^{-1}(\mathrm{K}+\mathrm{w}))}{\pi(\mathrm{f}^{-1}(\mathrm{K}))} \right) - \min(\psi(\mathrm{x}), \psi(\mathrm{w}_0)) \right| \tag{32}$$

$$+ \text{(IV)} \left| \psi(\mathrm{w}_0) - \frac{\pi(\mathrm{f}^{-1}(\mathrm{K}+\mathrm{w}))}{\pi(\mathrm{f}^{-1}(\mathrm{K}))} \right| (\psi(1) - 1) \tag{33}$$

$$+ \text{(V)} \left( \frac{\mathrm{O}(\mathrm{K})}{\mathrm{N}} + \frac{\mathrm{O}(\mathrm{f}^{-1}(0))}{\mathrm{N}} \right). \tag{34}$$

Now in the following we use Prime number theorem in the form

$$\pi(x) = \frac{x}{\log x}\theta(x), \quad \theta(x) \to 1, \text{ if } x \to \infty. \tag{35}$$

Using (see [8, Theorem 1])

$$\frac{t}{\log t}\left(1 + \frac{1}{2\log t}\right) < \pi(t) < \frac{t}{\log t}\left(1 + \frac{3}{2\log t}\right)$$

for $t \geq 59$, we have

$$1 + \frac{1}{2\log t} < \theta(t) < 1 + \frac{3}{2\log t}. \tag{36}$$

In our case

$$f(x) = \log_b x^r, \quad f^{-1}(x) = b^{\frac{x}{r}}, \quad \psi(x) = b^{\frac{x}{r}},$$

$$g_{w_0}(x) = \frac{1}{b^{\frac{w_0}{r}}} \cdot \frac{b^{\frac{x}{r}} - 1}{b^{\frac{1}{r}} - 1} + \frac{\min(b^{\frac{x}{r}}, b^{\frac{w_0}{r}}) - 1}{b^{\frac{w_0}{r}}}$$

and by (35)

$$\frac{\pi(f^{-1}(k+x))}{\pi(f^{-1}(k))} = b^{\frac{x}{r}}\left(\frac{k}{k+x}\right)\frac{\theta(b^{\frac{k+x}{r}})}{\theta(b^{\frac{k}{r}})}.$$

For an upper bound of (I)=(30) we use the inequality proved in Appendix

$$
\begin{aligned}
\mathrm{(I)}&\left|\frac{\sum_{\mathrm{k=0}}^{\mathrm{K-1}}(\pi(\mathrm{f}^{-1}(\mathrm{k+x}))-\pi(\mathrm{f}^{-1}(\mathrm{k})))}{\sum_{\mathrm{k=0}}^{\mathrm{K-1}}(\pi(\mathrm{f}^{-1}(\mathrm{k+1}))-\pi(\mathrm{f}^{-1}(\mathrm{k})))}-\frac{\psi(\mathrm{x})-1}{\psi(1)-1}\right| \\
&\qquad\le \Psi_{k_1}+\left|\frac{\pi(f^{-1}(k_1))-\pi(f^{-1}(0))}{\pi(f^{-1}(K))-\pi(f^{-1}(0))}\right|, \quad (37)
\end{aligned}
$$

where $K \geq k_1$, $\Psi_{k_1} = \sup_{k\geq k_1}\Phi_k$ and

$$
\Phi_k = \left|\ \frac{\pi(f^{-1}(k+x))\ -\ \pi(f^{-1}(k))}{\pi(f^{-1}(k+1))\ -\ \pi(f^{-1}(k))}\ -\ \frac{\psi(x)-1}{\psi(1)-1}\right|. \tag{38}
$$

Applying (35) to (38) we have

$$
\Phi_k = \left|\frac{b^{\frac{x}{r}}\left(\left(\frac{k}{k+x}\right)\frac{\theta(b^{\frac{k+x}{r}})}{\theta(b^{\frac{k}{r}})}-1\right)+b^{\frac{x}{r}}-1}{b^{\frac{1}{r}}\left(\left(\frac{k}{k+1}\right)\frac{\theta(b^{\frac{k+1}{r}})}{\theta(b^{\frac{k}{r}})}-1\right)+b^{\frac{1}{r}}-1}-\frac{b^{\frac{x}{r}}-1}{b^{\frac{1}{r}}-1}\right|. \tag{39}
$$

For (II)=(31) we have

$$
\begin{aligned}
\mathrm{(II)}\left|\frac{\pi(\mathrm{b}^{\frac{\mathrm{K}}{\mathrm{r}}})-\pi(1)}{\pi(\mathrm{b}^{\frac{\mathrm{K+w}}{\mathrm{r}}})}-\frac{1}{\mathrm{b}^{\frac{\mathrm{w_0}}{\mathrm{r}}}}\right| &= \left|\frac{1}{b^{\frac{w}{r}}}\Big(\frac{K+w}{K}\Big)\frac{\theta(b^{\frac{K}{r}})}{\theta(b^{\frac{K+w}{r}})}-\frac{1}{b^{\frac{w_0}{r}}}\right| \\
&\le \frac{1}{b^{\frac{w}{r}}}\left|\Big(\frac{K+w}{K}\Big)\frac{\theta(b^{\frac{K}{r}})}{\theta(b^{\frac{K+w}{r}})}-1\right|+\left|\frac{1}{b^{\frac{w}{r}}}-\frac{1}{b^{\frac{w_0}{r}}}\right|. \quad (40)
\end{aligned}
$$

For(III)=(32) we use

$$
\begin{aligned}
\mathrm{(III)} \le &\left|\min\left(\frac{\pi(f^{-1}(K+x))}{\pi(f^{-1}(K))},\frac{\pi(f^{-1}(K+w))}{\pi(f^{-1}(K))}\right)-\min(\psi(x),\psi(w))\right| \\
&+|\min(\psi(x),\psi(w))-\min(\psi(x),\psi(w_0))| \qquad (41)\\
\le& \max\left(\left|\frac{\pi(f^{-1}(K+x))}{\pi(f^{-1}(K))}-\psi(x)\right|,\quad \left|\frac{\pi(f^{-1}(K+w))}{\pi(f^{-1}(K))}-\psi(w)\right|\right) \\
&+|\psi(w)-\psi(w_0)| \\
=& \max\left(b^{\frac{x}{r}}\left|\left(\frac{K}{K+x}\right)\frac{\theta(b^{\frac{K+x}{r}})}{\theta(b^{\frac{K}{r}})}-1\right|,\quad b^{\frac{w}{r}}\left|\left(\frac{K}{K+w}\right)\frac{\theta(b^{\frac{K+w}{r}})}{\theta(b^{\frac{K}{r}})}-1\right|\right) \\
&+|b^{\frac{w}{r}}-b^{\frac{w_0}{r}}|. \qquad (42)
\end{aligned}
$$

For (IV)=(33) we have

$$(\mathrm{IV})\left|\mathrm{b}^{\frac{\mathrm{w}_0}{\mathrm{r}}}-\frac{\pi(\mathrm{b}^{\frac{\mathrm{K}+\mathrm{w}}{\mathrm{r}}})}{\pi(\mathrm{b}^{\frac{\mathrm{K}}{\mathrm{r}}})}\right|(\mathrm{b}^{\frac{1}{\mathrm{r}}}-1)$$
$$\leq\left(\left|b^{\frac{w_0}{r}}-b^{\frac{w}{r}}\right|+b^{\frac{w}{r}}\left|\left(\frac{K}{K+w}\right)\frac{\theta(b^{\frac{K+w}{r}})}{\theta(b^{\frac{K}{r}})}-1\right|\right)(b^{\frac{1}{r}}-1). \quad (43)$$

For (V)=(34) we have

$$(\mathrm{V})\left(\frac{\mathrm{O(K)}}{\mathrm{N}}+\frac{\mathrm{O}(\pi(\mathrm{f}^{-1}(0)))}{\mathrm{N}}\right)$$
$$\leq\frac{K+1+\pi(1)}{N}=\frac{K+1}{N}=\frac{[f(p_N)]+1}{N}\leq\frac{r\log_b p_N+1}{N}.$$

In all (I)=(39), (II)=(40), (III)=(42) and (IV)=(43) we have common factors

$$\left|\left(\frac{K}{K+w}\right)\frac{\theta(b^{\frac{K+w}{r}})}{\theta(b^{\frac{K}{r}})}-1\right| \quad \text{or} \quad \left|\left(\frac{K+w}{K}\right)\frac{\theta(b^{\frac{K}{r}})}{\theta(b^{\frac{K+w}{r}})}-1\right|.$$

Assuming $b^{\frac{k}{r}}\geq 59$, i.e., $k\geq r\log_b 59$ then (36) implies

$$\frac{\log b+\frac{r}{2(k+x)}}{\log b+\frac{3r}{2k}}<\frac{\theta(b^{\frac{k+x}{r}})}{\theta(b^{\frac{k}{r}})}<\frac{\log b+\frac{3r}{2(k+x)}}{\log b+\frac{r}{2k}}$$

and we have

$$\frac{-x\log b-\frac{r}{2}\cdot\frac{2k^2+6kx+3x^2}{k(k+x)}}{(k+x)\log b+\frac{3r}{2}\cdot\frac{k+x}{k}}<\left(\frac{k}{k+x}\right)\frac{\theta(b^{\frac{k+x}{r}})}{\theta(b^{\frac{k}{r}})}-1$$
$$<\frac{-x\log b+\frac{r}{2}\cdot\frac{2k^2-2kx-x^2}{k(k+x)}}{(k+x)\log b+\frac{k+x}{k}\cdot\frac{r}{2}}$$

which gives

$$\left|\left(\frac{k}{k+x}\right)\frac{\theta(b^{\frac{k+x}{r}})}{\theta(b^{\frac{k}{r}})}-1\right|\leq\frac{\log b+\frac{11}{2}r}{k\log b+\frac{r}{2}} \quad (44)$$

or

$$\left|\left(\frac{k+x}{k}\right)\frac{\theta(b^{\frac{k}{r}})}{\theta(b^{\frac{k+x}{r}})}-1\right|\leq\frac{\log b+\frac{11}{2}r}{k\log b+\frac{r}{4}}. \quad (45)$$

Applying (44) and (45) to (I)=(39), for $k \geq r \log_b 59$ we find

$$\Phi_k \leq \frac{2b^{\frac{1}{r}\frac{\log b+\frac{11}{2}r}{k\log b+\frac{r}{2}}}}{b^{\frac{1}{r}}-1-b^{\frac{1}{r}\frac{\log b+\frac{11}{2}r}{k\log b+\frac{r}{2}}}}, \quad \text{provided that } \ k > \frac{5rb^{\frac{1}{r}}+\frac{r}{2}+b^{\frac{1}{r}}\log b}{(b^{\frac{1}{r}}-1)\log b}. \tag{46}$$

In this case (46) is decreasing with respect to $k$ and in (37) it can be using (46) for $k = k_1$. Then for approximate (I) we need approximate second term in (37):

$$\left|\frac{\pi(f^{-1}(k_1))-\pi(f^{-1}(0))}{\pi(f^{-1}(K))-\pi(f^{-1}(0))}\right| = \frac{K}{k_1}\frac{1}{b^{\frac{K-k_1}{r}}}\frac{\theta(b^{\frac{k_1}{r}})}{\theta(b^{\frac{K}{r}})} < \frac{K}{k_1}\frac{1}{b^{\frac{K-k_1}{r}}}\frac{\log b+\frac{3}{2k_1}r}{\log b+\frac{1}{2K}r}.$$

Also applying (44) and (45) to (I)=(37), (II)=(40), (III)=(42) and (IV)=(43), we find

$$\begin{aligned}
&|F_N(x)-g_{w_0}(x)| \\
&\leq \text{(I)}\ \frac{2\mathrm{b}^{\frac{1}{\mathrm{r}}\frac{\log \mathrm{b}+\frac{11}{2}\mathrm{r}}{\mathrm{k}_1\log \mathrm{b}+\frac{1}{2}\mathrm{r}}}}{\mathrm{b}^{\frac{1}{\mathrm{r}}}-1-\mathrm{b}^{\frac{1}{\mathrm{r}}\frac{\log \mathrm{b}+\frac{11}{2}\mathrm{r}}{\mathrm{k}_1\log \mathrm{b}+\frac{1}{2}\mathrm{r}}}} + \frac{\mathrm{K}}{\mathrm{k}_1}\frac{1}{\mathrm{b}^{\frac{\mathrm{K}-\mathrm{k}_1}{\mathrm{r}}}}\cdot\frac{\log \mathrm{b}+\frac{3}{2\mathrm{k}_1}\mathrm{r}}{\log \mathrm{b}+\frac{1}{2\mathrm{K}}\mathrm{r}} \\
&\quad + \text{(II)}\ \frac{1}{\mathrm{b}^{\frac{\mathrm{w}}{\mathrm{r}}}}\cdot\frac{\log \mathrm{b}+\frac{11}{2}\mathrm{r}}{\mathrm{K}\log \mathrm{b}+\frac{\mathrm{r}}{4}} + \left|\frac{1}{\mathrm{b}^{\frac{\mathrm{w}_0}{\mathrm{r}}}}-\frac{1}{\mathrm{b}^{\frac{\mathrm{w}}{\mathrm{r}}}}\right| \\
&\quad + \text{(III)}\ \mathrm{b}^{\frac{1}{\mathrm{r}}}\cdot\frac{\log \mathrm{b}+\frac{11}{2}\mathrm{r}}{\mathrm{K}\log \mathrm{b}+\frac{\mathrm{r}}{2}} + |\mathrm{b}^{\frac{\mathrm{w}_0}{\mathrm{r}}}-\mathrm{b}^{\frac{\mathrm{w}}{\mathrm{r}}}| \\
&\quad + \text{(IV)}\ \mathrm{b}^{\frac{\mathrm{w}}{\mathrm{r}}}(\mathrm{b}^{\frac{1}{\mathrm{r}}}-1)\frac{\log \mathrm{b}+\frac{11}{2}\mathrm{r}}{\mathrm{K}\log \mathrm{b}+\frac{\mathrm{r}}{2}} + (\mathrm{b}^{\frac{1}{\mathrm{r}}}-1)\cdot|\mathrm{b}^{\frac{\mathrm{w}_0}{\mathrm{r}}}-\mathrm{b}^{\frac{\mathrm{w}}{\mathrm{r}}}| \\
&\quad + \text{(V)}\ \frac{\mathrm{r}\log_{\mathrm{b}}\mathrm{p}_{\mathrm{N}}+1}{\mathrm{N}}.
\end{aligned}$$

It can be simplified as

$$\text{(II)}+\text{(III)}+\text{(IV)} \leq 3(b^{\frac{1}{r}})^2\frac{\log b+\frac{11}{2}r}{K\log b+\frac{r}{4}} + 3(b^{\frac{1}{r}})^2\frac{|w-w_0|}{r}\log b$$

and

$$\text{the first term of (I)} \leq \frac{b^{\frac{1}{r}}(2\log b+11r)}{(b^{\frac{1}{r}}-1)(\log b)k_1-5rb^{\frac{1}{r}}-b^{\frac{1}{r}}\log b-\frac{r}{2}}.$$

$\square$

# 4. Appendix

**4.1. Proof of (37).**

Denote $\Psi_{k_1} = \sup_{k\ge k_1} \Phi_k$ and

$$\Phi_k = \left| \frac{\pi(f^{-1}(k+x)) - \pi(f^{-1}(k))}{\pi(f^{-1}(k+1)) - \pi(f^{-1}(k))} - \frac{\psi(x)-1}{\psi(1)-1} \right|.$$

For $k \ge k_1$ and $0 \le x \le 1$ we have

$$-\Psi_{k_1} \le \frac{\pi(f^{-1}(k+x)) - \pi(f^{-1}(k))}{\pi(f^{-1}(k+1)) - \pi(f^{-1}(k))} - \frac{\psi(x)-1}{\psi(1)-1} \le \Psi_{k_1}.$$

Then

$$\pi\big(f^{-1}(k+x)\big) - \pi(f^{-1}(k)) \le \left(\Psi_{k_1} + \frac{\psi(x)-1}{\psi(1)-1}\right)\Big(\pi\big(f^{-1}(k+1)\big) - \pi\big(f^{-1}(k)\big)\Big),$$

$$\Big(\pi\big(f^{-1}(k+1)\big) - \pi\big(f^{-1}(k)\big)\Big)\left(-\Psi_{k_1} + \frac{\psi(x)-1}{\psi(1)-1}\right) \le \pi\big(f^{-1}(k+x)\big) - \pi\big(f^{-1}(k)\big).$$

Thus we have

$$-\Psi_{k_1} + \frac{\psi(x)-1}{\psi(1)-1} \le \frac{\sum_{k=k_1}^{K-1}(\pi(f^{-1}(k+x)) - \pi(f^{-1}(k)))}{\sum_{k=k_1}^{K-1}(\pi(f^{-1}(k+1)) - \pi(f^{-1}(k)))} \le \Psi_{k_1} + \frac{\psi(x)-1}{\psi(1)-1}. \quad (47)$$

Abbreviated

$$A = \sum_{k=k_1}^{K-1}\Big(\pi\big(f^{-1}(k+x)\big) - \pi\big(f^{-1}(k)\big)\Big);$$

$$B = \sum_{k=k_1}^{K-1}\Big(\pi\big(f^{-1}(k+1)\big) - \pi\big(f^{-1}(k)\big)\Big);$$

$$C = \sum_{k=0}^{k_1-1}\Big(\pi\big(f^{-1}(k+x)\big) - \pi\big(f^{-1}(k)\big)\Big);$$

$$D = \sum_{k=0}^{k_1-1}\Big(\pi\big(f^{-1}(k+1)\big) - \pi\big(f^{-1}(k)\big)\Big);$$

Then

$$\frac{A+C}{B+D} - \frac{A}{B} = \frac{D\big(\frac{C}{D} - \frac{A}{B}\big)}{B+D}, \quad \text{where} \quad 0 \le \frac{C}{D} \le 1, \;\; 0 \le \frac{A}{B} \le 1,$$

and

$$D = \pi(f^{-1}(k_1)) - \pi(f^{-1}(0)), \;\; B + D = \pi(f^{-1}(K)) - \pi(f^{-1}(0)).$$

From it

$$\left|\frac{A+C}{B+D}-\frac{A}{B}\right|\leq\frac{\pi(f^{-1}(k_1))-\pi(f^{-1}(0))}{\pi(f^{-1}(K))-\pi(f^{-1}(0))},$$

using

$$\left|\frac{A+C}{B+D}-\frac{\psi(x)-1}{\psi(1)-1}\right|\leq\left|\frac{A+C}{B+D}-\frac{A}{B}\right|+\left|\frac{A}{B}-\frac{\psi(x)-1}{\psi(1)-1}\right|,$$

and (47) imply (39).

### 4.2. Proof of the inequality

$$A:=|\min\bigl(\psi(x),\psi(w)\bigr)-\min\bigl(\psi(x),\psi(w_0)\bigr)|\leq|\psi(w)-\psi(w_0)|$$

in (19) and (41).

We compute all cases:

$$\begin{aligned}
x<w<w_0;&\quad A=\psi(x)-\psi(x)=0,\\
x<w_0<w;&\quad A=\psi(x)-\psi(x)=0,\\
w_0<x<w;&\quad A=|\psi(x)-\psi(w_0)|<|\psi(w_0)-\psi(w)|,\\
w_0<w<x;&\quad A=|\psi(w)-\psi(w_0)|,\\
w<w_0<x;&\quad A=|\psi(w)-\psi(w_0)|,\\
w<x<w_0;&\quad A=|\psi(w)-\psi(x)|<|\psi(w_0)-\psi(w)|.
\end{aligned}$$

### 4.3. Discrepancy $D_N$ of the sequence

$$\{\log_b 1^r\},\quad\{\log_b 2^r\},\quad\{\log_b 3^r\},\dots,\{\log_b N^r\}.$$

Put

$$F_N(x)=\frac{1}{N}\#\{n\leq N;\{\log_b n^r\}\in[0,x)\}.$$

Then discrepancy is defined as

$$D_N=\sup_{x\in[0,1]}|F_N(x)-x|.$$

Clearly, we have

$$D_N\leq\sup_{x\in[0,1]}|F_N(x)-g_{w_0}(x)|+\sup_{x\in[0,1]}|g_{w_0}(x)-x|.\tag{48}$$

For the first part of (48) we use Theorem 5. Now we put $w_0 = 0$ and for the second part we need found upper bound of

$$x - g_0(x) = x - \frac{b^{\frac{x}{r}} - 1}{b^{\frac{1}{r}} - 1}.$$

By Lagrange's theorem

$$b^{\frac{x}{r}} - 1 = (x-0)b^{\frac{x_1}{r}}\frac{\log b}{r}, \quad x_1 \in (0,x),$$
$$b^{\frac{1}{r}} - 1 = (1-0)b^{\frac{x_2}{r}}\frac{\log b}{r}, \quad x_2 \in (0,1).$$

Thus

$$x - g_0(x) = x\bigl(1 - b^{\frac{x_1-x_2}{r}}\bigr) = x\bigl(0 - (x_1 - x_2)\bigr)b^{\frac{x_3}{r}}\frac{\log b}{r}, \quad x_3 \in (x_1, x_2).$$

The upper bound is

$$|x - g_0(x)| < b^{\frac{1}{r}}\frac{\log b}{r}$$

and applying Theorem 5 we have

$$D_N \le \frac{|w-0|}{r}\cdot \log b \cdot b^{\frac{1}{r}} \cdot (b^{\frac{1}{r}} + 1) + \frac{3}{N} + \frac{r\log_b N}{N} + b^{\frac{1}{r}}\frac{\log b}{r},$$

where $w = \{r \log_b N\}$.

## REFERENCES


[1] BAKER, R. C.—HARMAN, G.—PINTZ, J.: *The difference between consecutive primes, II*, Proc. London Math. Soc. (3) **83** (2001), no. 3, 532–562.

[2] BALÁŽ, V.—NAGASAKA, K.—STRAUCH, O.: *Benford's law and distribution functions of sequences in* $(0,1)$ (Russian), Mat. Zametki **88** (2010), no. 4, 485–501; (English translation) Math. Notes **88** (2010), no. 4, 449–463.

[3] BENFORD, F.: *The law of anomalous numbers*, Proc. Amer. Phil. Soc. **78** (1938), 551–572 (Zbl 18, 265; JFM 64.0555.03).

[4] DIACONIS, P.: *The distribution of leading digits and uniform distribution* mod 1, The Annals of Probability **5** (1977), no. 1, 72–81.

[5] ELIAHOU, S.—MASSÉ, B.—SCHNEIDER, D.: *On the Mantisa distribution of powers of natural and prime numbers*, Acta Math. Hungar. **139** (2013), no. 1–2, 49–63.

[6] NEWCOMB, S.: *Note on the frequency of use of the different digits in natural numbers*, Amer. J. Math. **4** (1881), 39–41 (JFM 13.0161.01).

[7] OHKUBO, Y.: *On sequences involving primes*, Uniform Distribution Theory, Vol. 6 (2011), no. 2, 221–238.

[8] ROSSER, J. B.—SCHOENFELD, L.: *Approximate formulas for some functions of prime numbers*, Illinois. J. Math. **6** (1962), 64–94.

[9] STRAUCH, O.—BLAŽEKOVÁ, O.: *Distribution of the sequence* $p_n/n$ mod 1, Uniform Distribution Theory **1** (2006), no. 1, 45-63.

[10] STRAUCH, O.—PORUBSKÝ, Š.: *Distribution of Sequences: A Sampler*, Peter Lang, Frankfurt am Main, 2005.

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