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TATRA MOUNTAINS
Mathematical Publications
DOI: 10.2478/tmmp-2018-0017
Tatra Mt. Math. Publ. **72** (2018), 1–15

PROBABILITY INTEGRAL AS A LINEARIZATION

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ABSTRACT. In fuzzified probability theory, a classical probability space $(\Omega, \mathbf{A}, p)$ is replaced by a generalized probability space $(\Omega, \mathcal{M}(\mathbf{A}), \int(.)\,\mathrm{d}p)$, where $\mathcal{M}(\mathbf{A})$ is the set of all measurable functions into [0,1] and $\int(.)\mathrm{d}p$ is the probability integral with respect to p. Our paper is devoted to the transition from p to $\int(.)\,\mathrm{d}p$. The transition is supported by the following categorical argument: there is a minimal category and its epireflective subcategory such that $\mathbf{A}$ and $\mathcal{M}(\mathbf{A})$ are objects, probability measures and probability integrals are morphisms, $\mathcal{M}(\mathbf{A})$ is the epireflection of $\mathbf{A}$, $\int(.)\,\mathrm{d}p$ is the corresponding unique extension of p, and $\mathcal{M}(\mathbf{A})$ carries the initial structure with respect to probability integrals.

We discuss reasons why the fuzzy random events are modeled by $\mathcal{M}(\mathbf{A})$ equipped with pointwise partial order, pointwise Łukasiewicz operations (logic) and pointwise sequential convergence. Each probability measure induces on classical random events an additive linear preorder which helps making decisions. We show that probability integrals can be characterized as the additive linearizations on fuzzy random events, i.e., sequentially continuous maps, preserving order, top and bottom elements.

Introduction

In [37], L. A. Zadeh has proposed to replace a classical probability space $(\Omega, \mathbf{A}, p)$ with a fuzzified probability space $\big(\Omega, \mathcal{M}(\mathbf{A}), \int(.)\,\mathrm{d}p\big)$, where $\mathcal{M}(\mathbf{A})$ is the set of all measurable functions into [0,1] and $\int(.)\,\mathrm{d}p$ is the probability integral with respect to p. Fundamental results on fuzzified probability theory (motivation, definitions of notions, technical results, applications, categorical

2010 Mathematics Subject Classification: 60A86, 60A05, 06D72, 18A20.
Keywords: generalized probability, measurable function, fuzzy random event, D-poset, effect algebra, A-poset, IA-poset, Łukasiewicz tribe, epireflection, probability integral, linearization.
This work was supported by the Slovak Scientific Grant Agency under the project VEGA 2/0031/15 and the Slovak Research and Development Agency under the contract No. APVV--16–0073. The author is a PhD student of Comenius University in Bratislava, Faculty of Mathematics, Physics and Informatics.

approach) can be found in [4], [5], [12]–[14], [16], [19]–[21], [24], [27], [32], [33], and in papers cited therein.

In [29], M. Navara observed that no justification to define the probability of a fuzzy event $f \in \mathcal{M}(\mathbf{A})$ by the formula $\int(f)\,\mathrm{d}p$ was given by Zadeh and he discussed two distinct approaches to generalized probability, probability on tribes and probability on MV-algebras with products [34]. In our contribution, we present another supportive argument for the transition from $(\Omega, \mathbf{A}, p)$ to $\big(\Omega, \mathcal{M}(\mathbf{A}), \int(.)\,\mathrm{d}p\big)$: categorical approach to generalized probability [17], [20]. Indeed, there is a minimal category and its epireflective subcategory such that $\mathbf{A}$ and $\mathcal{M}(\mathbf{A})$ are objects, probability measures and probability integrals are morphisms, $\mathcal{M}(\mathbf{A})$ is the epireflection [1] of $\mathbf{A}$ and $\int(.)\,\mathrm{d}p$ is the corresponding unique extension of p. Each object $\mathcal{M}(\mathbf{A})$ is equipped with the multivalued Łukasiewicz logic, carries the initial structure with respect to probability integrals, and each probability integral can be characterized as the additive linearization of fuzzy random events.

The idea of quantification of uncertainty about the future development (as a number p, $0 \le p \le 1$) goes back to Jacob Bernoulli: "The probability namely is the degree of certainty and differs from it as a part from the whole" (see [3]). The quantification of future events (assigning a number) induces a linear (pre)order on the events and helps to conjecture (make decisions). This explains our understanding of **linearization**. Besides having philosophical and methodological aspects, it has contributed to "mathematization" of probability.

Kolmogorov has "mathematized" probability theory (via axioms) in [25].

- At the beginning we have a **probability space** $(\Omega, \mathbf{A}, p)$, where Ω is the set of all outcomes of a random experiment, $\mathbf{A}$ is a σ-field of subsets of Ω, each $A \in \mathbf{A}$ is called an **event**, events of the form $A = \{\omega\}$, $\omega \in \Omega$, are called elementary events;
- $p : \mathbf{A} \to [0, 1]$ is a normalized σ-additive measure called **probability**, $p(A)$ measures how "big" is $A \in \mathbf{A}$ in comparison to Ω; the most important example is $(R, \mathbf{B}_R, p)$, where R are the real numbers, $\mathbf{B}_R$ is the real Borel σ-field, and p is a probability on $\mathbf{B}_R$.

Kolmogorov's axiomatization of probability was actually an attempt to solve the sixth problem of D. Hilbert: to axiomatize physics, because probability was considered as part of physics. From the viewpoint of category theory, Kolmogorov's probability has a weak point: it uses Boolean operations on events, but probability measures do not preserve these operations. The transition from $\mathbf{A}$ to $\mathcal{M}(\mathbf{A})$ is a minimal extension of the field of events so that basic maps become morphisms and the extended probability models the following quantum phenomenon: a classical outcome (point) can be mapped to a genuine probability measure. In fact (cf. [18], [22]), this is related to the divisibility of random events (each fuzzy random event $u \in \mathcal{M}(\mathbf{A})$ is divisible in $\mathcal{M}(\mathbf{A})$, i.e., for each positive natural number k we have $u/k \in \mathcal{M}(\mathbf{A})$, but classical random events from $\mathbf{A}$ fail to be divisible in $\mathbf{A}$).

In what follows, systems of functions $\mathcal{X} \subseteq [0,1]^\Omega$ are equipped with the natural pointwise partial order and pointwise convergence of sequences.

Observe that the Lebesgue Dominated Convergence Theorem, LDCT in short, implies that each probability integral $\int(.)\,\mathrm{d}p$ on $\mathcal{M}(\mathbf{A})$ (hence each probability measure p on $\mathbf{A}$) is sequentially continuous. Consequently, the σ-additivity of a normalized additive measure is equivalent to sequential continuity (not only to monotone continuity, as usually claimed). For this reason, we consider only sequentially continuous linearizations.

Definition 0.1. A sequentially continuous map $L\colon \mathcal{M}(\mathbf{A}) \longrightarrow [0,1]$, preserving order, top and bottom elements, is called a linearization of $\mathcal{M}(\mathbf{A})$. If for u, $v \in \mathcal{M}(\mathbf{A}), u(\omega)+v(\omega) \le 1, \omega \in \Omega$, we have $L(u+v) = L(u)+L(v)$, the L is said to be additive.

Phenomena in quantum physics motivate studies of generalized probability and mathematical quantum structures. In order to describe such phenomena, we seek suitable generalizations of classical models [4], [5], [24]. Random events in classical probability theory [25] can be generalized in different ways [4], [5], [24], [27], [31], [34]. For example, generalized random events are modeled by quantum logics, effect algebras, difference posets, etc. [7], [9], [26], [35].

We use another structure called A-posets which is isomorphic to effect algebras and D-posets [36]. It is defined in terms of a partial order and a partial operation "addition" which generalizes the original "disjoint disjunction" introduced by G. Boole (cf. [2]) and hence it has a more direct logical interpretation than the difference in D-posets [7], [26].

1. Why probability integral

In this section we outline arguments from which it follows that probability integral is the proper quantification of fuzzy random events. Technical details (definitions and propositions) will be presented in the last section.

L. A. Zadeh in his pioneering paper [37] has proposed to extend random events, represented by the indicator functions of a sigma-field $\mathbf{A}$ of sets, to fuzzy random events, represented by the set $\mathcal{M}(\mathbf{A}) \subset [0,1]^\Omega$ of all measurable fuzzy sets, and to consider the probability integral $\int(.)\,\mathrm{d}p$ as the extension of the probability measure p. Further, he proposed $\max, \min$ and the usual complementation as operations on fuzzy random events. In the follow-up papers, Zadeh concentrates on applications in engineering and soft computing.

A thorough study of fuzzified probability can be found in [29]. As stated by N a v a r a, $\int(.)\,\mathrm{d}p$ is a natural extension of p, but no justification was given by Z a d e h and N a v a r a discussed two distinct approaches to generalized probability, probability on tribes and probability on MV-algebras with products. Our goal is to describe a more complex reason based on the categorical approach to probability. It can be summarized as follows.

Claim. *There is a suitable category such that*

- $(\Omega, \mathbf{A}, p)$ *and* $\big(\Omega, \mathcal{M}(\mathbf{A}), \int(.)\,\mathrm{d}p\big)$ *are models of probability theory the basic notions of which are defined within the category in question;*
- $\mathbf{A}$*,* $\mathcal{M}(\mathbf{A})$ *are objects and* p*,* $\int(.)\,\mathrm{d}p$ *are morphisms. Moreover,* p *and* $\int(.)\,\mathrm{d}p$ *are linearizations characterized by a fundamental property of probability–additivity;*
- $\mathbf{A}$ *and* $\mathcal{M}(\mathbf{A})$ *carry the initial structure with respect to all probability measures on* $\mathbf{A}$ *and with respect to of all probability integrals on* $\mathcal{M}(\mathbf{A})$*, respectively.*
- $\big(\Omega, \mathcal{M}(\mathbf{A}), \int(.)\,\mathrm{d}p\big)$ *is a "minimal" extension of* $(\Omega, \mathbf{A}, p)$*. The minimality is based on natural properties of fuzzy random events and the extension can be characterized as an epireflection.*

Let us point out some requirements concerning the category in question.

Requirements.

- *Objects are sets equipped with a suitable structure.*
- *Morphisms are "structure preserving maps".*
- *Both* $\mathbf{A}$ *(boolean structure) and* $\mathcal{M}(\mathbf{A})$ *(fuzzy structure) have to be equipped with "the same" structure.*
- *Each probability measure* $p\colon \mathbf{A} \longrightarrow [0,1]$ *and each probability integral has to be a morphism, hence* $\mathbf{A}$*,* $\mathcal{M}(\mathbf{A})$ *and* $[0,1]$ *(equipped with a suitable structure) have to be objects of the corresponding category.*
- *Objects of the form* $\mathcal{M}(\mathbf{A})$ *have to form a distinguished subcategory.*

Let us recall (cf. [19]) why $\mathcal{M}(\mathbf{A})$ is a natural candidate to model fuzzy random events. Let $\mathbf{A}$ be a σ-field of subsets of Ω. Denote a_Ω, $a \in [0,1]$, the constant function such that $a_\Omega(\omega) = a$, $\omega \in \Omega$. Then $\mathcal{M}(\mathbf{A})$ is the smallest of all subsets $\mathcal{X} \subseteq [0,1]^\Omega$ containing $\mathbf{A}$ (indicator functions of sets in $\mathbf{A}$) and closed with respect to negations (if $u \in \mathcal{X}$, then $(1_\Omega - u) \in \mathcal{X}$), pointwise suprema, pointwise sequential limits, and divisible ($(1/n)_\Omega \in \mathcal{X}$, $n \in N^+$). So, $\mathcal{M}(\mathbf{A})$ has the necessary properties of a fuzzification of $\mathbf{A}$ and, as we shall see, it can be equipped with the appropriate structure (multivalued Łukasiewicz logic). Further, there is a one-to-one correspondence between σ-fields and measurable functions into [0,1] and a one-to-one correspondence between probability measures and probability integrals. As indicated above, the correspondence is functorial (epireflection). Finally, let $\mathbf{T} = \{\emptyset, \Omega\}$ be the trivial field of sets, where Ω is a singleton. Since each function in $\mathcal{M}(\mathbf{T})$ is determined by a single number in [0,1], hence [0,1] can be viewed as $\mathcal{M}(\mathbf{T})$.

To sum up, the transition from $(\Omega, \mathbf{A}, p)$ to $\big(\Omega, \mathcal{M}(\mathbf{A}), \int(.)\,\mathrm{d}p\big)$ has a categorical background: $\mathcal{M}(\mathbf{A})$ is a categorical extension of $\mathbf{A}$ and $\int(.)\,\mathrm{d}p$ is the corresponding unique categorical extension (epireflection) of p.

2. Why A-posets

As explicitly stated in [10], any generalized probability theory based on (algebraic) measure theory should be restricted to events for which "there are enough (generalized) probability measures". This leads to ID-posets, i.e., D-posets of functions, the structure of which is determined by sequentially continuous D-homomorphisms (see [16], [17], [30]). On the one hand, fuzzified probability theory by R. Frič and M. Papčo is based on the category of ID-posets, i.e., it uses the language of partial order and difference, but on the other hand, fuzzy random events are modeled via bold algebras and Łukasiewicz operations (generalizations of Boolean disjunction, negation and conjunction). Consequently, some mathematical and interpretational effort is needed to pass from "difference" to "plus".

Therefore, at ISCAMI 2014, V. Skřivánek has introduced A-posets and the corresponding category of fuzzy events which serves as an alternative reference category for the fuzzification of classical Kolmogorov's probability theory (see [36]). A-posets, D-posets and effect algebras are isomorphic structures (see [36]), but A-posets lead more directly to the Łukasiewicz logic. A-posets are defined in terms of a partial order and a partial operation "addition" and they are motivated by the original approach to logic via "disjoint disjunction" of G. Boole [2]. The resulting partial operations of disjunction and conjunction (along with negation) act on generalized random events and lead to a smooth transition from the classical to fuzzified probability: their extension to binary operations results in the usual Łukasiewicz operations on fuzzy random events.

Definition 2.1. An A-poset is a system $(S, \leq, 0, 1, \oplus)$ consisting of partial ordered set S with top element 1 and bottom element 0 and a partial binary operation $\oplus$ such that:

(A_1) If $a \oplus b$ is defined, then $b \oplus a$ is defined and $a \oplus b = b \oplus a$.

(A_2) If $(a \oplus b) \oplus c$ is defined, then $a \oplus (b \oplus c)$ is defined and $(a \oplus b) \oplus c = a \oplus (b \oplus c)$.

(A_3) For each $a \in S$ there exists a unique $a^c \in S$ such that $a \oplus a^c = 1$.

(A_4) If $a \oplus b$ is defined, $a_1 \leq a$ and $b_1 \leq b$, then $a_1 \oplus b_1$ is defined and $a_1 \oplus b_1 \leq a \oplus b$.

Observe that $a \oplus 0 = a$ and (A_4) is equivalent to "$a \oplus b$ is defined if and only if $a \leq b^c$".

If no confusion can arise, then an A-poset $(S, \leq, 0, 1, \oplus)$ will be condensed to S.

Definition 2.2. Let S_1 and S_2 be A-posets and let h be a map on S_1 into S_2 preserving the order, constants, and addition. Then h is said to be an A-homomorphism.

Example 2.3. Let $\mathbf{A}$ be a field of subsets of Ω. Then $\mathbf{A}$ can be reorganized into an A-poset as follows:

(i) $\mathbf{A}$ is partially ordered by inclusion.
(ii) $\emptyset$ and Ω represent the bottom element and the top element, respectively.
(iii) For $A \in \mathbf{A}$ define $A^c = \Omega \setminus A$.
(iv) For $A, B \in \mathbf{A}$ define $A \oplus B = A \cup B$ if and only if $A \cap B = \emptyset$.

Clearly, axioms (A_1)–(A_4) are satisfied.

Example 2.4. Let $\mathcal{X} \subseteq [0,1]^\Omega$ be a system of functions on Ω into [0,1] such that the constant functions $0_\Omega, 1_\Omega$ belong to $\mathcal{X}$, if $u \in \mathcal{X}$ then $1_\Omega - u \in \mathcal{X}$, if $u, v \in \mathcal{X}$ and $v \leq 1_\Omega - u$ then $u + v \in \mathcal{X}$.

(i) For $u \in \mathcal{X}$ define $u^c = 1_\Omega - u$.
(ii) For $u, v \in \mathcal{X}, v \leq 1_\Omega - u$, define $u \oplus v = u + v$.

Then $\mathcal{X}$ equipped with the pointwise partial order becomes an A-poset. Let $\mathbf{A}$ be a σ-field of subsets of Ω. Denote $s(\mathbf{A})$ the simple functions in $\mathcal{M}(\mathbf{A})$, i.e., functions of the form $\sum_{i=1}^n c_i \chi_{A_i}$, where $c_i \in [0,1]$, $A_1, A_2, \ldots, A_n$ are disjoint subsets in $\mathbf{A}$ covering Ω, and n is a natural number. Then $s(\mathbf{A})$ and $\mathcal{M}(\mathbf{A})$ (hence also [0,1] considered as $\mathcal{M}(\mathbf{T})$) can be viewed as A-posets. Observe that $s(\mathbf{A})$ is divisible. Let $\mathbf{A}$ be a σ-field of subsets of Ω. Denote e_1 the embedding of $\mathbf{A}$ into $s(\mathbf{A})$ and denote e_2 the embedding of $s(\mathbf{A})$ into $\mathcal{M}(\mathbf{A})$. Clearly, e_1, e_2, and their composition $e_2 \circ e_1$ are A-morphisms.

In what follows, the composition $e_2 \circ e_1$ will be denoted as *id*. Clearly, the A-homomorphisms e_1, e_2, and the composition $id = e_2 \circ e_1$ are sequentially continuous with respect to the pointwise convergence of functions.

Lemma 2.5. *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be fields of subsets of* Ω *and* Ξ*, respectively. Let* h *be an A-homomorphism of* $\mathbf{B}$ *into* $\mathbf{A}$*, considered as A-posets. Then* h *is a Boolean homomorphism.*

Proof. Clearly, $h(\Xi) = \Omega$ and $h(\emptyset) = \emptyset$. Let $B_1, B_2 \in \mathbf{B}$, $B_1 \cap B_2 = \emptyset$. Then $h(B_1 \cup B_2) = h(B_1 \oplus B_2) = h(B_1) \oplus h(B_2) = h(B_1) \cup h(B_2)$ and $h(B_1) \cap h(B_2) = \emptyset$. Consequently, $h\big(B \cup (\Xi \setminus B)\big) = \Omega = h(B) \cup h(\Xi \setminus B)$, where $h(B) \cap h(\Xi \setminus B) = \emptyset$ and hence $h(B^c) = h(B)^c$. Further, for $B_1, B_2 \in \mathbf{B}$ the set $B_1 \cup B_2$ is the union of three disjoint sets $B_1 \setminus B_2$, $B_1 \cap B_2$ and $B_2 \setminus B_1$, $h(B_1)$ is the disjoint union of $h(B_1) \setminus h(B_1 \cap B_2)$ and $h(B_1 \cap B_2)$, $h(B_2)$ is the disjoint union of $h(B_2) \setminus h(B_1 \cap B_2)$ and $h(B_1 \cap B_2)$. Necessarily, h preserves the union of two sets. From De Morgan's laws it follows that h preserves also the intersection of two sets. Thus h is a Boolean homomorphism. □

Recall the notion of a categorical product of two objects. An object A together with two morphisms (called projections) $pr_i \colon A \longrightarrow A_i$, $i = 1, 2$, is called the product of two objects A_1 and A_2, called factors, if for each object B and each two morphisms $h_i \colon B \longrightarrow A_i$, $i = 1, 2$, there exists a unique morphism $h \colon B \longrightarrow A$ such that $pr_i \circ h = h_i$, $i = 1, 2$. The product of an indexed family of factors is defined analogously. If the product exists, then it is unique (up to an isomorphism).

Denote $\mathbb{A}$ the category with A-posets as objects and A-homomorphisms as morphisms.

Lemma 2.6. *The category $\mathbb{A}$ has products.*

Proof. Let A_1 and A_2 be A-posets. Let A be the set of all pairs (a_1, a_2), $a_i \in A_i$, $i = 1, 2$. Define projections $pr_i \colon A \longrightarrow A_i$, $i = 1, 2$, in the usual way: $pr_1(a_1, a_2) = a_1$ and $pr_2(a_1, a_2) = a_2$. Define the A-poset structure on A pointwise. It is easy to see that A together with projections $pr_i, i = 1, 2$, is the categorical product of A_1 and A_2. The product of an indexed family of A-posets is constructed analogously. □

3. Generalized random events

In [36] R. Frič and V. Skřivánek introduced a fuzzified probability on A-posets of functions. The resulting generalized random events form a probability domain (cf. [16], [19], [20]) cogenerated by the closed unit interval $I = [0, 1]$, considered as an A-poset. Such probability domains are analogous to ID-posets (cf. [16], [30], [31]) but, unlike the partial operation difference $\ominus$ in an ID-poset $\mathcal{X} \subseteq I^X$, the partial operation addition $\oplus$ has a clear logical interpretation: "disjunction for disjoint fuzzy events".

The Boolean logic can be extended to fuzzy events in many ways. In particular, via the Łukasiewicz logic. As pointed out by D. Mundici in [28], among all continuous t-norms, Łukasiewicz conjunction is the only one yielding a logic with a continuous implication connective.

Definition 3.1. An A-poset of functions whose values are in $[0,1]$ is said to be an IA-poset. A sequentially continuous A-homomorphism of an IA-poset $\mathcal{X} \subseteq I^X$ into I is said to be a state. A sequentially continuous A-homomorphism of an IA-poset $\mathcal{X} \subseteq I^X$ into an IA-poset $\mathcal{Y} \subseteq I^Y$ is said to be an observable.

Definition 3.2. Łukasiewicz tribe is a system $\mathcal{X} \subseteq [0,1]^\Omega$ closed with respect to pointwise sequential limits, containing the constant functions 0_Ω, 1_Ω and closed with respect to the usual Łukasiewicz operations disjunction, conjunction, negation defined pointwise: for $u, v \in \mathcal{X}$ and $\omega \in \Omega$) we put

- $(u \oplus v)(\omega) = u(\omega) \oplus v(\omega) = \min\{1, u(\omega) + v(\omega)\}$;
- $(u \odot v)(\omega) = u(\omega) \odot v(\omega) = \max\{0, u(\omega) + v(\omega) - 1\}$;
- $u^*(\omega) = 1 - u(\omega)$.

Obviously, each σ-field $\mathbf{A}$ and the corresponding measurable functions $\mathcal{M}(\mathbf{A})$ are canonical examples of Łukasiewicz tribes (see also [11]). As pointed out in [22], the upgrading of classical probability lies in the divisibility of $\mathcal{M}(\mathbf{A})$.

Let $\mathcal{X} \subseteq [0,1]^\Omega$ be a Łukasiewicz tribe. Then there exists a unique σ-field $\mathbf{A}_\mathcal{X}$ of subsets of Ω such that $\mathbf{A}_\mathcal{X} \subseteq \mathcal{X} \subseteq \mathcal{M}(\mathbf{A}_\mathcal{X})$. Moreover, $\mathcal{X} = \mathcal{M}(\mathbf{A}_\mathcal{X})$ if and only if $\mathcal{X}$ contains all constant functions $r_\Omega, r \in [0,1]$ ([6], [35]). Łukasiewicz tribes of the form $\mathcal{M}(\mathbf{A})$ are said to be a **full**. We say that two Łukasiewicz tribes $\mathcal{X} \subseteq [0,1]^\Omega$ and $\mathcal{Y} \subseteq [0,1]^\Omega$ are equivalent whenever $\mathbf{A}_\mathcal{X} = \mathbf{A}_\mathcal{Y}$. Clearly, $\mathbf{A}_\mathcal{X}$ and $\mathcal{M}(\mathbf{A}_\mathcal{X})$ are equivalent. Further, $\mathbf{A}_\mathcal{X}$ and $\mathcal{M}(\mathbf{A}_\mathcal{X})$ are **extremal**, $\mathbf{A}_\mathcal{X}$ is the bottom element and $\mathcal{M}(\mathbf{A}_\mathcal{X})$ is the top element in the equivalence class in question.

If we identify $A \subseteq \Omega$ and its indicator function $\chi_A \in \{0,1\}^\Omega$, $\chi_A(\omega) = 1$ for $\omega \in A$ and $\chi_A(\omega) = 0$ for $\omega \in A^c$, then each indicator function can be viewed as a Boolean propositional function "ω belongs to A" and each measurable function can be viewed as a fuzzy propositional function.

Observe that G. Boole used partial union. He did not introduce Boolean algebra, it was introduced later [2]. Accordingly, the A-poset of fuzzy sets is a natural fuzzification of the original Boole's idea.

Denote by $\mathbb{IA}$ the category having IA-posets as objects and sequentially continuous A-homomorphisms as morphisms. To deal with the transition from $\mathbf{A}$ to $\mathcal{M}(\mathbf{A})$ in terms of category theory, we introduce the following subcategories of $\mathbb{IA}$: the objects of $\mathbb{LIA}$ are Łukasiewicz tribes, the objects of $\mathbb{ELIA}$ are extremal Łukasiewicz tribes (bottom or top elements in an equivalence class), and the objects of $\mathbb{FELIA}$ are full Łukasiewicz tribes.

Claims.

- *Basic notions of the classical probability theory: random events and Boolean logic operations, random variables, and probability measures can be defined within* $\mathbb{ELIA}$.

- *Via the epireflection, to each classical probabilistic notion there corresponds its "fuzzified" notion within* $\mathbb{FELIA}$.
- *All "stochastic maps" become morphisms in* $\mathbb{FELIA}$.
- *Basic constructions in probability theory become categorical.*
- $\mathcal{M}(\mathbf{A})$ *carries the initial A-poset structure with respect to states, i.e., morphisms into* $[0,1] = \mathcal{M}(\mathbf{T})$ *(cogenerator).*

The next lemma is a categorical bookkeeping.

Lemma 3.3. *Let* $\mathcal{X}_t, t \in T$, *be an indexed family of A-posets and let* $\mathcal{X}$ *be product of* $\mathcal{X}_t, t \in T$.

(i) *Let each* $\mathcal{X}_t$ *be an object of* $\mathbb{LIA}$. *Then* $\mathcal{X}$ *is an object of* $\mathbb{LIA}$.

(ii) *Let each* $\mathcal{X}_t$ *be an object of* $\mathbb{FELIA}$. *Then* $\mathcal{X}$ *is an object of* $\mathbb{FELIA}$.

Proof. (i) Each $\mathcal{X}_t$ is a Łukasiewicz tribe consisting of functions on a set Ω_t into [0,1], $t \in T$. Let Ω be their disjoint union. Then each $u \in \mathcal{X}$ is represented as a function on Ω into [0,1] "disjointly glued" of functions from $\mathcal{X}_t$, $t \in T$, and $\mathcal{X}$ is equipped with the pointwise A-structure. Clearly, $\mathcal{X}$ is an object of $\mathbb{LIA}$.

(ii) follows from (i). $\square$

4. Epireflection

As outlined in introductory sections, the transition from $(\Omega, \mathbf{A}, p)$ to $\big(\Omega, \mathcal{M}(\mathbf{A}), \int(.)\,\mathrm{d}p\big)$ can be described in terms of a categorical epireflection. In this section we state and prove the underlying assertions.

In [15], it has been proved that the category of full Łukasiewicz tribes is an epireflective subcategory of the category of bold algebras and sequentially continuous D-homomorphisms (see also [23]). This is a rather general assertion and the proof of it uses powerful machinery of abstract analysis. On the one hand, the transition from $\mathbf{A}$ to $\mathcal{M}(\mathbf{A})$ and from p to $\overline{p} = \int(.)\,\mathrm{d}p$ is a corollary of this general assertion, on the other hand, our assertions and their proofs are rather transparent and appropriate to describe the transition from the classical probability to its "minimal" fuzzification within $\mathbb{ELIA}$.

Lemma 4.1. *Let* $\mathbf{A}$ *be a* σ*-field of subsets of* Ω, *let* $p\colon \mathbf{A} \longrightarrow [0,1]$ *be a probability measure, and let* $\overline{p} = \int(.)\,\mathrm{d}p$ *be the corresponding probability integral on* $\mathcal{M}(\mathbf{A})$. *Then*

(i) $\overline{p}$ *can be viewed as a sequentially continuous A-homomorphism of* $\mathcal{M}(\mathbf{A})$ *into* $\mathcal{M}(\mathbf{T})$;

(ii) p *can be viewed as a sequentially continuous A-homomorphism of* $\mathbf{A}$ *into* $\mathcal{M}(\mathbf{T})$.

P r o o f.

(i) We consider [0,1] as $\mathcal{M}(\mathbf{T})$. Due to the LDCT $\overline{p}$ is sequentially continuous. Since each probability integral, as a mapping of $\mathcal{M}(\mathbf{A})$ into $\mathcal{M}(\mathbf{T})$, preserves order, constants and addition, the assertion holds true.

(ii) Since p is the restriction of $\overline{p}$ to the A-poset $\mathbf{A}$, (ii) follows from (i). □

Lemma 4.2. *Let $\mathbf{A}$ be a σ-field of subsets of Ω and let h be a sequentially continuous A-homomorphism of $\mathbf{A}$ into $\mathcal{M}(\mathbf{T})$. Then h is a probability measure.*

P r o o f. Consider h as a mapping of $\mathbf{A}$ into [0,1]. Clearly, $h(\emptyset) = 0$, $h(\Omega) = 1$ and $h(A \cup B) = h(A) + h(B)$ whenever $A \cap B = \emptyset$. Since h is sequentially continuous, it follows that h is σ-additive and hence a probability measure on $\mathbf{A}$. □

Consequently, probability measures are exactly sequentially continuous A-homomorphisms of σ-fields of sets into $\mathcal{M}(\mathbf{T})$.

Lemma 4.3. *Let $\mathbf{A}$ and $\mathbf{B}$ be fields of subsets of Ω and Ξ, respectively.*

(i) *Let g and h be a sequentially continuous A-homomorphisms of $s(\mathbf{A})$ into $\mathcal{M}(\mathbf{B})$. If $g(A) = h(A)$ for all $A \in \mathbf{A}$, then $g = h$.*

(ii) *Let g and h be a sequentially continuous A-homomorphisms of $\mathcal{M}(\mathbf{A})$ into $\mathcal{M}(\mathbf{B})$. If $g(A) = f(A)$ for all $A \in \mathbf{A}$, then $g = h$.*

(iii) *The sequentially continuous A-homomorphism $id\colon \mathbf{A} \longrightarrow \mathcal{M}(\mathbf{A})$ is an epimorphism.*

P r o o f.

(i) Let l be a positive natural number. Then for each natural number $k, k \le l$, and each $A \in \mathbf{A}$ we have $g\big((k/l)\chi_A\big) = (k/l)g(\chi_A) = (k/l)h(\chi_A) = h\big((k/l)\chi_A\big)$. Consequently h and g coincide on all $\sum_{i=1}^{n} c_i\chi_{A_i} \in s(\mathbf{A})$, where $c_i, i = 1, 2, \ldots, n$, are rational numbers in [0,1]. Since g and h are sequentially continuous, it follows that $g = h$.

(ii) It follows from (i) that g and h coincide on $s(\mathbf{A})$. The assertion follows from the fact that each $u \in \mathcal{M}(\mathbf{A})$ is a limit of a sequence $\{u_n\}_{n=1}^{\infty}$, where $u_n \in s(\mathbf{A})$ and $g(u_n) = h(u_n)$. Indeed, g and h are sequentially continuous and hence $g(u) = \lim_{n\to\infty} g(u_n) = \lim_{n\to\infty} h(u_n) = h(u)$.

(iii) Let g and h be a sequentially continuous A-homomorphisms of $\mathcal{M}(\mathbf{A})$ into $\mathcal{M}(\mathbf{B})$ such that $g(A) = f(A)$ for all $A \in \mathbf{A}$. Then $g \circ id = h \circ id$. We have to verify that $g = h$. But that is exactly what (ii) claims.

□

Lemma 4.4. *Let $\mathbf{A}$ be a σ-field of subsets of Ω and let h be a sequentially continuous A-homomorphism of $\mathbf{A}$ into $\mathcal{M}(\mathbf{T})$. Then there exists a unique sequentially continuous A-homomorphism h_s of $s(\mathbf{A})$ into $\mathcal{M}(\mathbf{T})$ extending h over $s(\mathbf{A})$.*

P r o o f. According to Lemma 4.2, h is a probability measure on $\mathbf{A}$. Denote $\overline{h} = \int(.)\,\mathrm{d}h$ the corresponding probability integral on $\mathcal{M}(\mathbf{A})$ and denote h_s the restriction of $\overline{h}$ to $s(\mathbf{A})$. It follows from Lemma 4.1 that $\overline{h}$ is a sequentially continuous A-homomorphism on $\mathcal{M}(\mathbf{A})$ into $\mathcal{M}(\mathbf{T})$ and hence h_s is a sequentially continuous A-homomorphism on $s(\mathbf{A})$ into $\mathcal{M}(\mathbf{T})$ which extends h. By Lemma 4.3, h_s is uniquely determined. □

Lemma 4.5. *Let $\mathbf{A}$ be a σ-field of subsets of Ω and let h_s be a sequentially continuous A-homomorphism of $s(\mathbf{A})$ into $\mathcal{M}(\mathbf{T})$. Then there exists a unique sequentially continuous A-homomorphism h_m of $\mathcal{M}(\mathbf{A})$ into $\mathcal{M}(\mathbf{T})$ extending h_s over $\mathcal{M}(\mathbf{A})$.*

P r o o f. It follows from Lemma 4.4 that there is a unique probability measure h on $\mathbf{A}$ such that h_s is the restriction of the probability integral $\int(.)\,\mathrm{d}h$ on $\mathcal{M}(\mathbf{A})$ to $s(\mathbf{A})$. It suffices to put $h_m = \int(.)\,\mathrm{d}h$. By Lemma 4.3, h_m is determined uniquely. □

Theorem 4.6. *Let $\mathbf{A}$ be a σ-field of subsets of Ω and let h be a sequentially continuous A-homomorphism of $\mathbf{A}$ into $\mathcal{M}(\mathbf{T})$. Then there exists a unique sequentially continuous A-homomorphism h_m of $\mathcal{M}(\mathbf{A})$ into $\mathcal{M}(\mathbf{T})$ extending h over $\mathcal{M}(\mathbf{A})$.*

P r o o f. It follows from the previous lemmas that h is a probability measure on $\mathbf{A}$ and h_m is exactly the probability integral $\overline{h} = \int(.)\,\mathrm{d}h$ on $\mathcal{M}(\mathbf{A})$, which is uniquely determined. □

Corollary 4.7. *Let $\mathbf{A}$ be a σ-field of sets and let L be a map of $\mathcal{M}(\mathbf{A})$ into $[0,1]$. Then the following are equivalent*

(i) *L is an additive linearization.*

(ii) *There exists a unique probability measure p on $\mathbf{A}$ such that $L = \int(.)\,\mathrm{d}p$.*

Theorem 4.8. *Let $\mathbf{A}$ and $\mathbf{B}$ be fields of subsets of Ω and Ξ, respectively. Let h be a sequentially continuous A-homomorphism of $\mathbf{A}$ into $\mathcal{M}(\mathbf{B})$. Then there exists a unique sequentially continuous A-homomorphism h_m of $\mathcal{M}(\mathbf{A})$ into $\mathcal{M}(\mathbf{B})$ extending h over $\mathcal{M}(\mathbf{A})$.*

P r o o f. Let $[0,1]^\Xi$ be the categorical power of $[0,1] = \mathcal{M}(\mathbf{T})$, let $pr_\xi, \xi \in \Xi$, be the projection of $[0,1]^\Xi$ to its ξth factor, let e be the embedding of $\mathcal{M}(\mathbf{B})$ into $[0,1]^\Xi$, and let id be the embedding of $\mathbf{A}$ into $\mathcal{M}(\mathbf{A})$. Then the composition $p_\xi = pr_\xi \circ e \circ h$ is a probability measure on $\mathbf{A}$ and, according to Theorem 4.6, p_ξ can be uniquely extended to a sequentially continuous A-homomorphism $\overline{p_\xi}$ over $\mathcal{M}(\mathbf{A})$, see Fig. 1. Since $[0,1]^\Xi$ is the categorical power of [0,1], there exists a unique sequentially continuous A-homomorphism h_Ξ of $\mathcal{M}(\mathbf{A})$ into $[0,1]^\Xi$ such that (for each $\xi \in \Xi$) the diagram in Fig. 2 commutes. Now, it suffices to prove

$$\mathbf{A} \xrightarrow{\ h\ } \mathcal{M}(\mathbf{B}) \xrightarrow{\ e\ } [0,1]^{\Xi} \xrightarrow{\ pr_\xi\ } [0,1]$$

$$\begin{tikzcd}
\mathbf{A} \arrow[rr, "p_\xi = pr_\xi \circ e \circ h"] \arrow[d, "id"'] \arrow[drr, phantom, "\#"] & & {[0,1]} \\
\mathcal{M}(\mathbf{A}) \arrow[urr, dashed, "\exists!\ \overline{p_\xi}"'] & & \forall \xi \in \Xi
\end{tikzcd}$$

Figure 1

that for each $u \in \mathcal{M}(\mathbf{A})$ we have $h_\Xi(u) \in \mathcal{M}(\mathbf{B})$. This yields the desired unique extension $h_m : \mathcal{M}(\mathbf{A}) \longrightarrow \mathcal{M}(\mathbf{B})$, see Fig. 3.

Let $A \in \mathbf{A}$. From Fig. 1 it follows that for all $\xi \in \Xi$ we have $\overline{p_\xi}(\chi_A) = pr_\xi\big(h(\chi_A)\big) = pr_\xi\big(h_\Xi(\chi_A)\big)$ and hence $h(\chi_A) = h_\Xi(\chi_A)$. Thus $h_m(\chi_A) = h(\chi_A)$ and h_m is a sequentially continuous A-homomorphism of $\mathcal{M}(\mathbf{A})$ into $[0,1]^\Xi$ such that $h_m(\chi_A) \in \mathcal{M}(\mathbf{B})$. Let $u = \sum_{i=1}^n c_i \chi_{A_i} \in s(\mathbf{A})$, where all c_i are rational numbers in [0,1]. Then $h_m(u) = \sum_{i=1}^n c_i h(\chi_{A_i}) \in \mathcal{M}(\mathbf{B})$ and hence $h_m(u) \in \mathcal{M}(\mathbf{B})$ for all $u \in \mathcal{M}(\mathbf{A})$. Finally, it follows from Lemma 4.3 that h_m is uniquely determined. $\square$

$$\begin{tikzcd}
{[0,1]^{\Xi}} \arrow[rr, "pr_\xi"] \arrow[drr, phantom, "\#"] & & {[0,1]} \\
\mathcal{M}(\mathbf{A}) \arrow[u, dashed, "\exists!\ h_\Xi"'] \arrow[urr, "\overline{p_\xi}"'] & & \forall \xi \in \Xi
\end{tikzcd}$$

Figure 2

$$\begin{tikzcd}
\mathbf{A} \arrow[r, "h"] \arrow[dr, "id"'] \arrow[dr, phantom, "\#"] & \mathcal{M}(\mathbf{B}) \arrow[r, "e"] \arrow[d, phantom, "\#"] & {[0,1]^{\Xi}} \\
& \mathcal{M}(\mathbf{A}) \arrow[u, dashed, "?\ h_m"] \arrow[ur, "h_\Xi"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{A} \arrow[rr, "h"] \arrow[d, "id"'] \arrow[drr, phantom, "\#"] & & \mathcal{M}(\mathbf{B}) \\
\mathcal{M}(\mathbf{A}) \arrow[urr, dashed, "\exists! h_m"'] & & h_m(.) = h_\Xi(.)
\end{tikzcd}$$

Figure 3.

Corollary 4.9. *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be fields of subsets of* Ω *and* Ξ*, respectively. Let* h *be a sequentially continuous A-homomorphism of* $\mathbf{A}$ *into* $\mathbf{B}$*. Then there exists a unique sequentially continuous A-homomorphism* h_m *of* $\mathcal{M}(\mathbf{A})$ *into* $\mathcal{M}(\mathbf{B})$ *such that* $h(A) = h_m(A)$ *for all* $A \in \mathbf{A}$*.*

Theorem 4.10. $\mathbb{FELIA}$ *is an epireflective subcategory of the category* $\mathbb{ELIA}$*, where* $\mathcal{M}(\mathbf{A})$ *is the epireflection of* $\mathbf{A}$*.*

Proof. Let $\mathbf{O}$ be an object of $\mathbb{ELIA}$, let $\mathcal{M}(\mathbf{B})$ be an object of $\mathbb{FELIA}$, and let $h\colon \mathbf{O} \longrightarrow \mathcal{M}(\mathbf{B})$ be a morphism. Then $\mathbf{O}$ is either of the form $\mathbf{A}$ or $\mathcal{M}(\mathbf{A})$ for some σ-field of sets $\mathbf{A}$. Since (cf. (iii) in Lemma 4.3) the embedding of $\mathbf{O}$ into $\mathcal{M}(\mathbf{A})$ is an epimorphism, we have to prove that h can be uniquely extended over $\mathcal{M}(\mathbf{A})$. In the first case the assertion follows by Theorem 4.8 and in the second case the assertion is trivial. $\square$

Conclusion.

- *Observables are morphisms in* $\mathbb{ELIA}$*. To each classical observable* $h\colon \mathbf{A} \longrightarrow \mathbf{B}$ *there corresponds a unique observable* $h_m\colon \mathcal{M}(\mathbf{A}) \longrightarrow \mathcal{M}(\mathbf{B})$ *which extends* h*.*
- *Probability measures and probability integrals are exactly observables into* $\mathcal{M}(\mathbf{T})$*.*
- $\mathcal{M}(\mathbf{A})$ *is the epireflection of* $\mathbf{A}$ *and* $\mathcal{M}(\mathbf{A})$ *carries the initial A-poset structure with respect to probability integrals.*
- *Probability integrals are exactly additive linearizations.*

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Received October 20, 2017

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