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CALCULATION OF OPTIMAL PATH FOR PARALLEL CAR PARKING

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The problem of calculation of optimal path for reverse parallel parking is examined. Elementary mathematical model of car movement on plane is described. Optimality criterion and restriction on a possible path for parallel parking are formulated. It is offered a way of choice of a quasi-optimal path for reverse parallel parking. Corresponding numerical calculations and graphic example are presented.

Keywords: parking a car, mathematical model, system of differential equations, Cauchy problem, conditional optimisation

1. Introduction

Lately many key manufacturers of cars pay much attention to work out the subsystems that can help a driver in driving, both at road movement and parking. Such companies as Volkswagen, Toyota, Citroen, BMW, Mercedes, Fords, have even provided automatic parking subsystems that is car parking without participation of a person (hands free). However such subsystems have still cost quite expensive and the parking process itself takes a lot of time because the system constantly scans surrounding territory to except collisions. Wherefore, it is possible to work out the subsystems which do not replace the person completely but only prompt an optimal path, displaying it on the screen of any device (for example, GPS). Such parking subsystems should be much cheaper than automatic ones because the main work is made by a driver. Besides, widespread introduction of such subsystems would also promote economy of fuel, and so less environmental contamination by exhaust gases because they allow to cut time of car parking.

It is natural that you need a mathematical model of the car movement to create subsystems of car control. Recently many researchers have been engaged in the development and improvement of such a model [1–7]. Meanwhile, computer simulators and robots that realize mathematical models of car movement control in various operating conditions are developed [4–7]. One of the research directions is a logical design of parallel parking [6, 7].

2. Modelling and Research of Parallel Parking Process

2.1. Elementary mathematical model of car movement

Position of a standard car on plane is definitely determined by its sizes (length and width) and coordinates (x, y, θ), where (x, y) are coordinates of point M on the plane xOy, θ is a corner between a car longitudinal axle and axis Ox (Figure 1).

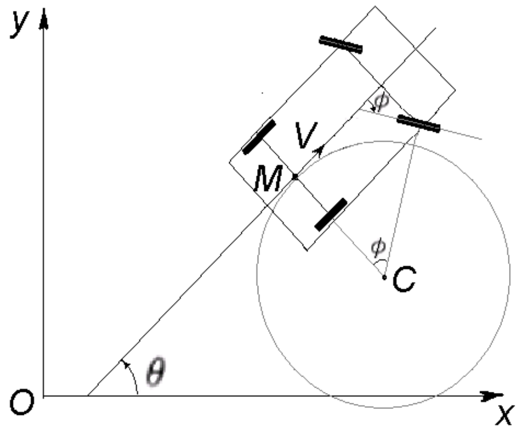

Figure 1. Model of car movement on plane

The repositioning of a car depends on its speed V and steering angle of front wheels relative to the longitudinal axle of a car ϕ. In this case the turning radius R for the point M is $W\,Ctg\phi + \frac{w}{2}$, where W is a distance between front axle and back axle (wheel base), w is a distance between car wheels on the back axle (rear track).

As the rate of angle θ change is equal-in-magnitude to the angular rate of the point M and it is opposite to it in direction, that is $\frac{\partial\theta}{\partial t} = -\frac{V}{R}$, the car movement process on plane in the most elementary case can be described with the following differential equation system:

$$\frac{\partial x}{\partial t} = V\cos\theta\,,$$
$$\frac{\partial y}{\partial t} = V\sin\theta\,, \qquad (1)$$
$$\frac{\partial\theta}{\partial t} = -\frac{V}{W\,Ctg\phi + \frac{w}{2}}\,.$$

The examined model of car movement on plane (1) slightly differs from the kinematics model offered in the work of P. Mellodge [1]. The basic difference is that in the model (1) we use the distance between car wheels on back axle w, which isn't considered in the model of P. Mellodge.

2.2. Statement of problem of optimal parallel car parking

Now we will examine the problem of parallel parking in case of right driving (Figure 2).

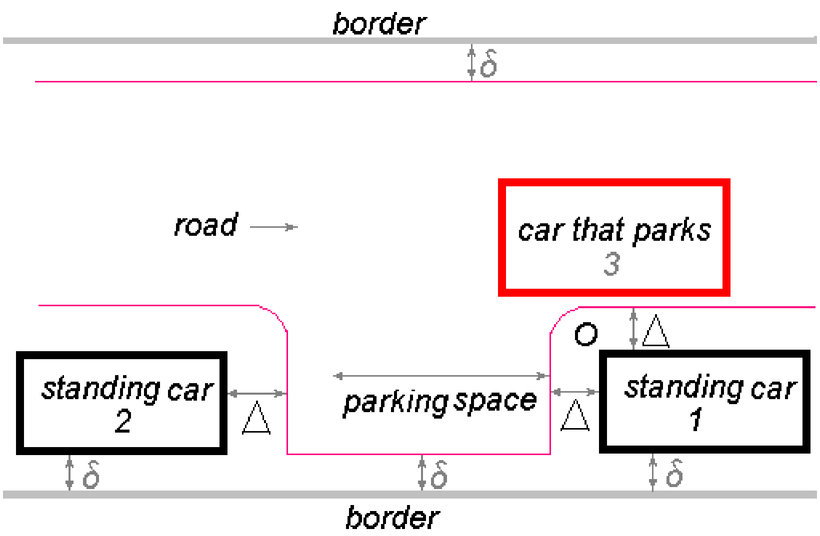

Figure 2. Parallel car parking

It is necessary to park a car in a parking space without approaching to the standing cars and right border nearer than at a distance of Δ and δ, correspondingly, and without going outside the left border. The area which is not allowed to leave is limited on the Figure 2 with lines of lilac colour.

We will understand as optimal parking such a parking when the length of car 3 passing in a parking space, concerning the standing car 1, is minimum. Here it is supposed that the parking car backs up, because (as is known from practical experience) it is more optimal than passing forward. In such formulation of parking optimality it is also determined the minimum possible distance between standing cars 1 and 2 that supposes possibility of car 3 parking.

In the statement formulated above the problem of optimal parking is a problem of optimal control [8], where the control actions are car speed $V = V(t)$ and steering angle of front wheels relative to the car longitudinal axle $\phi = \phi(t)$ For mathematical notation of optimal parking problem we will examine t he coordinate system xOy, oriented in such a way that the reference point O is in back corner point of the standing car 1, and the axle Ox is parallel to the road (Figure 2). At that we introduce the following signs:

$$z(t) = \begin{pmatrix} x(t) \\ y(t) \\ \theta(t) \end{pmatrix},\ u(t) = \begin{pmatrix} V(t) \\ \phi(t) \end{pmatrix},\ F(z(t), u(t), t) = \begin{pmatrix} u_1(t)\cos(t) \\ u_1(t)\sin(t) \\ -\dfrac{u_1(t)}{W\,Ctg(u_2(t)) + \dfrac{w}{2}} \end{pmatrix}.$$

Then the system (1) will be written in such a way:

$$\frac{\partial z(t)}{\partial t} = F(z(t), u(t), t)\,. \tag{2}$$

The objective function for the formulated problem of optimal parking is:

$$-z_1(T) \to \min, \tag{3}$$

where T is a moment of control time when the car 3stands on the parking space (Figure 2). Then $t \in [0, T]$ and $t = 0$ stands for the point of time when the car 3 starts to back upstanding next to the car 1.

It is necessary to notice that in the problem (2)–(3) the limits of motion path which secure parking (you cannot leave the admissible area (Figure 2)) should be observed.

2.3. How to choose a quasi-optimal path for parallel car parking

Considering the position and motion of the car in abounded space in whole depends on its physical sizes and technical features, for the further consideration of the problem of parking path calculation it is necessary to introduce the following parameters (in addition to the above introduced ones): L is a car length (m), B is a car width (m), ϕ_{max} is a maximum steering angle of front wheels (radian), b is a distance from the point M to the edge of back bumper (m). Also, with Bs we mark the width of standing car 1, and with S we mark the road width.

In case of reverse parallel parking the *S*–shaped motion path of the point M is obvious. To provide it the analogous conduct must be in case of change of angle ϕ, therefore we will consider a variant of path calculation when the change of angle ϕ is described with the function

$$\phi_1(t) = \frac{\phi_{max}}{\frac{\pi}{2}} \operatorname{arctg}(a_1 t + a_0)\,, \tag{4}$$

where a_0, a_1 are function parameters $\phi_1(t)$ .In such a variant of calculation the angle ϕ is near to ϕ_{max} at the initial time, and it is near to the angle $-\phi_{max}$ at the end time. Here it is also supposed that the parking car is parallel to the standing car 1 at the initial time that is $z_3(0) = \theta(0) = 0$.

When choosing the optimal parking path the initial position of the parking car is important, that is the distance from the standing car 1 (we mark it a_3) and the value of its passing ahead concerning the car 1. Therefore with a_2 we mark *x*-coordinate of the point *M* at the initial time. There by the initial situation in the Cauchy problem for the system (2) is the following:

$$z(0) = \begin{pmatrix} a_2 \\ a_3 + \dfrac{B}{2} \\ 0 \end{pmatrix}. \tag{5}$$

The final time T of the Cauchy problem solution is determined with the position when the car is in the parking space parallel to the border that is when $z_3(T) = \theta(T) = 0$.

The speed of the car V at normal (that is non-extreme) parking should not be high and there fore its change cannot be decisive at short time, and so we will consider it constant and mark with a_4.

Considering the suppositions above the choice of optimal path in parallel parking comes to the determination of parameters a_0, a_1 of the function $\phi_1(t)$, parameters of the initial car position a_2, a_3 and car speed a_4, that provide condition (2), criterion optimality (3) and satisfaction of limits for safe motion path. The latter limits we can write as following:

$$z_1(T) + (L - b) \le -\Delta, \tag{6}$$

$$\min_{t\in[0,T]}\left\{\sqrt{(z_1(t) + (L-b)\cos(z_3(t)) + \frac{B}{2}\sin(z_3(t)))^2 + (z_2(t) + (L-b)\sin(z_3(t)) - \frac{B}{2}\cos(z_3(t)))^2}\right\} \ge \Delta, \tag{7}$$

$$\min_{t\in[0,T]}\left\{z_2(t) + (-b)\sin(z_3(t)) + \frac{B}{2}\cos(z_3(t))\right\} \ge -Bs, \tag{8}$$

$$\max_{t\in[0,T]}\left\{z_2(t) + (L-b)\sin(z_3(t)) + \frac{B}{2}\cos(z_3(t))\right\} \le S - Bs - \delta. \tag{9}$$

The limit (6) provides that after parking termination the front bumper of the parking car will be at a distance not less than Δ from the standing car 1. The limit (7) provides that in the parking process the right front wing of the parking car will not approach at a distance less than Δ to the left back wing of the standing car 1. The limit (8) provides that in the parking process the right back wing of the parking car will not approach at a distance less than δ to the right road border. The limit (9) provides that in the parking process the left front wing of the parking car will not approach at a distance less than δ to the left road border.

With suppositions above the problem (2), (3), (6)–(9) can be considered as a problem of conditional optimisation relative to 5 optimisation parameters $a = (a_0, a_1, a_2, a_3, a_4)^T$ [9].

2.4. Example of calculation of optimal parking path

For example we will examine three cars of different classes which have the following technical features that are interesting for us:

- Class A – Volkswagen Polo Hatchback (length $L = 3.97$ m, width $B = 1.682$ m, wheelbase $W = 2.47$ m, track of back wheels $w = 1.456$ m, maximum angle of front wheels turning $\phi_{max} = 0.785$ radian (45^0, approximately),distance from the point M to the edge of back bumper $b = 0.6$ m);
- Class B – Seat Leon (length $L = 4.315$ m, width $B = 1.768$ m, wheelbase $W = 2.578$ м, track of back wheels $w = 1.517$ m, maximum angle of front wheels turning $\phi_{max} = 0.785$ radian (45^0, approximately), distance from the point M to the edge of back bumper $b = 0.796$ m);
- Class C – Skoda Superb Sedan (length $L = 4.838$ м, width $B = 1.817$ m, wheelbase $W = 2.761$ м, track of back wheels $w = 1.518$ m, maximum angle of front wheels turning $\phi_{max} = 0.785$ radian (45^0, approximately), distance from the point M to the edge of back bumper $b = 1.119$ m).

At that we will consider the following values of safety parameters when parking: $\Delta = 0.5$ m, $\delta = 0.2$ m and width of the standing car $Bs = 1.75$ m. We will examine the road width S for two cases: $S = 5.5$ m и $S = 8$ m to research influence of the available space on the left from the parking car.

The calculations we remade with the mathematical packet MATLAB.

The results of calculation of optimal path for parallel parking are in the Table 1and on the Figures 3–6. At that on the Figures 3–4 we can see the motion paths only for Volkswagen Polo, because for other cars the type of these paths does not differ in the main.

Table 1. Results of calculations

Road width	Car model	ΔX, m	$\frac{\Delta X}{L}$	a_2, m	a_3, m	a_4, km/h
5.5 m	VW Polo	5.846	1.472	-0.2968	0.638	-1
	Seat Leon	6.437	1.492	-0.1876	0.6102	-1
	Skoda Superb	7.147	1.477	-0.6	0.45	-1-
8 m	VW Polo	5.817	1.465	0.6213	3.1171	-1
	Seat Leon	6.261	1.451	0.9744	2.9864	-1
	Skoda Superb	6.947	1.436	-0.4	0.5	-1-

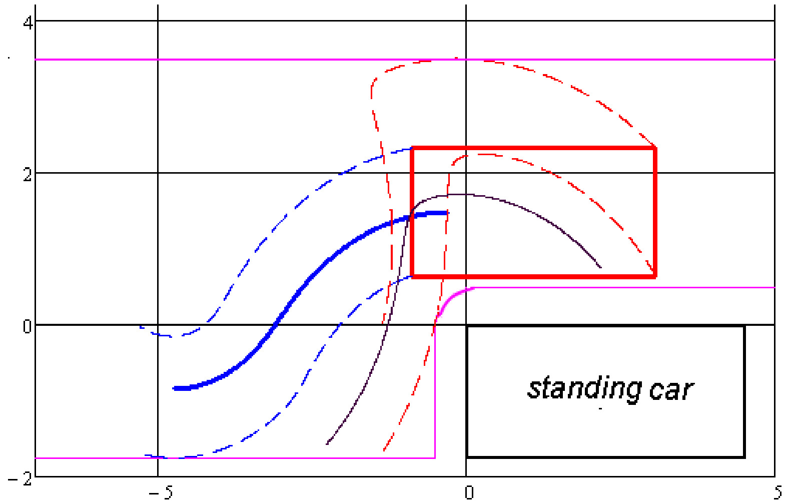

Figure 3. Parking path of Volkswagen Polo at the road width 5.5m

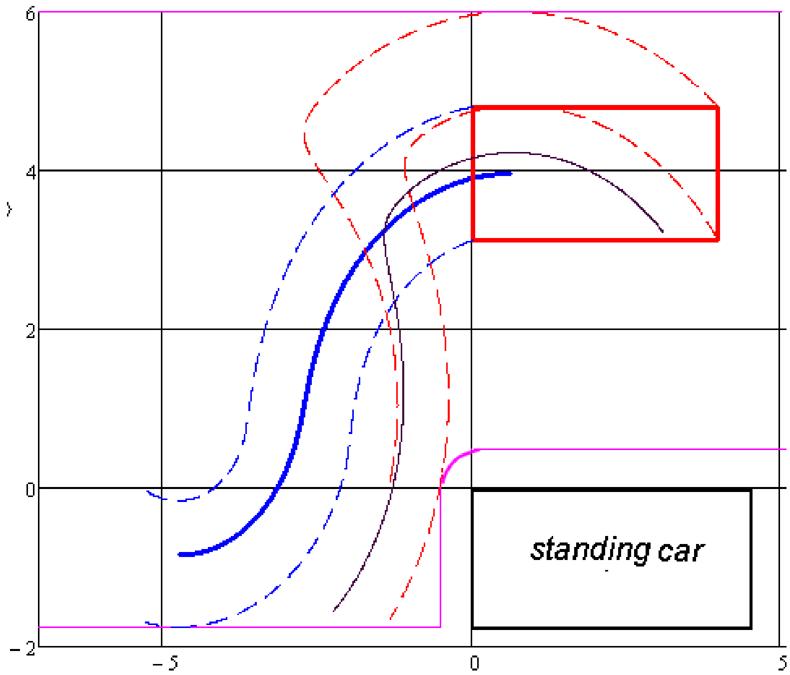

Figure 4. Parking path of Volkswagen Polo at the road width 8 m

On the Figures 3–4 the blue solid thick line shows the path of the point M, the black solid line shows the path of the front right wheel, the red dashed line shows the path of front overall points of the car, the blue dashed line shows the path of back overall points of the car.

In the Table 1 the following signs are used: ΔX– is a minimum distance between the standing cars that is necessary for parking in view of the safety parameter Δ; $\frac{\Delta X}{L}$ is a relation of this distance to the car length.

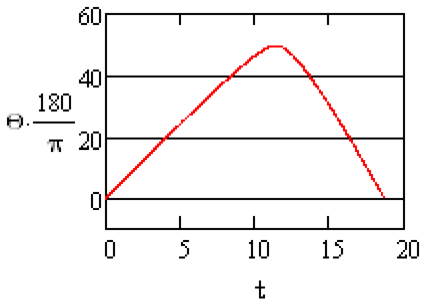

Figure 5. Graphof the angle θ change when parking Volkswagen Polo and at the road width of 5.5 meters

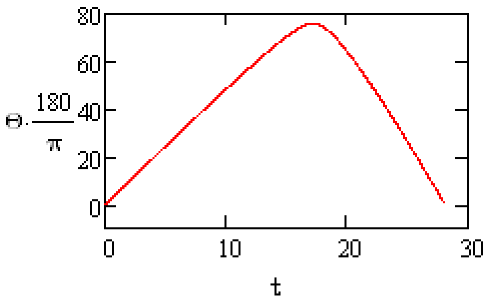

Figure 6. Graphof the angle θ change when parking Volkswagen Polo and at the road width of 8 meters

The results of calculation have showed that the car speed must be very slow, about 1km/hour. Therefore in the calculations, finally, it was fixed in the value of $a_4 = -1$ km/hour.

On the figures 5–6 you can see the graphs of the angle θ change when parking Volkswagen Polo and at the road width of 5.5m and 8m. As it is seen in the graph 5, at first the angle θ grows slowly almost till 50 degrees, and then with high speed it falls till zero. At the road width of 8m the value of the angle θ reaches almost 80 degrees.

2.5. Interpretation of results

Results of calculation allow us to draw conclusions about adequacy of the created mathematical model of car motion at low speeds.

The analysis of calculation results (Table 1) allows drawing the following theoretical conclusions for parallel car parking:

- length of a parking space should be practically not less than 1.5 of car length;
- for big cars, the more space for manoeuvre on the left, the better. The availability of large space on the left from the standing car is not of great importance for cars of A class, because it does not give any essential benefit (only some centimetres);
- if the space on the left is limited the passing forward concerning the standing car should be little;
- speed of the parking car should be as minimum as possible;
- inclination of the car axle relative to the road in optimal parking reaches 50 degrees and more.

3. Conclusions

It is proposed an algorithm of optimal path of car movement in the parallel parking on the basis of functions (4).

We want to notice that the path of car movement in the parallel parking calculated with the function (4) won't be optimal generally and it is only its approximation. Besides, in practice it is possible to make the movement path in two stages: at first backing and then forward and to the right. The calculation of such combined path also can be realized using the method described above.

On the basis of the offered approach a computer simulator could be developed for beginning (inexperienced) drivers. Using this simulator a driver can choose for himself an algorithm of optimal behaviour when parking.

The proposed algorithm for choosing an optimal path in the parallel parking can be used in design of inexpensive car systems that can help a driver to choose the best path for parallel parking.

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