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# REMARKS ON SUPER-ADDITIVE AND SUB-ADDITIVE TRANSFORMATIONS OF AGGREGATION FUNCTIONS

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ABSTRACT. In this contribution we modify the definitions of the super-additive and sub-additive transformations of aggregation functions. Firstly, we define $k$-bounded transformations that represent only finite decompositions with at most $k$ elements. Secondly, we introduce two other transformations that preserve the super-additivity property in some sense. Also, a remark on continuity of the classical super-additive transformation of an aggregation function is presented for one-dimensional case.

## 1. Introduction

Aggregation functions have proved to be a prolific topic of investigation both from a theoretic as well as from an application points of view. As a tool for furnishing a single value out of multiple inputs, such functions have proven to be useful in various branches of research, science and technology, notably in statistics, decision-making, data mining, artificial intelligence and economics, to list just a few. In order to avoid being repetitious we refer for particulars about the theory and applications of aggregation functions to the monographs [1] and [2] and to the references therein.

This contribution focuses on certain features of transformations of aggregation functions that seem not to have been studied in the literature. It builds on two basic notions of super- and sub-additive transformations of aggregation functions introduced recently in [3] and studies their further properties and variations. We remark that transformations of aggregation functions as defined in [3]


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have been extensively studied as well, especially in connection with existence of aggregation functions with given super- and sub-additive transformations; see, e.g., [4]–[7], [10]–[12].

The paper is organized as follows. In Section 2 we introduce some preliminaries and basic building blocks of this paper. Section 3 introduces $k$-bounded transformations of aggregation functions and properties of these transformations are discussed. The main proposition of Section 4 is the proof of the continuity of the classical super-additive transformation of a continuous aggregation function. Lastly, in the Section 5, we introduce two other transformations of aggregation functions.

## 2. Preliminaries

For the purpose of this article, by an *aggregation function* we will understand any function

$$A\colon [0,\infty[^n \to [0,\infty[$$

such that $A(\mathbf{0}) = 0$ and $A$ is increasing in every coordinate. This is consistent with [2] and one usually assumes that $n \geq 2$ in applications. As their name suggests, such functions are used to aggregate a certain number of inputs into a single output. Although this motivation of aggregation function cannot be used in one dimension, analysing the case $n = 1$ sometimes proves to be helpful before going into higher dimensions. For completeness we note that there are different approaches to the very definition of an aggregation function; for example, in the survey [8] the domain is restricted to $[0,1]^n$.

We say that an aggregation function as defined above is *super-additive* if $A(\mathbf{x}+\mathbf{y}) \geq A(\mathbf{x}) + A(\mathbf{y})$, and *sub-additive* if $A(\mathbf{x}+\mathbf{y}) \leq A(\mathbf{x}) + A(\mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in [0,\infty[^n$.

Motivated by applications in economics, the authors of [3] introduced the following two important concepts. For an aggregation function $A$ its *super-additive transformation* $A^*$ is given by

$$A^*(\mathbf{x}) = \sup\left\{\sum_{i=1}^{k} A(\mathbf{x}_i)\colon \sum_{i=1}^{k} \mathbf{x}_i = \mathbf{x}, \mathbf{x}_i \geq \mathbf{0},\ k \in \mathbb{N}\right\}.$$

Analogously, the *sub-additive transformation* $A_*$ of $A$ is defined by

$$A_*(\mathbf{x}) = \inf\left\{\sum_{i=1}^{k} A(\mathbf{x}_i)\colon \sum_{i=1}^{k} \mathbf{x}_i = \mathbf{x}, \mathbf{x}_i \geq \mathbf{0},\ k \in \mathbb{N}\right\}.$$

Again, the terminology reflects the fact that $A^*$ is always a super-additive and $A_*$ is always a sub-additive aggregation function. The properties of these transformations have been extensively studied in the literature, see, e.g., [3]–[7], [10]–[12]. Also note that $A^*$ might assume the value of $\infty$; if this happens at some point then we will say that $A^*$ *escapes locally.* If $A^*(\mathbf{x}) = \infty$ for all $\mathbf{x} \in ]0,\infty[^n$ we will say that $A^*$ *escapes globally.* There is no need to define escapeness for sub-additive transformations because $A_*$ is bounded below by 0 and bounded above by $A$.

## 3. Transformations of finite decomposition

The way super- and sub-additive transformations have been introduced assumes that a given point $\mathbf{x} \in [0,\infty[^n$ may be written as a finite sum of non--negative points in an arbitrary way. In practice, however, it may happen that such decompositions of $\mathbf{x}$ are available only with a bounded number of summands. This restriction motivates us to define a new type of transformation that takes this into the account.

**Definition 3.1.** Let $k \in \mathbb{N}$ and let $A\colon [0,\infty[^n \to [0,\infty[$ be any aggregation function. An *upper $k$-bounded transformation* of $A$ is the function $A^{(k)}$ defined by

$$A^{(k)}\colon [0,\infty[^n \to [0,\infty[\colon \mathbf{x} \mapsto \sup\left\{\sum_{i=1}^{k} A(\mathbf{x}_i)\colon \sum_{i=1}^{k} \mathbf{x}_i = \mathbf{x},\, \mathbf{x}_i \geq \mathbf{0}\right\}.$$

Similarly, a *lower $k$-bounded transformation* of $A$ is the aggregation function

$$A_{(k)}\colon [0,\infty[^n \to [0,\infty[\colon \mathbf{x} \mapsto \inf\left\{\sum_{i=1}^{k} A(\mathbf{x}_i)\colon \sum_{i=1}^{k} \mathbf{x}_i = \mathbf{x},\, \mathbf{x}_i \geq \mathbf{0}\right\}.$$

Example 3.2. Let $F, G, H\colon [0,\infty[ \to [0,\infty[$ be aggregation functions given by $F(x) = \min\{x, 1\}$, $G(x) = \sqrt{x}$, and $H(x) = x^2$. Then

$$\begin{aligned} F^{(k)}(x) &= \min\{x, k\}, & F_{(k)}(x) &= F(x),\\ G^{(k)}(x) &= \sqrt{kx}, & G_{(k)}(x) &= G(x),\\ H^{(k)}(x) &= H(x), & H_{(k)}(x) &= x^2/k. \end{aligned}$$

**Remark 3.3.** Note that the upper and lower $k$-bounded transformations of an aggregation functions are again aggregation functions.

**Remark 3.4.** In contrast with super- and sub-additive transformations, an upper or a lower $k$-bounded transformation of an aggregation function need not be super- or sub-additive, respectively; this can be seen by the functions $F$ and $H$ from Example 3.2.

Nevertheless, one has the following interesting property of our bounded transformations.

**Proposition 3.5.** *Let $A$ be an aggregation function and let $k, l \in \mathbb{N}$. Then $(A^{(k)})^{(l)} = A^{(kl)}$ and $(A_{(k)})_{(l)} = A_{(kl)}$.*

Proof. Note that

$$\begin{aligned}(A^{(k)})^{(l)}(\mathbf{x}) &= \sup\left\{\sum_{i=1}^{l} A^{(k)}(\mathbf{x}_i) \colon \sum_{i=1}^{l} \mathbf{x}_i = \mathbf{x}\right\}\\ &= \sup\left\{\sum_{i=1}^{l} \sup\left\{\sum_{j=1}^{k} A(\mathbf{x}_{i,j}) \colon \sum_{j=1}^{k} \mathbf{x}_{i,j} = \mathbf{x}_i\right\} \colon \sum_{i=1}^{l} \mathbf{x}_i = \mathbf{x}\right\}\\ &= \sup\left\{\sum_{i=1}^{l}\sum_{j=1}^{k} A(\mathbf{x}_{i,j}) \colon \sum_{j=1}^{k} \mathbf{x}_{i,j} = \mathbf{x}_i, \sum_{i=1}^{l} \mathbf{x}_i = \mathbf{x}\right\},\end{aligned}$$

where $\mathbf{x}_i$ and $\mathbf{x}_{i,j}$ are all non-negative vectors, and also note that

$$A^{(kl)}(\mathbf{x}) = \sup\left\{\sum_{m=1}^{kl} A(\mathbf{x}_m) \colon \sum_{m=1}^{kl} \mathbf{x}_m = \mathbf{x},\ \mathbf{x}_m \geq \mathbf{0}\right\}.$$

From these equations it easily follows that $(A^{(k)})^{(l)}(\mathbf{x}) = A^{(kl)}(\mathbf{x})$ for all $\mathbf{x} \geq \mathbf{0}$. The proof for lower $k$-bounded transformations is analogous. □

Even though some properties of super- and sub-additive transformations do not carry over to upper and lower $k$-bounded transformations, some of them do. These are summarized in the following theorem.

**Theorem 3.6.** *Let $A$ be an aggregation function. Then*

- *if $A \leq B$ then $A^{(k)} \leq B^{(k)}$ and $A_{(k)} \leq B_{(k)}$;*
- *if $A$ is super-additive (sub-additive) then $A^{(k)} = A$ ($A_{(k)} = A$);*
- *$A^{(k)}$ and $A_{(k)}$ are positively homogeneous of degree $1$, i.e., $(\alpha A)^{(k)} = \alpha A^{(k)}$ and $(\alpha A)_{(k)} = \alpha A_{(k)}$ for all $\alpha \geq 0$.*

Proof. We will prove the theorem only for upper $k$-bounded transformation, leaving the similar arguments for lower $k$-bounded transformations to the reader. Let us assume that $A \leq B$. Then

$$\sum_{i=1}^{k} A(\mathbf{x}_i) \leq \sum_{i=1}^{k} B(\mathbf{x}_i)$$

holds for all decompositions $\{\mathbf{x}_i\}_{i=1}^{k}$ of $\mathbf{x}$ and thus $A^{(k)}(\mathbf{x}) \leq B^{(k)}(\mathbf{x})$. Note that if $A$ is super-additive, then

$$A(\mathbf{x}) \geq \sum_{i=1}^{k} A(\mathbf{x}_i)$$

for all decompositions $\{\mathbf{x}_i\}_{i=1}^k$ of $\mathbf{x}$ and thus $A(\mathbf{x}) \geq A^{(k)}(\mathbf{x})$. The fact that $\mathbf{x}$ is a singleton decomposition of $\mathbf{x}$ implies that $A^{(k)}(\mathbf{x}) = A(\mathbf{x})$. Lastly, let $\alpha \geq 0$. Then

$$\begin{aligned}(\alpha A)^{(k)}(\mathbf{x}) &= \sup\left\{\sum_{i=1}^k \alpha A(\mathbf{x}_i) \colon \sum_{i=1}^k \mathbf{x}_i = \mathbf{x}, \mathbf{x}_i \geq \mathbf{0}\right\} \\ &= \alpha \sup\left\{\sum_{i=1}^k A(\mathbf{x}_i) \colon \sum_{i=1}^k \mathbf{x}_i = \mathbf{x}, \mathbf{x}_i \geq \mathbf{0}\right\} = \alpha A^{(k)}(\mathbf{x})\end{aligned}$$

which completes the proof of the theorem. □

Our next theorem relates the upper and lower $k$-bounded transformations by certain inequalities.

**Theorem 3.7.** *Let $A$ be an aggregation function. Then $A^{(1)} = A$ and $A^{(k+1)} \geq A^{(k)}$ for all $k \in \mathbb{N}$. Similarly, $A_{(1)} = A$ and $A_{(k+1)} \leq A_{(k)}$ for all $k \in \mathbb{N}$.*

P r o o f. This theorem will again be proved only for upper $k$-bounded transformation. Note that

$$A^{(1)}(\mathbf{x}) = \sup\bigl\{A(\mathbf{x}_1) \colon \mathbf{x}_1 = \mathbf{x}, \mathbf{x}_1 \geq \mathbf{0}\bigr\} = A(\mathbf{x})$$

and thus $A^{(1)} = A$. Now let $\{\mathbf{x}_i\}_{i=1}^k$ be any decomposition of $\mathbf{x}$. Then $\{\mathbf{x}_i\}_{i=1}^{k+1}$ with $\mathbf{x}_{k+1} = \mathbf{0}$ is also a decomposition of $\mathbf{x}$ and thus $A^{(k)}(\mathbf{x}) \leq A^{(k+1)}(\mathbf{x})$. □

We continue with a study of inheritance of continuity of upper and lower $k$-bounded transformations.

**Proposition 3.8.** *If $A$ is continuous then so are $A^{(k)}$ and $A_{(k)}$.*

P r o o f. Restricting ourselves to the case of upper $k$-bounded transformations again, note that the definition of $A^{(k)}(\mathbf{x})$ can be rewritten to the form

$$A^{(k)}(\mathbf{x}) = \sup\bigl(A(\mathbf{x}_1) + \dots + A(\mathbf{x}_{k-1}) + A(\mathbf{x} - \mathbf{x}_1 - \dots - \mathbf{x}_{k-1})\bigr),$$

where the supremum is taken subject to inequalities

$$\mathbf{0} \leq \mathbf{x}_i \leq \mathbf{x} - \sum_{j=1}^{i-1} \mathbf{x}_j$$

for $i = 1, 2, \dots, k-1$. Note that this area deforms continuously when the point $\mathbf{x}$ is changed continuously which implies that $A^{(k)}$ is continuous. □

## 4. Remarks on super-additive transformations

**Theorem 4.1.** *Let $A\colon [0,\infty[^n \to [0,\infty[$ be any aggregation function. Then $A^*$ escapes locally if and only if $A^*$ escapes globally.*

Proof. The proof of $\Leftarrow$ is trivial. Now let us assume that there exists an $\overline{\mathbf{x}}$ such that $A^*(\overline{\mathbf{x}}) = \infty$. From monotonicity we trivially obtain that $A^*(\mathbf{y}) = \infty$ for all $\mathbf{y} \geq \overline{\mathbf{x}}$. Let $\overline{\mathbf{y}}$ be a point such that $\overline{\mathbf{y}} > \mathbf{0}$ and $\overline{\mathbf{y}} \geq \mathbf{x}$. From the previous discussion we conclude that $A^*(\overline{\mathbf{y}}) = \infty$. Now, from the definition of $A^*(\overline{\mathbf{y}})$ it follows that there exists a sequence $\{\mathbf{y}_n\}_{n\in\mathbb{N}}$ converging to $\mathbf{0}$ such that

$$\sum_{n\in\mathbb{N}} \mathbf{y}_n = \overline{\mathbf{y}} \quad \text{and} \quad \sum_{n\in\mathbb{N}} A(\mathbf{y}_n) = \infty.$$

The only limit point of $\{\mathbf{y}_n\}_{n\in\mathbb{N}}$ is the point $\mathbf{0}$. Now let $\mathbf{y}$ be any point such that $\mathbf{0} < \mathbf{y} \leq \overline{\mathbf{y}}$. Then we can remove a finite number, say $k$, elements of $\{\mathbf{y}_n\}_{n\in\mathbb{N}}$ so that

$$\sum_{n=k+1}^{\infty} \mathbf{y}_n \leq \mathbf{y}.$$

From this we obtain that

$$A^*(\mathbf{y}) \geq \sum_{n=k+1}^{\infty} A^*(\mathbf{y}_n) \geq \sum_{n=k+1}^{\infty} A(\mathbf{y}_n) = \sum_{n\in\mathbb{N}} A(\mathbf{y}_n) - \sum_{n=1}^{k} A(\mathbf{y}_n) = \infty$$

and thus $A^*(\mathbf{y}) = \infty$ for all $\mathbf{y}$ such that $\mathbf{0} < \mathbf{y} \leq \overline{\mathbf{y}}$. Now let $\mathbf{x} > \mathbf{0}$ be any point. Then there exists a point $\mathbf{z} > \mathbf{0}$ such that $\mathbf{z} \leq \mathbf{x}$ and $\mathbf{z} \leq \overline{\mathbf{y}}$. Finally, from monotonicity we obtain that $A^*(\mathbf{x}) \geq A^*(\mathbf{z}) = \infty$ which completes the proof. □

In general one cannot say what happens on the "edges", i.e., at the points $\mathbf{x}$ that have at least one coordinate equal to zero. This is shown in the following example.

Example 4.2. Let $A\colon [0,\infty[^n \to [0,\infty[$ be an aggregation function such that

$$A(\mathbf{x}) = \left(\prod_{i=1}^{n} x_i\right)^{1/2n}.$$

Then

$$A^*(\mathbf{x}) = \begin{cases} \infty & \text{if } \mathbf{x} > \mathbf{0}, \\ 0, & \text{otherwise.} \end{cases}$$

Continuity inheritance for super- and sub-additive transformations have, to the best of the authors' knowledge, not been studied. In this contribution we present a corresponding result only for one-dimensional aggregation functions, deferring the more complex multi-dimensional case to [9].

**Theorem 4.3.** *Let $A\colon [0,\infty[\to [0,\infty[$ be a continuous aggregation function. If $A^*$ does not escape globally then $A^*$ is continuous.*

Proof. It is sufficient to prove that the super-additive transformation $A^*$ is uniformly continuous on $[0,\overline{x}]$ for any $\overline{x}\in ]0,\infty[$. Let $\varepsilon>0$ and let $n\geq 2A^*(\overline{x})/\varepsilon$ be such that

$$|y-y'|<\frac{\overline{x}}{n}\quad\text{implies}\quad |A(y)-A(y')|<\frac{\varepsilon}{4}. \tag{1}$$

This can be done based on the uniform continuity of $A$ on $[0,\overline{x}]$. Suppose now that $x,x'\in ]0,\overline{x}[$ and, without loss of generality, $x'<x$. We will show that

$$x-x'<\frac{\overline{x}}{n}\quad\text{implies}\quad A^*(x)-A^*(x')<\varepsilon.$$

Let us thus assume that $x-x'<\overline{x}/n$. For the given $\varepsilon>0$, by definition of $A^*$, there exists a $k$-tuple $\{x_i\}_{i=1}^k$, $x_i\geq 0$, such that

$$\sum_{i=1}^k x_i = x\quad\text{and}\quad A^*(x)-\sum_{i=1}^k A(x_i)<\frac{\varepsilon}{4}. \tag{2}$$

*Case 1:* Suppose that one of the $x_i$'s, say, without loss of generality, $x_k$, is such that $x_k\geq x-x'$. Let $\{y_i\}_{i=1}^{k+1}$ be such that $y_i=x_i$, $1\leq i\leq k-1$, $y_{k+1}=x-x'$, and $y_k=x_k-y_{k+1}$. Note that $x_k-y_k=x-x'$ and

$$\sum_{i=1}^k y_i = x'.$$

From the definition of $A^*(x)$ we obtain the inequality

$$\sum_{i=1}^k A(y_i)\leq A^*(x)$$

from which we obtain, for $x-x'<\overline{x}/n$, that

$$\begin{aligned}
A^*(x)-A^*(x') &\leq A^*(x)-\sum_{i=1}^k A(y_i)\\
&=\left(A^*(x)-\sum_{i=1}^k A(x_i)\right)+\left(\sum_{i=1}^k A(x_i)-\sum_{i=1}^k A(y_i)\right)\\
&=\left(A^*(x)-\sum_{i=1}^k A(x_i)\right)+A(x_k)-A(y_k).
\end{aligned}$$

Note that by (1) we get that $A(x_k) - A(y_k) < \varepsilon/4$ and that by (2) we get that

$$A^*(x) - \sum_{i=1}^{k} A(x_i) < \frac{\varepsilon}{4}$$

and thus

$$A^*(x) - A^*(x') < \frac{\varepsilon}{4} + \frac{\varepsilon}{4} < \varepsilon.$$

*Case 2:* Suppose now that $x_i < x - x'$ for all $i \leq k$. Without loss of generality we can assume that $x_1 \geq x_2 \geq \cdots \geq x_k$. Let $l \leq k$ be such that

$$\sum_{i=1}^{l-1} x_i < x' \quad \text{but} \quad \sum_{i=1}^{l} x_i \geq x'.$$

Let $x_l = u + v$ so that

$$\sum_{i=1}^{l-1} x_i + u = x' \quad \text{and note that} \quad \sum_{i=l+1}^{k} x_i + v = x - x'.$$

Using the fact that

$$\sum_{i=1}^{l-1} A(x_i) + A(u) \leq A^*(x')$$

we obtain that

$$\begin{aligned} A^*(x) - A^*(x') &\leq A^*(x) - \left( \sum_{i=1}^{l-1} A(x_i) + A(u) \right) \\ &= \left( A^*(x) - \sum_{i=1}^{k} A(x_i) \right) + \left( \sum_{i=1}^{k} A(x_i) - \sum_{i=1}^{l-1} A(x_i) - A(u) \right) \\ &= \left( A^*(x) - \sum_{i=1}^{k} A(x_i) \right) + \left( \sum_{i=l+1}^{k} A(x_i) + A(x_l) - A(u) \right). \end{aligned}$$

Again, using (2), we get that

$$A^*(x) - \sum_{i=1}^{k} A(x_i) < \frac{\varepsilon}{4},$$

from the definition of the super-additive transformation, the fact that $x - x' < \frac{\overline{x}}{n}$, and the way we chose $n$ it follows that

$$\sum_{i=l+1}^{k} A(x_i) \leq A^*(x - x') \leq \frac{1}{n} A^*(\overline{x}) \leq \frac{\varepsilon}{2},$$

and from the uniform continuity of $A$ and the fact that $x_l - u < x - x'$ we get that
$$A(x_l) - A(u) < \frac{\varepsilon}{4}.$$
Combining these results we obtain that
$$A^*(x) - A^*(x') < \frac{\varepsilon}{4} + \frac{\varepsilon}{2} + \frac{\varepsilon}{4} = \varepsilon$$
which completes the proof. □

Note that thanks to continuity of $A^*$ for continuous aggregation functions $A$ we can prove a uniform convergence of the upper $k$-bounded transformations.

**Theorem 4.4.** *Let $A\colon [0,\infty[\to [0,\infty[$ be continuous aggregation function that does not escape globally. Then $\{A^{(k)}\}_{k\in\mathbb{N}}$ converges to $A^*$ uniformly.*

Proof. Follows from the monotonicity of the sequence $\{A^{(k)}\}_{k\in\mathbb{N}}$ and Dini's theorem. □

## 5. Other types of transformations of aggregation

### 5.1. Super-additive ray transformation

**Definition 5.1.** Let $A\colon [0,\infty[^n\to [0,\infty[$ be an aggregation function. Its super-additive ray transformation is a mapping $A^-\colon [0,\infty[^n\to [0,\infty[$ given by
$$A^-(\mathbf{x}) = \sup\left\{\sum_{i=1}^{k} A(\lambda_i\mathbf{x})\colon \lambda_i \geq 0, \sum_{i=1}^{k}\lambda_i = 1, k\in\mathbb{N}\right\}.$$

**Theorem 5.2.** *Let $A\colon [0,\infty[^n\to [0,\infty[$ be an aggregation function. Then $A^-$ is also an aggregation function.*

Proof. Note that $A^-(\mathbf{0}) = \mathbf{0}$. Now let $\mathbf{x}\geq\mathbf{y}$. The monotonicity of $A$ implies that $A(\lambda\mathbf{x})\geq A(\lambda\mathbf{y})$ for all $\lambda\geq 0$ and thus for every sequence $\{\lambda_i\}_{i=1}^{k}$ such that $\lambda_1+\cdots+\lambda_k = 1$ we obtain that
$$\sum_{i=1}^{k} A(\lambda_i\mathbf{x}) \geq \sum_{i=1}^{k} A(\lambda_i\mathbf{y})$$
which implies that $A^-(\mathbf{x})\geq A^-(\mathbf{y})$ and thus $A^-$ is an aggregation function. □

**Theorem 5.3.** *Let $A\colon [0,\infty[^n\to [0,\infty[$ be an aggregation function. Then $A^-$ is super-additive on rays starting at $\mathbf{0}$, i.e., for all $\mathbf{x}\in[0,\infty[^n$ and all $\alpha,\beta\geq 0$ we have*
$$A^-\big((\alpha+\beta)\mathbf{x}\big) \geq A^-(\alpha\mathbf{x}) + A^-(\beta\mathbf{x}).$$

P r o o f. Let us fix $\mathbf{x} \in [0,\infty[^n$ and let us define

$$B\colon [0,\infty[\to [0,\infty[\colon t \mapsto A(t\mathbf{x}).$$

Note that $B$ is an aggregation function and also note that

$$\begin{aligned} B^*(t) &= \sup\left\{\sum_{i=1}^k B(t_i)\colon \sum_{i=1}^k t_i = t, t_i \geq 0, k \in \mathbb{N}\right\} \\ &= \sup\left\{\sum_{i=1}^k A(t_i\mathbf{x})\colon \sum_{i=1}^k \frac{t_i}{t} = 1, t_i \geq 0, k \in \mathbb{N}\right\} \\ &= \sup\left\{\sum_{i=1}^k A\big(\beta_i(t\mathbf{x})\big)\colon \sum_{i=1}^k \beta_i = 1, \beta_i \geq 0, k \in \mathbb{N}\right\} = A^-(t\mathbf{x}), \end{aligned}$$

where $\beta_i = t_i/t$. From the super-additivity of $B^*$ we obtain that

$$A^-\big((\alpha+\beta)\mathbf{x}\big) = B^*(\alpha+\beta) \geq B^*(\alpha) + B^*(\beta) = A^-(\alpha\mathbf{x}) + A^-(\beta\mathbf{x})$$

which completes the proof. □

Example 5.4. Note that, in general, $A^-$ might not be super-additive. Let $A$ be such that $A(\mathbf{x}) = \sqrt{x_1^2 + x_2^2}$. Then $A^-(\mathbf{x}) = A(\mathbf{x})$ but

$$A^-(3,4) = 5 \ngeq 3 + 4 = A^-(3,0) + A^-(0,4)$$

which implies that $A^-$ is not super-additive and thus $A^- \neq A^*$.

From the previous example and the properties of the super-additive transformation we obtain:

**Theorem 5.5.** *Let $A$ be an aggregation function. Then $A \leq A^- \leq A^*$.*

P r o o f. To see that $A \leq A^-$ is trivial. Note that in $A^-$ we restrict ourselves to special type of decompositions which implies that $A^- \leq A^*$. □

### 5.2. Linear super-additive transformation

**Definition 5.6.** Let $A\colon [0,\infty[^n\to [0,\infty[$ be an aggregation function. Its linear super-additive transformation is a mapping $A^\dagger\colon [0,\infty[^n\to [0,\infty[$ given by

$$A^\dagger(\mathbf{x}) = \sup\left\{\sum_{i=1}^k \alpha_i A(\mathbf{x}_i)\colon \alpha_i \geq 0, \mathbf{x}_i \geq \mathbf{0}, \sum_{i=1}^k \alpha_i\mathbf{x}_i = \mathbf{x}, k \in \mathbb{N}\right\}.$$

**Theorem 5.7.** *Let $A\colon [0,\infty[^n\to [0,\infty[$ be an aggregation function. Then $A^\dagger$ is a super-additive aggregation function.*

P r o o f. The proof is analogous to the proof of the super-additivity of the super-additive transformation of aggregation function. □

**Theorem 5.8.** *Let $A$ be an aggregation function. Then $A \leq A^* \leq A^\dagger$.*

Proof. The first inequality $A \leq A^*$ follows from the definition of the super-additive transformation of aggregation function. Note that the super-additive transformation uses a subclass of decompositions that $A^\dagger$ uses which implies the second inequality. $\square$

EXAMPLE 5.9. Let us consider an aggregation function $A\colon [0,\infty[^n \to [0,\infty[$ defined by

$$A(\mathbf{x}) = \prod_{i=1}^{n} x_i,$$

where $\mathbf{x} = (x_1, x_2, \dots, x_n)$ and $n \geq 2$. Then $A^* = A$ but

$$A^\dagger(\mathbf{x}) = \begin{cases} \infty, & \text{if } \mathbf{x} > 0, \\ 0, & \text{otherwise.} \end{cases}$$

## 6. Concluding remarks

For each aggregation function $A$ we have introduced the upper and the lower $k$-bounded transformations $A^{(k)}$ and $A_{(k)}$ of $A$ that are modifications of super-additive and sub-additive transformations of aggregation functions. These newly defined transformations take into account that only finite decompositions might be available. Some properties of these transformations were discussed.

We established a criterion of the super-additive transformation to be well-defined, i.e., we proved that if the super-additive transformation $A^*$ escapes locally then it escapes everywhere. Moreover, we proved the continuity inheritance for the super-additive transformation. Lastly, we proved that the sequence of upper $k$-bounded transformations of continuous aggregation function converges to the super-additive transformation uniformly.

In the last section of this contribution we constructed two new types of transformations, namely the super-additive ray transformation $A^-$ and the linear super-additive transformation $A^\dagger$-. Properties of these transformations are also discussed.

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