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CONVERGENCE OF THE NUMERICAL SCHEME FOR REGULARISED RIEMANNIAN MEAN CURVATURE FLOW EQUATION

Matúš Tibenský — Angela Handlovičová

Slovak University of Technology in Bratislava, Bratislava, SLOVAKIA

ABSTRACT. The aim of the paper is to study problem of image segmentation and missing boundaries completion introduced in [Mikula, K.—Sarti, A.–Sgallarri, A.: *Co-volume method for Riemannian mean curvature flow in subjective surfaces multiscale segmentation,* Comput. Vis. Sci. **9** (2006), 23–31], [Mikula, K.—Sarti, A.—Sgallari, F.: *Co-volume level set method in subjective surface based medical image segmentation,* in: Handbook of Medical Image Analysis: Segmentation and Registration Models (J. Suri et al., eds.), Springer, New York, 583–626, 2005], [Mikula, K.—Ramarosy, N.: *Semi-implicit finite volume scheme for solving nonlinear diffusion equations in image processing,* Numer. Math. **89** (2001), 561–590] and [Tibenský, M.: *Využitie Metód Založených na Level Set Rovnici v Spracovaní Obrazu,* Faculty of mathematics, physics and informatics, Comenius University, Bratislava, 2016]. We generalize approach presented in [Eymard, R.—Handlovičová, A.—Mikula, K.: *Study of a finite volume scheme for regularised mean curvature flow level set equation,* IMA J. Numer. Anal. **31** (2011), 813–846] and apply it in the field of image segmentation. The so called regularised Riemannian mean curvature flow equation is presented and the construction of the numerical scheme based on the finite volume method approach is explained. The principle of the level set, for the first time given in [Osher, S.—Sethian, J. A.: *Fronts propagating with curvature-dependent speed: Algorithms based on Hamilton-Jacobi formulations,* J. Comput. Phys. **79** (1988), 12–49] is used. Based on the ideas from [Eymard, R.—Handlovičová, A.–Mikula, K.: *Study of a finite volume scheme for regularised mean curvature flow level set equation,* IMA J. Numer. Anal. **31** (2011), 813–846] we prove the stability estimates on the numerical solution and the uniqueness of the numerical solution. In the last section, there is a proof of the convergence of the numerical scheme to the weak solution of the regularised Riemannian mean curvature flow equation and the proof of the convergence of the approximation of the numerical gradient is mentioned as well.

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1. Studied equation and assumptions on the data

We consider the following problem arising in image segmentation as a generalisation of the approach given in [1], find an approximate solution to the equation

$$u_t - f_1(|\nabla u|)\nabla \cdot \left(g(|\nabla G_S * I^0|) \frac{\nabla u}{f(|\nabla u|)} \right) = r, \quad a.e.\ (x,t) \in \Omega \times [0,T], \quad (1)$$

where $u(x,t)$ is an unknown (segmentation) function defined in $Q_T \equiv [0,T] \times \Omega$, where Ω is a finite connected open subset of $\mathbb{R}^d$, $d \in \mathbb{N}$, $[0,T]$ is a time interval and I^0 is a given image, typically on this image is an object we want to segment.

We consider zero Dirichlet boundary condition

$$u = 0, \qquad a.e.\ (x,t) \in \partial\Omega \times [0,T] \qquad (2)$$

and initial condition

$$u(x,0) = u_0(x), \qquad a.e.\ x \in \Omega. \qquad (3)$$

The assumptions on the data in (1)–(3) are similar as in [1] and [3]. We can summarize them into the following hypothesis:

Hypothesis H

- **(H1):** Ω is a finite connected open subset of $\mathbb{R}^d$, $d \in \mathbb{N}$, with boundary $\partial\Omega$ defined by a finite union of subsets of hyperplanes of $\mathbb{R}^d$,
- **(H2):** $u_0 \in L^\infty(\Omega)$,
- **(H3):** $r \in L^2(\Omega \times [0,T])$ for all $T > 0$,
- **(H4):** $f \in C^0(\mathbb{R}_+; [a,b])$ is a Lipschitz continuous (non-strictly) increasing function, such that the function $x \mapsto x/f(x)$ is strictly increasing on $\mathbb{R}_+$. For practical application we are using $f(s) = \min(\sqrt{s^2 + a^2}, b)$, where a and b are given positive parameters,
- **(H5):** $f_1 \in C^0(\mathbb{R}_+; [a_1, b_1])$, in general $a_1 \neq a$, $b_1 \neq b$, but for now in our model we consider the case $a_1 = a$ and $b_1 = b$,
- **(H6):** $g \in C^0(\mathbb{R}_+; [0,1])$ is decreasing function, $g(0) = 1$, $g(s) \to 0$ for $s \to \infty$. For practical numerical computation we use $g(s) = \frac{1}{1+Ks^2}$, where K is constant of sensitivity of function g and we choose it,
- **(H7):** $G_S \in C^\infty(\mathbb{R}^d)$ is a smoothing kernel (Gauss function), with width of the convolution mask S and such that $\int_{\mathbb{R}^d} G_S(x)\,\mathrm{d}x = 1$, $\int_{\mathbb{R}^d} |G_S|\,\mathrm{d}x \leq C_S, C_S \in \mathbb{R}$, $G_S(x) \to \delta_x$ for $S \to 0$, where δ_x is Dirac measure at point x and

$$(\nabla G_S * I^0)(x) = \int_{\mathbb{R}^d} \nabla G_S(x - \xi) \tilde{I^0}(\xi)\,\mathrm{d}\xi, \qquad (4)$$

where $\tilde{I^0}$ is extension of image I^0 to $\mathbb{R}^d$ given by periodic reflection through boundary of Ω and for which

$$1 \geq g^S(x) = g(|\nabla G_S * I^0|)(x) \geq \nu_S > 0 \tag{5}$$

for $\forall x \in \Omega$ due to properties of the convolution.

Definition of the numerical scheme and the space discretisation of the equation we are generalising in this paper could be found in [1]. We apply method presented in [1] in the field of image segmentation, but in addition, we have function g and convolution of the initial image with smoothing kernel in our approach (see [3] or [4]).

For now just remark that discretisation of Ω, denoted by $\mathcal{D}$, is defined as the triplet $\mathcal{D} = (\mathcal{M}, \mathcal{E}, \mathcal{P})$, where $\mathcal{M}$ is a finite family of non-empty connected open disjoint subsets of Ω (the "control volumes") with measure marked by $|p|$, h_p denote the diameter of p and $h_\mathcal{D}$ denote the maximum value of $(h_p)_{m\in\mathcal{M}}$, $\mathcal{E}$ is a finite family of disjoint subsets of $\overline{\Omega}$ (the "edges" of the mesh) with measure marked by $|\sigma|$ and $\mathcal{P}$ is a family of points of Ω indexed by $\mathcal{M}$, denoted by $\mathcal{P} = (x_p)_{p\in\mathcal{M}}$, such that for all $p \in \mathcal{M}$, $x_p \in p$ and p is assumed to be x_p-star--shaped so for all $x \in p$ the inclusion $[x_p, x] \subset p$ holds.

We say that $(\mathcal{D}, \tau)$ is a space-time discretisation of $\Omega \times [0, T]$ if $\mathcal{D}$ is a space discretisation of Ω in the sense we mentioned above and if there exists $N_T \in \mathbb{N}$ with $T = (N_T + 1)\tau$, where τ is a symbol for the time step.

Another important assumption on the discretisation we make is that

$$d_{p\sigma} n_{p,\sigma} = x_\sigma - x_p, \qquad \forall p \in \mathcal{M}, \quad \forall \sigma \in \mathcal{E}_p, \tag{6}$$

where $\mathcal{E}_p$ denotes the set of the edges of the control volume p, $x_\sigma \in \sigma$, $d_{p\sigma}$ is a symbol for the Euclidean distance between x_p and hyperplane including σ (it is assumed that $d_{p\sigma} > 0$) and $n_{p,\sigma}$ denotes the unit vector normal to σ outward to p.

We define the set $H_\mathcal{D} \subset \mathbb{R}^{|\mathcal{M}|} \times \mathbb{R}^{|\mathcal{E}|}$ such that $u_\sigma = 0$ for all $\sigma \in \mathcal{E}_{\text{ext}}$ (the set of boundary interfaces). We define the following functions on $H_\mathcal{D}$:

$$N_p(u)^2 = \frac{1}{|p|} \sum_{\sigma\in\mathcal{E}_p} \frac{|\sigma|}{d_{p\sigma}} (u_\sigma - u_p)^2, \qquad \forall p \in \mathcal{M}, \quad \forall u \in H_\mathcal{D}, \tag{7}$$

where u_p is defined as $u_p = u(x_p)$ and u_σ is defined as $u_\sigma = u(x_\sigma)$.

Let us recall that

$$\|u\|^2_{1,\mathcal{D}} = \sum_{p\in\mathcal{M}} |p| N_p(u)^2 \tag{8}$$

defines a norm on $H_\mathcal{D}$ (see [9]).

Under the above mentioned assumptions and notations the semi-implicit scheme is defined by

$$u^0_p = u_0(x_p), \qquad \forall p \in \mathcal{M}, \tag{9}$$

$$u_\sigma^0 = u_0(x_\sigma), \qquad \forall \sigma \in \mathcal{E}, \tag{10}$$

$$r_p^{n+1} = \int_{n\tau}^{(n+1)\tau} \int_p r(x,t)\,\mathrm{d}x\,\mathrm{d}t, \qquad \forall p \in \mathcal{M}, \quad \forall n \in \mathbb{N}, \tag{11}$$

$$u_\sigma^{n+1} = 0, \qquad \forall \sigma \in \mathcal{E}_{\mathrm{ext}}, \quad \forall n \in \mathbb{N}, \tag{12}$$

and

$$\begin{aligned}\frac{|p|}{\tau\ f_1(N_p(u^n))}\big(u_p^{n+1} - u_p^n\big) - \frac{1}{f(N_p(u^n))} \sum_{\sigma \in \mathcal{E}_p} g_p^S \frac{|\sigma|}{d_{p\sigma}}\big(u_\sigma^{n+1} - u_p^{n+1}\big) \\ = \frac{r_p^{n+1}}{\tau\ f_1(N_p(u^n))}, \qquad \forall p \in \mathcal{M}, \quad \forall n \in \mathbb{N},\end{aligned} \tag{13}$$

where the following relation is given for the interior edges

$$\frac{u_\sigma^{n+1} - u_p^{n+1}}{f(N_p(u^n))\ d_{p\sigma}} + \frac{u_\sigma^{n+1} - u_q^{n+1}}{f(N_q(u^n))\ d_{q\sigma}} = 0, \tag{14}$$

$\forall n \in \mathbb{N}$, $\forall \sigma \in \mathcal{E}_{\mathrm{int}}$ (the set of interior interfaces) where σ is the edge between p and q.

For the explanation of the selection of u_p^0 and u_σ^0, which impacts the assumptions given on function u_0 in (H2) see [7].

The g_p is in (13) $\forall p \in \mathcal{M}$ defined by

$$g_p^S := g^S(x_p) = g\left(\left|\int_{\mathbb{R}^d} \nabla G_S(x_p - \xi)\tilde{I}^0(\xi)\,\mathrm{d}\xi\right|\right). \tag{15}$$

Now we define some symbols we will be using in the next sections:

$$w_{\mathcal{D},\tau}(x,t) = -\frac{u_p^{n+1} - u_p^n}{\tau f_1(N_p(u^n))} + \frac{r_p^{n+1}}{|p|\tau f_1(N_p(u^n))} \tag{16}$$

$$\delta u_{\mathcal{D},\tau}(x,t) = \frac{u_p^{n+1} - u_p^n}{\tau}, \quad N_{\mathcal{D},\tau}(x,t) = N_p(u^{n+1}), \quad g_{\mathcal{D}}^S(x) = g_p^S, \tag{17}$$

for a.e. $x \in p$, for a.e. $t \in [n\tau, (n+1)\tau]$, $\forall p \in \mathcal{M}$, $\forall n \in \mathbb{N}$,

$$G_{\mathcal{D},\tau}(x,t) = d\,\frac{u_\sigma^{n+1} - u_p^{n+1}}{d_{p\sigma}} n_{p\sigma}, \quad H_{\mathcal{D},\tau}(x,t) = d\,g_p^S\,\frac{u_\sigma^{n+1} - u_p^{n+1}}{d_{p\sigma} f(N_p(u^n))} n_{p\sigma}, \tag{18}$$

for a.e. $x \in D_{p\sigma}$, for a.e. $t \in [n\tau, (n+1)\tau]$, $\forall p \in \mathcal{M}$, $\forall \sigma \in \mathcal{E}_p$, $\forall n \in \mathbb{N}$, where $D_{p\sigma}$ is the cone with vertex x_p and basis σ.

Finally, we define function F (see [1]), which we need to use in the following sections. Let F be function defined by

$$F(s)=\int\limits_{0}^{s}\frac{z}{f(z)}\,\mathrm{d}z,\qquad \forall s\in\mathbb{R}_{+}. \tag{19}$$

Definition of function f implies that

$$F(s)\in\left[\frac{s^2}{2\;b},\frac{s^2}{2\;a}\right], \tag{20}$$

where constants a and b are the same as in the definition of the function f in (H4).

2. Stability estimates

Lemma 2.1 (L^∞ stability of the scheme)**.** *Under Hypothesis (H), let $(\mathcal{D},\tau)$ be a space-time discretisation of $\Omega\times[0,T]$. Now we denote by*

$$|u_0|_{\mathcal{D},\infty}=\max_{p\in\mathcal{M}}|u_p^0|, \tag{21}$$

and by

$$|r|_{\mathcal{D},\tau,\infty}=\max\left\{\frac{|r_p^{n+1}|}{\tau\;|p|},p\in\mathcal{M},n=0,\ldots,N_T\right\} \tag{22}$$

(note that, if $u_0\in L^\infty(\Omega)$ and $r\in L^\infty(\Omega\times\mathbb{R}_+)$, then $|u_0|_{\mathcal{D},\infty}\le\|u_0\|_{L^\infty(\Omega)}$ and $|r|_{\mathcal{D},\tau,\infty}\le\|r\|_{L^\infty(\Omega\times(0,T))}$). Let $(u_p^n)_{p\in\mathcal{M},n\in\mathbb{N}}$ be a solution of (13)–(14)*. Then following holds:*

$$\begin{aligned}|u_p^n|&\le|u_0|_{\mathcal{D},\infty}+|r|_{\mathcal{D},\tau,\infty}n\tau\\&\le|u_0|_{\mathcal{D},\infty}+|r|_{\mathcal{D},\tau,\infty}T,\qquad\forall p\in\mathcal{M},\quad\forall n=0,\ldots,\mathbb{N}_T.\end{aligned}$$

P r o o f. Now follow the ideas from [1] and suppose that for fixed time step $(n+1)$ the maximum of all u_p^{n+1} is achieved at the finite volume p. We can write (13) in the following way

$$u_p^{n+1}+\frac{\tau\;f_1(N_p(u^n))}{|p|\;f(N_p(u^n))}\sum_{\sigma\in\mathcal{E}_p}g_p^S\frac{|\sigma|}{d\;\;_{p\sigma}}\big(u_p^{n+1}-u_\sigma^{n+1}\big)=u_p^n+\frac{r_p^{n+1}}{|p|}. \tag{23}$$

On the other hand, from (14) we know that the value u_σ^{n+1} satisfies the equality

$$u_\sigma^{n+1}f\big(N_p(u^n)d_{p\sigma}\big)+f\big(N_q(u^n)d_{q\sigma}\big)=u_p^{n+1}f\big(N_q(u^n)d_{q\sigma}\big)+u_q^{n+1}f\big(N_p(u^n)d_{p\sigma}\big).$$

Let us write this equation in the following way

$$u_\sigma^{n+1}=\frac{u_p^{n+1}f(N_q(u^n))d_{q\sigma}+u_q^{n+1}f(N_p(u^n))d_{p\sigma}}{f(N_p(u^n))d_{p\sigma}+f(N_q(u^n))d_{q\sigma}},$$

which is a convex linear combination of points u_p^{n+1} and u_q^{n+1}.

We obtain

$$u_p^{n+1} - u_\sigma^{n+1} = u_p^{n+1} - \frac{u_p^{n+1} f(N_q(u^n)) d_{q\sigma} + u_q^{n+1} f(N_p(u^n)) d_{p\sigma}}{f(N_p(u^n)) d_{p\sigma} + f(N_q(u^n)) d_{q\sigma}}$$
$$= \frac{f(N_p(u^n)) d_{p\sigma} (u_p^{n+1} - u_q^{n+1})}{f(N_p(u^n)) d_{p\sigma} + f(N_q(u^n)) d_{q\sigma}}. \tag{24}$$

Because u_p^{n+1} is the maximum of u_p for the fixed time step $(n+1)$ is $u_p^{n+1} - u_q^{n+1}$ non-negative, thanks to definition of function f are $f\big(N_p(u^n)\big)$ and $f\big(N_q(u^n)\big)$ positive and $d_{p\sigma}$ and $d_{q\sigma}$ are positive. So $u_p^{n+1} - u_\sigma^{n+1}$ must be non-negative, too.

This implies the following inequality

$$\frac{\tau}{|p|} \sum_{\sigma \in \mathcal{E}_p} g_p^S \frac{|\sigma|}{d_{p\sigma}} \big(u_p^{n+1} - u_\sigma^{n+1}\big) \geq 0. \tag{25}$$

If we look back to the equality (23), we can see that it leads to

$$u_p^{n+1} \leq u_p^n + \frac{r_p^{n+1}}{|p|} \leq u_p^n + |r|_{\mathcal{D},\tau,\infty}\ \tau. \tag{26}$$

Applying this method recursively for n we get

$$u_p^{n+1} \leq u_p^0 + |r|_{\mathcal{D},\tau,\infty} n\tau \leq |u_0|_{\mathcal{D},\infty} + |r|_{\mathcal{D},\tau,\infty} T. \tag{27}$$

So we get our estimate. Proof for the minimum values is similar. □

Remark 2.1 (Uniqueness of the discrete solution)**.** The consequence is that there exists one and only one solution to the semi-implicit scheme (13), (14).

Lemma 2.2 ($L^2(\Omega \times [0,T])$ estimate on u_t and $L^\infty(0,T;H_{\mathcal{D}})$ estimate)**.** *Let Hypothesis* (H) *be fulfilled. Let* $(\mathcal{D},\tau)$ *be a space-time discretisation of* $\Omega \times [0,T]$ *and let* $\theta \in (0,\theta_{\mathcal{D}}]$*, where* $\theta_{\mathcal{D}} = \min_{p\in\mathcal{M}} \min_{\sigma\in\mathcal{E}_p} \frac{d_{p\sigma}}{h_p}$ *and let* ν_S *be defined in* (H7)*. Let* $(u_p^n)_{p\in\mathcal{M},n\in\mathbb{N}}$ *be the solution of* (13), (14)*. Then there exists* $C_\theta > 0$*, only depending on* θ*, such that it holds:*

$$\frac{1}{2b\nu_S} \sum_{n=0}^{m-1} \tau \sum_{p\in\mathcal{M}} |p| \left(\frac{u_p^{n+1} - u_p^n}{\tau}\right)^2$$
$$+ \sum_{p\in\mathcal{M}} |p|\ F\big(N_p(u^m)\big) + \frac{1}{2b} \sum_{n=0}^{m-1} \sum_{p\in\mathcal{M}} |p| \big(N_p(u^{n+1}) - N_p(u^n)\big)^2$$
$$\leq \frac{C_\theta \|u^0\|_{H^1(\Omega)}^2 + \frac{1}{\nu_S} \|r\|_{L^2(\Omega\times(0,T))}^2}{2\ a}, \qquad \forall m = 1,\dots,N_T. \tag{28}$$

P r o o f. Based on idea from [1] we multiply the scheme (13) by $u_p^{n+1} - u_p^n$ and sum over p. We obtain

$$\begin{aligned}
&\sum_{p\in\mathcal{M}} \frac{|p|}{\tau\ f_1(N_p(u^n))}\big(u_p^{n+1} - u_p^n\big)^2 \\
&\quad - \sum_{p\in\mathcal{M}} \frac{(u_p^{n+1} - u_p^n)}{f(N_p(u^n))} \sum_{\sigma\in\mathcal{E}_p} g_p^S \frac{|\sigma|}{d_{p\sigma}}\big(u_\sigma^{n+1} - u_p^{n+1}\big) \\
&= \sum_{p\in\mathcal{M}} \frac{r_p^{n+1}}{\tau\ f_1(N_p(u^n))}\big(u_p^{n+1} - u_p^n\big), \qquad \forall p\in\mathcal{M}, \quad \forall n\in\mathbb{N}.
\end{aligned} \tag{29}$$

We can write it in the form

$$T_1 + T_2 = T_3, \tag{30}$$

where

$$\begin{aligned}
T_1 &= \sum_{p\in\mathcal{M}} \frac{|p|}{\tau f_1(N_p(u^n))}\big(u_p^{n+1} - u_p^n\big)^2, \\
T_2 &= \sum_{p\in\mathcal{M}} \frac{g_p^S}{f(N_p(u^n))} \sum_{\sigma\in\mathcal{E}_p} \frac{|\sigma|}{d_{p\sigma}}\big(u_\sigma^{n+1} - u_p^{n+1}\big)\big(u_p^n - u_p^{n+1}\big) \\
&= \sum_{p\in\mathcal{M}} \frac{1}{f(N_p(u^n))} \sum_{\sigma\in\mathcal{E}_p} g_p^S \frac{|\sigma|}{d_{p\sigma}}\big(u_\sigma^{n+1} - u_p^{n+1}\big)\big(u_\sigma^{n+1} - u_p^{n+1} - (u_\sigma^n - u_p^n)\big), \\
T_3 &= \sum_{p\in\mathcal{M}} \frac{r_p^{n+1}}{\tau\ f_1(N_p(u^n))}\big(u_p^{n+1} - u_p^n\big).
\end{aligned}$$

In term T_2 we have used the property (14) of the finite volume scheme.

At first we remark that, thanks to Young's inequality and to the Cauchy-Schwarz inequality,

$$\begin{aligned}
T_3 &\le \frac{1}{2}\left(\sum_{p\in\mathcal{M}} \frac{|p|}{\tau f_1(N_p(u^n))}\big(u_p^{n+1} - u_p^n\big)^2 + \sum_{p\in\mathcal{M}} \frac{(r_p^{n+1})^2}{|p|\tau f_1(N_p(u^n))}\right) \\
&= \frac{1}{2}T_1 + \sum_{p\in\mathcal{M}} \frac{(r_p^{n+1})^2}{2|p|\ \ \tau\ f_1(N_p(u^n))} \le \frac{1}{2}T_1 + \frac{1}{2a}\sum_{p\in\mathcal{M}} \frac{(r_p^{n+1})^2}{|p|\ \ \tau} \\
&\le \frac{1}{2a}\int\limits_{n\tau}^{(n+1)\tau}\int\limits_{\Omega} r(x,t)^2\,\mathrm{d}x\,\mathrm{d}t + \frac{1}{2}T_1,
\end{aligned} \tag{31}$$

where we have in last step used (11).

In the study of the T_2 we apply approach presented in [1]. Using (19)—the definition of function F, we have

$$F\big(N_p(u^{n+1})\big) - F\big(N_p(u^n)\big) = \int\limits_{N_p(u^n)}^{N_p(u^{n+1})} \frac{z\mathrm{d}z}{f(z)}.$$

Now, we set for $c, d \in \mathbb{R}_+$,

$$\Phi_c(d) = \frac{d}{f(c)}(d-c) - \frac{(d-c)^2}{2f(c)} - \int\limits_c^d \frac{z\mathrm{d}z}{f(z)}.$$

We have $\Phi_c(c) = 0$, and $\Phi'_c(d) = \frac{2d-c}{f(c)} - \frac{2d-2c}{2f(c)} - \frac{d}{f(d)} = \frac{d}{f(c)} - \frac{d}{f(d)}$.

Thanks to Hypothesis (H4) is f (non-strictly) increasing so we get

$$\forall c, d \in \mathbb{R}_+, \quad \int\limits_c^d \frac{z\mathrm{d}z}{f(z)} + \frac{(d-c)^2}{2f(c)} \leq \frac{d}{f(c)}(d-c). \tag{32}$$

Thanks to (7) we know we can set $d = N_p(u^{n+1})$ and $c = N_p(u^n)$ to (32), realize (5) - $\nu_S \leq g_p^S$ for all $p \in \mathcal{M}$ and definition of the function F stated in (19) to get

$$\begin{aligned} g_p^S\Big(F\big(N_p(u^{n+1})\big) - F\big(N_p(u^n)\big)\Big) + \frac{\nu_S}{2b}\big(N_p(u^{n+1}) - N_p(u^n)\big)^2 \leq \\ g_p^S\Big(F\big(N_p(u^{n+1})\big) - F\big(N_p(u^n)\big)\Big) + \frac{g_p^S}{2}\frac{\big(N_p(u^{n+1}) - N_p(u^n)\big)^2}{f\big(N_p(u^n)\big)} \leq \\ g_p^S\frac{N_p(u^{n+1})}{f(N_p(u^n))}\big(N_p(u^{n+1}) - N_p(u^n)\big). \end{aligned} \tag{33}$$

Now we multiply (33) by $|p|$ and sum it over all $p \in \mathcal{M}$

$$\begin{aligned} \sum_{p\in\mathcal{M}} |p| g_p^S\Big(F\big(N_p(u^{n+1})\big) - F\big(N_p(u^n)\big)\Big) + \frac{\nu_S}{2b}\sum_{p\in\mathcal{M}} |p| \ \big(N_p\big(u^{n+1}\big) - N_p\big(u^n\big)\big)^2 \leq \\ \sum_{p\in\mathcal{M}} |p| g_p^S\frac{N_p(u^{n+1})^2}{f(N_p(u^n))} - \sum_{p\in\mathcal{M}} |p| g_p^S\frac{N_p(u^{n+1})N_p(u^n)}{f(N_p(u^n))}. \end{aligned} \tag{34}$$

Note that the Cauchy-Schwarz inequality implies

$$\sum_{\sigma\in\mathcal{E}_p} \frac{|\sigma|}{d_{p\sigma}}\big(u_\sigma^{n+1} - u_p^{n+1}\big)\big(u_\sigma^n - u_p^n\big) \leq |p| \ \ N_p\big(u^n\big)N_p\big(u^{n+1}\big), \tag{35}$$

which in turn gives us

$$\sum_{p\in\mathcal{M}} |p| g_p^S \frac{N_p(u^{n+1})^2}{f(N_p(u^n))} - \sum_{p\in\mathcal{M}} |p| g_p^S \frac{N_p(u^{n+1}) N_p(u^n)}{f(N_p(u^n))} \leq$$

$$\sum_{p\in\mathcal{M}} \frac{g_p^S}{f(N_p(u^n))} \sum_{\sigma\in\mathcal{E}_p} \frac{|\sigma|}{d_{p\sigma}} (u_\sigma^{n+1} - u_p^{n+1}) \big(u_\sigma^{n+1} - u_p^{n+1} - (u_\sigma^n - u_p^n)\big) = T_2. \quad (36)$$

Putting the above together we obtain

$$T_1 + \sum_{p\in\mathcal{M}} |p| g_p^S \Big(F\big(N_p(u^{n+1})\big) - F\big(N_p(u^n)\big)\Big) +$$

$$\frac{\nu_S}{2b} \sum_{p\in\mathcal{M}} |p| \, \big(N_p(u^{n+1}) - N_p(u^n)\big)^2 \leq$$

$$T_1 + T_2 = T_3 \leq \frac{1}{2} T_1 + \frac{1}{2a} \int\limits_{n\tau}^{(n+1)\tau} \int\limits_{\Omega} r(x,t)^2 \mathrm{d}x\mathrm{d}t. \quad (37)$$

After rewriting, summing this inequality over $n = 0, \ldots, m-1$ for all $m = 1, \ldots, N_T$ and using (5) and **(H5)**, we get that

$$\frac{1}{2b} \sum_{n=0}^{m-1} \tau \sum_{p\in\mathcal{M}} |p| \left(\frac{u_p^{n+1} - u_p^n}{\tau} \right)^2 + \nu_S \sum_{p\in\mathcal{M}} |p| \; F\big(N_p(u^m)\big) +$$

$$\frac{\nu_S}{2b} \sum_{n=0}^{m-1} \sum_{p\in\mathcal{M}} |p| \, \big(N_p(u^{n+1}) - N_p(u^n)\big)^2 \leq$$

$$\sum_{p\in\mathcal{M}} |p| g_p^S \; F\big(N_p(u^0)\big) + \frac{1}{2a} \int\limits_0^{m\tau} \int\limits_{\Omega} r(x,t)^2 \mathrm{d}x\mathrm{d}t \leq$$

$$\sum_{p\in\mathcal{M}} |p| g_p^S \; F\big(N_p(u^0)\big) + \frac{1}{2a} \|r\|^2_{L^2(\Omega\times(0,T))},$$

where we define u_σ^0 by (10).

As the last step we use the following inequality, proven in [8]: there exists $C_\theta > 0$, only depending on θ, such that

$$|p| \; N_p(u^0)^2 \leq C_\theta \|u_0\|^2_{H^1(p)}, \qquad \forall p \in \mathcal{M} \quad (38)$$

and we get

$$\sum_{p\in\mathcal{M}} |p|\; F\big(N_p(u^0)\big) = \sum_{p\in\mathcal{M}} |p| \int\limits_0^{N_p(u^0)} \frac{z}{f(z)}\,\mathrm{d}z \le$$

$$\frac{1}{a}\sum_{p\in\mathcal{M}} |p|\frac{(N_p(u^0))^2}{2} \le \frac{1}{2a}C_\theta \sum_{p\in\mathcal{M}} ||u_0||^2_{H^1(p)} = \frac{1}{2a}C_\theta ||u_0||^2_{H^1(\Omega)}, \quad (39)$$

which concludes the estimate (28). □

3. Convergence of the scheme

This section is based on the approach presented in [1], so we list few lemmas without proofs as there are no changes from [1]. The others are given with proofs as the generalisation and new point of view were needed.

Lemma 3.1. *Let Ω be an open bounded connected subset of $\mathbb{R}^d$, with $d \in \mathbb{N}$ and let $T > 0$. Let $(\mathcal{D}_m, \tau_m)_{m\in\mathbb{N}}$ denote a sequence of space-time discretisations such that $h_{\mathcal{D}_m}$ and τ_m tends to 0 as $m \to \infty$. Let $(u_m)_{m\in\mathbb{N}}$ be such that $u_m \in \mathcal{D}_m, \tau_m$, such that $||u_m||_{1,\mathcal{D}_m,\tau_m} \le C$ for all $m \in \mathbb{N}$ and such that there exists $\bar{u} \in L^2(\Omega \times [0,T])$ such that sequence of functions $u_{\mathcal{D}_m,\tau_m}$ defined for $u = u_m$, $\mathcal{D} = \mathcal{D}_m$ and $\tau = \tau_m$ by*

$$u_{\mathcal{D},\tau}(x,t) = u_m^{n+1},\ for\ a.e.\ x \in p,\ t \in \big[n\tau,(n+1)\tau\big],\ \forall p \in \mathcal{M},\ \forall n = 0,\dots,N_T,$$

satisfies $u_{\mathcal{D}_m,\tau_m} \to \bar{u}$ in $L^2(\Omega \times [0,T])$ as $m \to \infty$.

Then $\bar{u} \in L^2\big(0,T;H^1_0(\Omega)\big)$. Moreover, defining $G_m \in L^\infty\big(0,T;L^2(\Omega)\big)$ by

$$G_m(x,t) = d\frac{u_\sigma^{n+1} - u_p^{n+1}}{d_{p\sigma}} n_{p\sigma}$$

for a.e. $x \in D_{p\sigma}$ and $t \in \big[n\tau,(n+1)\tau\big]$, then $G_m \rightharpoonup \nabla\bar{u}$ in $L^2(\Omega\times[0,T])^d$ as $m \to \infty$.

Proof. See proof of the Lemma 4.1 in [1]. □

Lemma 3.2 (Strong convergence of the approximate gradient norm)**.** *Let Ω be an open bounded connected subset of $\mathbb{R}^d$, with $d \in \mathbb{N}$ and let $T > 0$. Let $(\mathcal{D}_m, \tau_m)_{m\in\mathbb{N}}$ denote a sequence of space-time discretisations such that $h_{\mathcal{D}_m}$ and τ_m tends to 0 as $m \to \infty$. For any $\phi \in C^\infty_C(\Omega \times [0,T])$ we define the discrete interpolation of ϕ, denoted $v \in H_{(\mathcal{D},\tau)}$, by $v_p^n = \phi(x_p, n\tau)$ and $v_\sigma^n = \phi(x_\sigma, n\tau)$ and we define $\mathcal{N}_{\mathcal{D},\tau}$ by*

$$\mathcal{N}_{\mathcal{D}_m,\tau_m}(x,t) = N_p(v^{n+1}),\ for\ a.e.\ x \in p,\ t \in \big[n\tau,(n+1)\tau\big],\ \forall p \in \mathcal{M},\ \forall n = 0,\dots,N_T.$$

Then $\mathcal{N}_{\mathcal{D}_m,\tau_m} \to |\nabla\phi|$ *in* $L^\infty(\Omega \times [0,T])$ *as* m *tends to* ∞.

P r o o f. See proof of the Lemma 4.2 in [1]. □

Lemma 3.3 (Strong approximate of the gradient of ϕ)**.** *For all* $\phi \in C_C^\infty(\Omega \times [0,T])$ *we denote by* $v_p^n = \phi(x_p, n\tau)$ *and* $v_\sigma^n = \phi(x_\sigma, n\tau)$*. We introduce the approximation*

$$\begin{aligned}\nabla_{p\sigma}^{n+1}\phi = {} & \frac{v_\sigma^{n+1} - v_p^{n+1}}{d_{p\sigma}} n_{p\sigma} \\ & + \nabla\phi\big(x_p,(n+1)\tau\big) - \Big(\nabla\phi\big(x_p,(n+1)\tau\big)\cdot n_{p\sigma}\Big) n_{p\sigma},\end{aligned} \tag{40}$$

and $\nabla_{\mathcal{D}_m,\tau_m}\phi(x,t) = \nabla_{p\sigma}^{n+1}\phi$ *for* $x \in D_{p\sigma}$ *and* $t \in \big[n\tau,(n+1)\tau\big]$*.*

Then $\nabla_{\mathcal{D}_m,\tau_m}\phi \to \nabla\phi$ *in* $L^\infty(\Omega \times [0,T])$ *as* m *tends to* ∞*.*

P r o o f. See proof of the Lemma 4.3 in [1]. □

As the next we prove short lemma, which will be needed later in the proof of the convergence of the scheme.

Lemma 3.4. *For all* $u, v \in H_{\mathcal{D}}$*:*

$$\sum_{p\in\mathcal{M}}\sum_{\sigma\in\mathcal{E}_p} g_p^S \frac{|\sigma|}{d_{p\sigma}}\left(\frac{u_\sigma - u_p}{f(N_p(u))} - \frac{v_\sigma - v_p}{f(N_p(v))}\right)(u_\sigma - u_p - v_\sigma + v_p) \geq 0.$$

P r o o f. Applying the Cauchy-Schwartz inequality, we get

$$\begin{aligned}&\sum_{\sigma\in\mathcal{E}_p}\frac{|\sigma|}{d_{p\sigma}}\left(\frac{u_\sigma - u_p}{f(N_p(u))} - \frac{v_\sigma - v_p}{f(N_p(v))}\right)(u_\sigma - u_p - v_\sigma + v_p) \\ &= \sum_{\sigma\in\mathcal{E}_p}\Bigg(\frac{|\sigma|}{d_{p\sigma}}\frac{(u_\sigma - u_p)^2}{f(N_p(u))} + \frac{|\sigma|}{d_{p\sigma}}\frac{(v_\sigma - v_p)^2}{f(N_p(v))} \\ &\quad - \frac{|\sigma|}{d_{p\sigma}}\frac{(u_\sigma - u_p)(v_\sigma - v_p)}{f(N_p(u))} - \frac{|\sigma|}{d_{p\sigma}}\frac{(u_\sigma - u_p)(v_\sigma - v_p)}{f(N_p(v))}\Bigg) \\ &\geq \frac{|p|N_p(u)^2}{f(N_p(u))} + \frac{|p|N_p(v)^2}{f(N_p(v))} - \frac{|p|N_p(u)N_p(v)}{f(N_p(u))} - \frac{|p|N_p(u)N_p(v)}{f(N_p(v))} \\ &= |p|\left(\frac{N_p(u)}{f(N_p(u))} - \frac{N_p(v)}{f(N_p(v))}\right)\big(N_p(u) - N_p(v)\big) \geq 0.\end{aligned} \tag{41}$$

The last expression is non-negative thanks to the (H4). On the other hand, from (H7), it is clear that

$$\begin{aligned}&\sum_{p\in\mathcal{M}} g_p^S \sum_{\sigma\in\mathcal{E}_p} \frac{|\sigma|}{d_{p\sigma}}\left(\frac{u_\sigma-u_p}{f(N_p(u))}-\frac{v_\sigma-v_p}{f(N_p(v))}\right)(u_\sigma-u_p-v_\sigma+v_p)\\ &\geq \nu_S\sum_{p\in\mathcal{M}}\sum_{\sigma\in\mathcal{E}_p} \frac{|\sigma|}{d_{p\sigma}}\left(\frac{u_\sigma-u_p}{f(N_p(u))}-\frac{v_\sigma-v_p}{f(N_p(v))}\right)(u_\sigma-u_p-v_\sigma+v_p),\end{aligned} \tag{42}$$

where ν_S is independent on the discretisation. Which concludes proof of the lemma. □

Lemma 3.5 (Convergence properties). *Let Hypothesis* (H) *be fulfilled and for all $m\in\mathbb{N}$ the function $u_{\mathcal{D}_m,\tau_m}$ is defined by $u_{\mathcal{D}_m,\tau_m}(x,t)=u_p^{n+1}$ for a.e. $x\in p$, $\forall t\in(n\tau,(n+1)\tau]$, $\forall p\in\mathcal{M}$, $\forall n\in\mathbb{N}$. Let $(\mathcal{D}_m,\tau_m)_{m\in\mathbb{N}}$ denotes a sequence of space-time discretisations such that $h_{\mathcal{D}_m}$ and τ_m tend to 0 as $m\to\infty$, $\theta_{\mathcal{D}_m}$ remains bounded away from 0. Then there exists a subsequence of $(\mathcal{D}_m,\tau_m)_{m\in\mathbb{N}}$, again denoted $(\mathcal{D}_m,\tau_m)_{m\in\mathbb{N}}$, there exists a function*

$$\bar{u}\in L^\infty\big(0,T;H_0^1(\Omega)\big)\bigcap C^0\big(0,T;L^2(\Omega)\big),$$

such that $\bar{u}_t\in L^2(\Omega\times[0,T])$, $u(.,0)=u_0$ and $u_{\mathcal{D}_m,\tau_m}$ tend to $\bar{u}$ in $L^2\big(0,T;H_0^1(\Omega)\big)$ and there exists functions $\bar{H}\in L^2(\Omega\times[0,T])^d$, $\bar{w}\in L^2(\Omega\times[0,T])$ such that $H_{\mathcal{D}_m,\tau_m}\rightharpoonup\bar{H}$ weakly in $L^2(\Omega\times[0,T])^d$ (see (18)*) and such that $w_{\mathcal{D}_m,\tau_m}\rightharpoonup\bar{w}$ and $\delta u_{\mathcal{D}_m,\tau_m}\rightharpoonup\bar{u}_t$ weakly in $L^2(\Omega\times[0,T])$ as $m\to\infty$. Moreover, $G_{\mathcal{D}_m,\tau_m}\rightharpoonup\nabla\bar{u}$ weakly in $L^2(\Omega\times[0,T])^d$ and the following relations holds:*

$$\lim_{m\to\infty}\int_0^T\!\!\int_\Omega g^S_{\mathcal{D}_m}(x)\frac{N_{\mathcal{D}_m,\tau_m}(x,t)^2}{f(N_{\mathcal{D}_m,\tau_m}(x,t))}\,\mathrm{d}x\,\mathrm{d}t=\int_0^T\!\!\int_\Omega \bar{H}(x,t)\cdot\nabla\bar{u}(x,t)\,\mathrm{d}x\,\mathrm{d}t. \tag{43}$$

P r o o f. From the definition of F and (20) $u_{\mathcal{D}_m,\tau_m}(\cdot,t)$ is uniformly bounded in $H_{\mathcal{D}}$ for all $t\in[0,T]$. Hence we can apply a generalisation of Arzela-Ascoli's theorem (see [1, Theorem 6.1]), which implies that the convergence property $u_{\mathcal{D}_m,\tau_m}(\cdot,t)\to\bar{u}\in C^0\big(0,T;L^2(\Omega)\big)$ holds in $L^\infty\big(0,T;L^2(\Omega)\big)$. Moreover, thanks to (9), we have $\bar{u}(\cdot,t)=u_0$ and, thanks to Lemma 3.1, that $\bar{u}\in L^\infty\big(0,T;H_0^1(\Omega)\big)$ and that $G_{\mathcal{D}_m,\tau_m}\rightharpoonup\nabla\bar{u}$ in $L^2(\Omega\times[0,T])^d$.

From definition of $w_{\mathcal{D},\tau}$ in (16) and from Lemma 2.2 we get that $w_{\mathcal{D}_m,\tau_m}$ remains bounded in $L^2(\Omega\times[0,T])$ for all $m\in\mathbb{N}$. Therefore there exists a function $\bar{w}\in L^2(\Omega\times[0,T])$ such that $w_{\mathcal{D}_m,\tau_m}\rightharpoonup\bar{w}$ up to a subsequence in $L^2(\Omega\times[0,T])$.

Similarly, we have $\delta u_{\mathcal{D}_m,\tau_m}\rightharpoonup\bar{u}_t$ in $L^2(\Omega\times[0,T])$, which shows that $\bar{u}_t\in L^2(\Omega\times[0,T])$ and $H_{\mathcal{D}_m,\tau_m}\rightharpoonup\bar{H}$ in $L^2(\Omega\times[0,T])^d$ up to a subsequence.

Let rewrite (13) to the form

$$-\frac{1}{f(N_p(u^n))}\sum_{\sigma\in\mathcal{E}_p} g_p^S\frac{|\sigma|}{d_{p\sigma}}\big(u_\sigma^{n+1}-u_p^{n+1}\big)=|p|w_p^{n+1},\quad \forall\, p\in\mathcal{M},\ \ \forall n\in\mathbb{N} \tag{44}$$

and turn to study (43). Let $\phi\in C_C^\infty(\Omega\times[0,T])$ be given. We denote by $v_p^n=\phi(x_p,n\tau)$ and $v_\sigma^n=\phi(x_\sigma,n\tau)$. Multiplying (44) by τv_p^{n+1} and summing over n and p we get $T_{1m}=T_{2m}$ with

$$T_{1m}=\sum_{n=0}^{N_T}\tau\sum_{p\in\mathcal{M}} g_{\mathcal{D}_m}^S\sum_{\sigma\in\mathcal{E}_p}\frac{|\sigma|}{d_{p\sigma}}\frac{u_\sigma^{n+1}-u_p^{n+1}}{f(N_p(u^n))}\big(v_\sigma^{n+1}-v_p^{n+1}\big) \tag{45}$$

and

$$T_{2m}=\sum_{n=0}^{N_T}\tau\sum_{p\in\mathcal{M}}|p|w_p^{n+1}v_p^{n+1}. \tag{46}$$

Using the approximation $\nabla_{\mathcal{D}_m,\tau_m}\phi$ of $\nabla\phi$ introduced in Lemma 3.3 we obtain

$$T_{1m}=\int_0^T\!\!\int_\Omega H_{\mathcal{D}_m,\tau_m}\cdot\nabla_{\mathcal{D}_m,\tau_m}\phi\,\mathrm{d}x\,\mathrm{d}t. \tag{47}$$

As we mentioned above $H_{\mathcal{D}_m,\tau_m}\rightharpoonup\bar{H}$ in $L^2(\Omega\times[0,T])^d$ and thanks to the Lemma 3.3 we know that $\nabla_{\mathcal{D}_m,\tau_m}\phi\to\nabla\phi$ in $L^\infty(\Omega\times[0,T])$. From (H6) we know that function g is bounded and continuous, together with properties of convolution mentioned in (H7) it gives us that $g_{\mathcal{D}_m}^S\to g^S$ in $L^\infty(\Omega)$. So we get

$$\lim_{m\to\infty}T_{1m}=\int_0^T\!\!\int_\Omega\bar{H}\cdot\nabla\phi\,\mathrm{d}x\,\mathrm{d}t. \tag{48}$$

On the other hand, we have

$$\lim_{m\to\infty}T_{2m}=\int_0^T\!\!\int_\Omega\bar{w}\phi\,\mathrm{d}x\,\mathrm{d}t. \tag{49}$$

Hence

$$\int_0^T\!\!\int_\Omega\bar{H}\cdot\nabla\phi\,\mathrm{d}x\,\mathrm{d}t=\int_0^T\!\!\int_\Omega\bar{w}\phi\,\mathrm{d}x\,\mathrm{d}t. \tag{50}$$

Since this equality holds for all $\phi\in C_C^\infty(\Omega\times[0,T])$ it also holds, thanks to the density, for all $v\in L^2\big(0,T;H_0^1(\Omega)\big)$ and especially for $\bar{u}$

$$\int_0^T\!\!\int_\Omega\bar{H}\cdot\nabla\bar{u}\mathrm{d}x\mathrm{d}t=\int_0^T\!\!\int_\Omega\bar{w}\bar{u}\,\mathrm{d}x\,\mathrm{d}t. \tag{51}$$

We now multiply (44) by τu_p^{n+1}, summing over n and p and we get $T_{3m} = T_{4m}$ with

$$\begin{aligned} T_{3m} &= \sum_{n=0}^{N_T} \tau \sum_{p\in\mathcal{M}} g_p^S \sum_{\sigma\in\mathcal{E}_p} \frac{|\sigma|}{d_{p\sigma}} \frac{(u_\sigma^{n+1} - u_p^{n+1})^2}{f(N_p(u^n))} \\ &= \sum_{n=0}^{N_T} \tau \sum_{p\in\mathcal{M}} g_p^S |p| \frac{N_p(u^{n+1})^2}{f(N_p(u^n))} \\ &= \int_0^T\!\!\int_\Omega g_{\mathcal{D}_m}^S(x) \frac{N_{\mathcal{D}_m,\tau_m}(x,t)^2}{f(N_{\mathcal{D}_m,\tau_m}(x,t))} \,\mathrm{d}x\,\mathrm{d}t \end{aligned} \tag{52}$$

and

$$T_{4m} = \sum_{n=0}^{N_T} \tau \sum_{p\in\mathcal{M}} |p| w_p^{n+1} u_p^{n+1} = \int_0^T\!\!\int_\Omega w_{\mathcal{D}_m,\tau_m} u_{\mathcal{D}_m,\tau_m} \,\mathrm{d}x\,\mathrm{d}t. \tag{53}$$

As we mentioned above

$$w_{\mathcal{D}_m,\tau_m} \rightharpoonup \bar{w} \in L^2(\Omega\times[0,T])$$

and

$$u_{\mathcal{D}_m,\tau_m} \to \bar{u} \in L^\infty\big(0,T;L^2(\Omega)\big)$$

so we get

$$\lim_{m\to\infty} T_{4m} = \int_0^T\!\!\int_\Omega \bar{w}\bar{u} \,\mathrm{d}x\,\mathrm{d}t. \tag{54}$$

(51) and (54) lead to

$$\lim_{m\to\infty} T_{3m} = \int_0^T\!\!\int_\Omega \bar{w}\bar{u}\,\mathrm{d}x\,\mathrm{d}t = \int_0^T\!\!\int_\Omega \bar{H}\cdot\nabla\bar{u}\,\mathrm{d}x\,\mathrm{d}t, \tag{55}$$

which completes the proof of (43). □

Lemma 3.6. *Let Hypothesis* (H) *be fulfilled and for all* $m \in \mathbb{N}$ *the function* $u_{\mathcal{D}_m,\tau_m}$ *is defined by*

$$u_{\mathcal{D}_m,\tau_m}(x,t) = u_p^{n+1} \ \text{ for a.e. } x\in p,\ \ \forall t\in(n\tau,(n+1)\tau],\ \ \forall p\in\mathcal{M},\ \ \forall n\in\mathbb{N}.$$

Let $(\mathcal{D}_m,\tau_m)_{m\in\mathbb{N}}$ *denotes an extracted subsequence (the existence is provided by Lemma 3.5). Let* $\phi\in C_C^\infty(\Omega\times[0,T])$ *be given. We denote by*

$$v_p^n = \phi(x_p, n\tau) \quad \text{and} \quad v_\sigma^n = \phi(x_\sigma, n\tau)$$

and by

$$T_m = \sum_{n=0}^{N_T} \tau \sum_{p\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}_p} g_p^S \frac{|\sigma|}{d_{p\sigma}} \Bigg(\frac{u_\sigma^{n+1} - u_p^{n+1}}{f(N_p(u^{n+1}))} - \frac{v_\sigma^{n+1} - v_p^{n+1}}{f(N_p(v^{n+1}))} (u_\sigma^{n+1} - u_p^{n+1} - v_\sigma^{n+1} + v_p^{n+1}) \Bigg). \qquad (56)$$

Then the following holds

$$\lim_{m\to\infty} T_m = \int_0^T\!\!\int_\Omega \left(\bar{H} - g^S \frac{\nabla\phi}{f(|\nabla\phi|)} \right) (\nabla\bar{u} - \nabla\phi)\,\mathrm{d}x\,\mathrm{d}t, \qquad (57)$$

and

$$\int_0^T\!\!\int_\Omega \bar{H}\cdot\nabla v\,\mathrm{d}x\,\mathrm{d}t = \int_0^T\!\!\int_\Omega g^S \frac{\nabla\bar{u}}{f(|\nabla\bar{u}|)} \cdot \nabla v\,\mathrm{d}x\,\mathrm{d}t \qquad \forall v \in L^2\big(0,T;H_0^1(\Omega)\big). \qquad (58)$$

P r o o f. Remark that T_m we can rewrite in the form $T_m = T_{3m} - T_{5m} - T_{6m} + T_{7m}$, where T_{3m} is the same as in (52) in Lemma 3.5 and

$$T_{5m} = \sum_{n=0}^{N_T} \tau \sum_{p\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}_p} g_p^S \frac{|\sigma|}{d_{p\sigma}} \left(\frac{u_\sigma^{n+1} - u_p^{n+1}}{f(N_p(u^{n+1}))} (v_\sigma^{n+1} + v_p^{n+1}) \right),$$

$$T_{6m} = \sum_{n=0}^{N_T} \tau \sum_{p\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}_p} g_p^S \frac{|\sigma|}{d_{p\sigma}} \left(\frac{v_\sigma^{n+1} - v_p^{n+1}}{f(N_p(u^{n+1}))} (u_\sigma^{n+1} + u_p^{n+1}) \right),$$

$$T_{7m} = \sum_{n=0}^{N_T} \tau \sum_{p\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}_p} g_p^S \frac{|\sigma|}{d_{p\sigma}} \left(\frac{v_\sigma^{n+1} - v_p^{n+1}}{f(N_p(u^{n+1}))} (v_\sigma^{n+1} + v_p^{n+1}) \right).$$

From (H6) we know that function g is bounded and continuous, together with properties of convolution mentioned in (H7) it gives us that $g_{\mathcal{D}_m}^S \to g^S$ in $L^\infty(\Omega)$. On the other hand, thanks to the strong convergence of the $\mathcal{N}_{\mathcal{D},\tau}$ to $|\nabla\phi|$ in $L^\infty(\Omega\times[0,T])$ provided in Lemma 3.2, strong convergence of the $\nabla_{\mathcal{D},\tau}\phi$ to $\nabla\phi$ in $L^\infty(\Omega\times[0,T])$ and weak convergence of the $H_{\mathcal{D}_m,\tau_m}$ to $\bar{H}$ in $L^2(\Omega\times[0,T])^d$ we have (55) and

$$\lim_{m\to\infty} T_{5m} = \int_0^T\int_\Omega \bar{H}\cdot\nabla\phi \,\mathrm{d}x\,\mathrm{d}t,$$

$$\lim_{m\to\infty} T_{6m} = \int_0^T\int_\Omega g^S \frac{\nabla\phi}{f(|\nabla\phi|)}\cdot\nabla\bar{u} \,\mathrm{d}x\,\mathrm{d}t,$$

$$\lim_{m\to\infty} T_{7m} = \int_0^T\int_\Omega g^S \frac{\nabla\phi}{f(|\nabla\phi|)}\cdot\nabla\phi \,\mathrm{d}x\,\mathrm{d}t.$$

Gathering these results together we get (57).

Thanks to Lemma 3.4 we know that $T_m \geq 0$, which provides

$$\int_0^T\int_\Omega \left(\bar{H} - g^S \frac{\nabla\phi}{f(|\nabla\phi|)}\right)(\nabla\bar{u} - \nabla\phi)\,dx\,dt \geq 0, \qquad \forall\,\phi \in C_C^\infty(\Omega\times[0,T]). \quad (59)$$

Density of the functions from $C_C^\infty(\Omega\times[0,T])$ in $L^2\big(0,T;H_0^1(\Omega)\big)$ implies that the above inequality holds for all $v \in L^2\big(0,T;H_0^1(\Omega)\big)$.

Now we apply the so called Minty trick taking $v = \bar{u} - \lambda\psi$, with $\lambda > 0$ and $\psi \in C_C^\infty(\Omega\times[0,T])$ and use Lebesgue's dominated convergence theorem to obtain

$$\int_0^T\int_\Omega \left(\bar{H} - g^S \frac{\nabla\bar{u}}{f(|\nabla\bar{u}|)}\right)\nabla\psi\,dx\,dt \geq 0, \qquad \forall\,\psi \in C_C^\infty(\Omega\times[0,T]). \quad (60)$$

The same trick we can apply for $-\psi$, so actually

$$\int_0^T\int_\Omega \left(\bar{H} - g^S \frac{\nabla\bar{u}}{f(|\nabla\bar{u}|)}\right)\nabla\psi\,dx\,dt = 0, \qquad \forall\,\psi \in C_C^\infty(\Omega\times[0,T]). \quad (61)$$

As this equality could be extended for all $v \in L^2\big(0,T;H_0^1(\Omega)\big)$ we achieve proof of (58). □

Lemma 3.7. *Under the same assumptions as in Lemma 3.6 $\mathcal{N}_{\mathcal{D}_m,\tau_m} \to |\nabla\bar{u}|$ in $L^2(\Omega\times[0,T])$ as $m\to\infty$.*

Proof. See proof of the Lemma 4.7 in [1]. □

Theorem 3.1. *Let Hypothesis (H) be fulfilled and for all $m \in \mathbb{N}$ the function $u_{\mathcal{D}_m,\tau_m}$ is defined by $u_{\mathcal{D}_m,\tau_m}(x,t) = u_p^{n+1}$ for a.e. $x \in p$, $\forall t \in (n\tau,(n+1)\tau]$, $\forall p \in \mathcal{M}$, $\forall n \in \mathbb{N}$. Let $(\mathcal{D}_m,\tau_m)_{m\in\mathbb{N}}$ denotes a sequence of space-time discretisations such that $h_{\mathcal{D}_m}$ and τ_m tends to 0 as $m\to\infty$, $\theta_{\mathcal{D}_m}$ remains bounded away* *from* 0*. We assume that sequence* $(\mathcal{D}_m, \tau_m)_{m\in\mathbb{N}}$ *denotes an extracted subsequence (the existence is provided by Lemma 3.4).*

Then the function

$$\bar{u} \in L^\infty\big(0,T; H_0^1(\Omega)\big), \quad \text{such that} \quad u_{\mathcal{D}_m,\tau_m} \to \bar{u} \quad \text{in } L^2\big(0,T; H_0^1(\Omega)\big),$$

is a weak solution of (1)–(3)*. Moreover, if we define:*

$$\hat{G}_{\mathcal{D},\tau}(x,t) = \frac{1}{|p|} \sum_{\sigma\in\mathcal{E}_p} \big(u_\sigma^{n+1} - u_p^{n+1}\big) n_{p\sigma}, \tag{62}$$

for a.e. $x \in p$, $t \in (n\tau, (n+1)\tau]$, $\forall p \in \mathcal{M}$, $\forall n \in \mathbb{N}$, *it holds that* $\hat{G}_{\mathcal{D}_m,\tau_m} \to \nabla\bar{u}$ *in* $L^2(\Omega\times[0,T])^d$ *and* $N_{\mathcal{D},\tau}(x,t) \to |\nabla\bar{u}|$ *in* $L^2(\Omega\times[0,T])$*.*

$\hat{G}_{\mathcal{D},\tau}$ *define strongly convergent approximation for the gradient of the* $\bar{u}$ *(recall that* $G_{\mathcal{D},\tau}(x,t)$ *defined in* (18) *is only weak convergent).*

P r o o f. Following arguments presented in [1] from Lemma 3.6 we know that

$$\int_0^T\!\!\int_\Omega g^S \frac{\nabla\bar{u}}{f(|\nabla\bar{u}|)} \cdot \nabla v \,\mathrm{d}x\,\mathrm{d}t = \int_0^T\!\!\int_\Omega \bar{H}\cdot\nabla v\,\mathrm{d}x\,\mathrm{d}t, \qquad \forall v \in L^2\big(0,T; H_0^1(\Omega)\big). \tag{63}$$

On the other hand, from Lemma 3.5 we know that

$$\int_0^T\!\!\int_\Omega \bar{H}\cdot\nabla v = \int_0^T\!\!\int_\Omega \bar{w} v\,\mathrm{d}x\,\mathrm{d}t, \qquad \forall v \in L^2\big(0,T; H_0^1(\Omega)\big). \tag{64}$$

Combination of these two equations with the definition of w in (16), which give us that $\bar{w} = \frac{r-u_t}{f(|\nabla u|)}$ completes the proof that $\bar{u}$ is a weak solution of (1)–(3).

Proof of the strong convergence of $\hat{G}_{\mathcal{D},\tau}$ to $\nabla\bar{u}$ in $L^2(\Omega\times[0,T])^d$ and of the strong convergence of $N_{\mathcal{D},\tau}$ to $|\nabla\bar{u}|$ in $L^2(\Omega\times[0,T])$ is, compared to proof of the [1, Theorem 4.1], without changes, so we do not repeat it here. □

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Department of Mathematics
Slovak University of Technology
in Bratislava
Radlinského 11
SK–810-05 Bratislava
SLOVAKIA

E-mail: matus.tibensky@stuba.sk
angela.handlovicova@stuba.sk