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uniform
distribution
theory

DOI: 10.1515/udt-2016–0002
Unif. Distrib. Theory **11** (2016), no.1, 15–21

UNIFORM DISTRIBUTION OF THE SEQUENCE OF BALANCING NUMBERS MODULO m

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Dedicated to the memory of Professor Pierre Liardet

ABSTRACT. The balancing numbers and the balancers were introduced by Behera et al. in the year 1999, which were obtained from a simple diophantine equation. The goal of this paper is to investigate the moduli for which all the residues appear with equal frequency with a single period in the sequence of balancing numbers. Also, it is claimed that, the balancing numbers are uniformly distributed modulo 2, and this holds for all other powers of 2 as well. Further, it is shown that the balancing numbers are not uniformly distributed over odd primes.

Communicated by Oto Strauch

1. Introduction

The concept of balancing numbers were first introduced by Behera and Panda [1] when they considered the integer solutions of the Diophantine equation

$$1+2+\cdots+(n-1)=(n+1)+(n+2)+\cdots+(n+r)$$

for some positive integers n and r, calling r the balancer corresponding to the balancing number n. The nth balancing number is denoted by B_n which satisfy the recurrence relation

$$B_{n+1}=6B_n-B_{n-1},\ \text{for n}\geq 1,$$

with initial values $B_0=0$, $B_1=1$. The closed form of balancing numbers which is also called as Binet's formula for balancing number is given by $B_n=\frac{\lambda_1^n-\lambda_2^n}{\lambda_1-\lambda_2}$, where $\lambda_1=3+\sqrt{8}$ and $\lambda_2=3-\sqrt{8}$ are the roots of the equation $\lambda^2-6\lambda+1=0$.

2010 Mathematics Subject Classification: 11B37, 11B39.
Keywords: Balancing numbers; Balancers; Uniform distribution; Periodicity of balancing numbers.

In the year 1960, Wall studied the periodicity of Fibonacci numbers modulo arbitrary natural numbers [8]. He explicitly determined the period length of $a_{n+2} = a_{n+1} + a_n \pmod m$ in terms a_0, a_1 and m. By Niederreiter [3], the Fibonacci sequence

$$F_0 = 0, \quad F_1 = 1, \quad F_{n+2} = F_{n+1} + F_n$$

is uniformly distributed modulo m if and only if $m = 5^k$, $k = 1, 2, \ldots$ Recently, Panda et al. established the periodicity of balancing numbers modulo primes and studied the periods of sequence of balancing numbers modulo balancing, The Pell and the associated Pell numbers [6]. In addition, the authors also shown that the period of this sequence coincides with the modulus of congruence if there exists the modulus in any power of 2. In [4], it is given that for the sequence of rational integers $\{a_n\}$, define $A(j, m; N)$ as the number of terms among $a_1, a_2, a_3, \ldots, a_N$ that satisfy the congruence $a_n \equiv j \ (mod\ m)$. Then $\{a_n\}$ is uniformly distributed mod m in case

$$\lim_{N\to\infty} \frac{A(j, m; N)}{N} = \frac{1}{m}, \quad \text{for each} \quad j = 0, 1, \ldots, m-1. \tag{1}$$

In this article, we survey how frequently each residue is excepted to appear. It is proved that, the balancing numbers are uniformly distributed modulo even prime 2, and are not uniformly distributed over odd primes. Further, it is also shown that the balancing numbers attains uniform values from each residue class modulo 2^k for any natural number k.

2. Distribution of sequence of balancing numbers modulo primes

Panda [5] and Ray [7] established some interesting results of divisibility and the greatest common divisor of balancing numbers. We summarize few of them.

Lemma 2.1. *If n and k are natural numbers, then B_k divides B_{nk}.*

Lemma 2.2. *If m and n are natural numbers, then $(B_m, B_n) = B_{(m,n)}$.*

Lemma 2.3. *If m and n are any natural numbers which $m > 1$, then $B_m | B_n$ if and only if $m | n$.*

The following result can be easily verified.

Lemma 2.4. *Every positive integer m divides some balancing number whose index does not exceed m^2.*

The following result is proved in [6] .

Lemma 2.5. *If p is a prime of the form $8x \pm 1$, then $B_{p-1} \equiv 0 \pmod p$, $B_p \equiv 0 \pmod p$; further if the prime p is of the form $8x \pm 3$, then $B_p \equiv -1 \pmod p$, $B_{p+1} \equiv 0 \pmod p$.*

The following result shows that the sequence of balancing number is uniformly distributed modulo even prime.

Theorem 2.6. *The sequence of balancing numbers is uniformly distributed modulo 2.*

Proof. If all the balancing numbers are reduced modulo 2, then the sequence of the least residues is obtained where the initial string is

$$\text{“ } 0, 1, 0, 1, 0, 1, \dots \text{”},$$

which is periodic with the period length 2. Using (1), it would seem that

$$\lim_{n\to\infty} \frac{1}{N} A\,(j,\, 2;\, N) = \frac{1}{2} \quad \text{for } \; j = 0, 1,$$

which means that the sequence of balancing numbers is uniformly distributed modulo 2. □

For any odd prime, say for $p = 5$, it is observed that the sequence of balancing numbers is not uniformly distributed modulo 5. Because, if all balancing numbers are reduced modulo 5, then the sequence $0, 1, 1, 0, 4, 4, 0, 1, 1, 0, 4, 4, 0, \dots$ of the least residues is obtained which is periodic with the period length 6. But it would seem that among the first six elements of the sequence, it does not occur at the most in one element from each residue class modulo 5. We generalize this result as follows:

Theorem 2.7. *The sequence of balancing numbers is not uniformly distributed modulo p for any odd prime p.*

Proof. Let p be an odd prime. Let $\lambda > 0$ be the minimal such that $B_\lambda \equiv 0 \pmod p$. Then $B_{k\lambda} \equiv 0 \pmod p$ for all $k = 1, 2, \dots$ If $B_\gamma \equiv 0 \pmod p$ and $k\lambda < \gamma < (k+1)\lambda$, then for n.s.d.$(\lambda, \gamma) = max\{n \in \mathbb{N} : n|\lambda \wedge n|\gamma\} = \beta$, we have $B_\beta \equiv 0 \pmod p$ and $0 < \beta < \lambda$, which is a contradiction. Thus all $B_n \equiv 0 \pmod p$ are $B_{k\lambda} \equiv 0 \pmod p$ for $k = 1, 2, \dots$ It follows that,

$$A\,(0,\, p;\, N) = \frac{N}{\lambda} + O(1),$$

which ends the proof. □

For each natural number m, $\pi(m)$ denotes the periodic length of the sequence of balancing numbers modulo m. In [6], Panda et al. shown some divisibility properties concerning the periodicity of balancing numbers. Using those results, we have the following observations.

Observation 2.8. We know that for any natural number $m > 1$, $\pi(m) = m$ if and only if $m = 2^k$ for some natural number k. Therefore, if the sequence of balancing numbers modulo m is uniform, then evidently $m|\pi(m)$.

Observation 2.9. Let p be a prime. If $p = 8x \pm 1$, then the sequence of balancing numbers modulo p cannot be uniform for such p, because $\pi(p)|p-1$ but $p \nmid p-1$ and so, $p \nmid \pi(p)$.

Observation 2.10. If $p = 8x \pm 3$, then the sequence of balancing numbers modulo p cannot be uniform. As $\pi(p)|p+1$ and $p \nmid p+1$, therefore $p \nmid \pi(p)$.

Observation 2.11. If the sequence of balancing numbers modulo m is uniform and n divides m, then the sequence of balancing numbers modulo n is also uniform.

Observation 2.12. If $\pi(n)$ divides $\pi(m)$ is b, then with in $\pi(n)$ terms each residue appears b divides x times. Then, the sequence of balancing numbers modulo n is uniform.

The following result is useful while proving the subsequent result.

Lemma 2.13. *Let B_n be the nth balancing number, then*

$$B_n = 3^{n-1} \sum_{j=0}^{\infty} \left(\frac{\sqrt{8}}{3} \right)^{2j} \binom{n}{2j+1}.$$

P r o o f. Using Binet's formula for balancing numbers and expanding by binomial theorem, we get

$$\begin{aligned} B_n &= \frac{\lambda_1^n - \lambda_2^n}{2\sqrt{8}} \\ &= \frac{3^n}{2\sqrt{8}} \left[\left(1 + \frac{\sqrt{8}}{3}\right)^n - \left(1 - \frac{\sqrt{8}}{3}\right)^n \right] \\ &= \frac{3^n}{2\sqrt{8}} \left[\left(1 + \binom{n}{1} \left(\frac{\sqrt{8}}{3}\right) + \binom{n}{2} \left(\frac{\sqrt{8}}{3}\right)^2 + \cdots \right) \right. \\ &\qquad \left. - \left(1 - \binom{n}{1} \left(\frac{\sqrt{8}}{3}\right) + \binom{n}{2} \left(\frac{\sqrt{8}}{3}\right)^2 - \cdots \right) \right] \end{aligned}$$

$$= 3^{n-1}\left[\binom{n}{1} + \binom{n}{3}\left(\frac{\sqrt{8}}{3}\right)^2 + \cdots\right]$$

$$= 3^{n-1}\sum_{j=0}^{\infty}\left(\frac{\sqrt{8}}{3}\right)^{2j}\binom{n}{2j+1},$$

which completes the proof. □

Theorem 2.14. *The sequence of balancing numbers is uniformly distributed modulo 2^k for all integers $k \geq 1$.*

Proof. In [6], Panda et al. shown that for any natural number $n > 1$, $\pi(n) = n$ if and only if $n = 2^k$ for some natural number k. It will be adequate to show that among the first 2^k elements of the sequence, we found at most one element from each residue class $\bmod 2^k$. The basis step is clear by Theorem 2.6. Assume that the sequence of balancing numbers is uniform modulo 2^{r-1}, where r is any positive integer less than or equal to k. Now we proceed by induction on r.

Let $0 \leq m, n < 2^r$, and without loss of generality, say $m \leq n$. We will prove that, in fact, $m = n$. By virtue of Lemma 2.13, the expression $B_n \equiv B_m \pmod{2^r}$ becomes

$$\sum_{j=0}^{r-1} 2^{3j}\cdot 3^{-2j}\binom{n}{2j+1} \equiv 3^{-(n-m)}\sum_{j=0}^{\infty} 2^{3j}\cdot 3^{-2j}\binom{m}{2j+1} \pmod{2^r}. \quad (2)$$

As $3^{-(2^{r-1})} \equiv 1 \pmod{2^r}$ by the Euler-Fermat theorem, the equation (2) reduces to the following:

$$\sum_{j=0}^{2^{r-1}} 2^{3j}\cdot 3^{-2j}\left[\binom{n}{2j+1} - \binom{m}{2j+1}\right] \equiv 0 \pmod{2^r}. \quad (3)$$

Using the Cauchy combinatorial formula

$$\binom{a+b}{c} = \sum_{d=0}^{a}\binom{a}{d}\binom{b}{c-d},$$

we obtain

$$\binom{n}{2j+1} = \binom{m}{2j+1} + \sum_{i=1}^{2j+1}\binom{n-m}{i}\binom{m}{2j+1-i}.$$

Substituting this result into (3) gives

$$(n-m)\sum_{j=0}^{2^{r-1}}\left[\sum_{i=1}^{2j+1}2^{3j}3^{-2j}\binom{n-m}{i}\frac{1}{(n-m)}\binom{m}{2j+1-i}\right]\equiv 0 \pmod{2^r}. \tag{4}$$

The term $j=0$ and $i=1$ in (4) is equal to 1 and every other terms for $j\geq 1$ and $1\leq i\leq 2j+1$ is even. This follows from the fact that the maximal α for which $2^\alpha|i!$ is $\alpha<i$. Since $i\leq 2j+1$, then $\alpha\leq 2j$ and

$$\frac{8^j}{2^\alpha}\geq\frac{2^{3j}}{2^{2j}}=2^j.$$

Thus

$$2^{3j}3^{-2j}\frac{(n-m-1)(n-m-2)\dots(n-m-i+1)}{i!}\binom{m}{2j+1-i}\quad\text{is even.}$$

Now, rewrite the formula (4) as $(n-m)(1+A)$.

To prove $B_n\equiv B_m \pmod{p^r}$ implies $n\equiv m \pmod{p^r}$, there can be used the following implication

$$2^r|(n-m)(1+A)\quad\text{and}\quad 2|A\Rightarrow 2^r|(n-m).$$

We know that $0\leq m,\ n<2^r$, and so we must conclude that $n=m$. Therefore any residue modulo 2^r appears once, which completes the induction. □

Theorem 2.15. *The sequence of balancing numbers is not uniformly distributed modulo m for any composite integer $m\neq 2^k$ for every value of $k\in\mathbb{N}$.*

Proof. Suppose the sequence of balancing numbers is uniformly distributed modulo m for any composite integer $m\neq 2^k$. In [4] Niven proved that a sequence is uniformly distributed modulo any positive integer n, then it is uniformly distributed modulo every divisor of n. Since, every composite number can be written as the product of two or more (not necessarily distinct) primes, we have that the sequence of balancing numbers is uniformly distributed modulo p, where p is some prime factors of $m\neq 2^k$. This contradicts Theorem 2.7. □

Remark 2.16. Note that, our proof is similar but different as the proof in Niederreiter [3]. Also Nathanson [2] studied the distribution of a sequence $x_{n+2}=ax_{n+1}+bx_n$ and proved that if p is an odd prime, then the sequence x_n is uniformly distributed modulo p if and only if $p|(a^2+4b)$, $p\nmid a$ and $p\nmid(2x_2-ax_1)$. The sequence x_n is uniformly distributed modulo 2 if and only if $2|a$, $2\nmid b$ and $2\nmid(x_2-x_1)$. The balancing numbers satisfy such conditions.

Acknowledgement. The authors are grateful to the anonymous reviewers for their useful and helpful remarks and suggestions to improve this article.

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Received December 24, 2014
Accepted June 26, 2015

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