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# AN OPTIMAL CONTROL PROBLEM FOR A VISCOELASTIC PLATE IN A DYNAMIC CONTACT WITH AN OBSTACLE

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ABSTRACT. We deal with an optimal control problem governed by a nonlinear hyperbolic initial-boundary value problem describing the perpendicular vibrations of a simply supported anisotropic viscoelastic plate against a rigid obstacle. A variable thickness of a plate plays the role of a control variable. We verify the existence of an optimal thickness function.

## 1. Introduction

Shape design optimization problems belong to frequently solved problems with many engineering applications. We deal here with an optimal design problem for a viscoelastic anisotropic plate vibrating against a rigid foundation. A variable thickness of a plate plays the role of a control variable. The considered initial-boundary value state problem represents one of the most natural engineering problem not frequently solved because of the hyperbolic character of the presented evolutional variational inequality. We deal here with a plate made of a short memory viscoelastic material. It characterizes constructions made of concrete for example [5]. The dynamic contact for a viscoelastic bridge in a contact with a fixed road has been solved in [2]. The similar problems for beams in a boundary and inner contact are investigated in [3] and [4]. Due to the contact between a bottom of the plate and the rigid obstacle the state problem for the dynamics of the plate has the form of the initial-boundary value problem for the hyperbolic variational inequality. We substitute the variational inequality firstly by the penalized nonlinear equation and solve it by the Galerkin method in the same way as in [1], where the problem for a viscoelastic von Kármán

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plate vibrating against a rigid obstacle has been solved. The state variational inequality in its weak form is formulated without an acceleration term using the integration by parts in the time domain. The optimal control problem is formulated for solutions of the state problem achieved using the penalization method. It enables to find the optimal thickness as a limit of a sequence of thicknesses solving penalized control problems.

## 2. Solving the state problem

### 2.1. Setting of the state problem

We consider an anisotropic short memory viscoelastic plate with the middle surface $\Omega \subset \mathbb{R}^2$ and its Lipschitz continuous boundary $\partial\Omega$. The variable thickness of the plate is expressed by a positive function $x \mapsto e(x)$, $x = (x_1, x_2) \in \bar{\Omega}$, the positive constant $\rho$ is the material density, $A_{ijk\ell}$, $B_{ijk\ell}$ are the symmetric and positively definite tensors expressing the viscoelastic and elastic properties of the material respectively. The plate is simply supported on its boundary. Let $F\colon (0,T] \times \Omega \mapsto \mathbb{R}$ be a perpendicular load per a square unit acting on the plate, $w\colon \partial\Omega \to \mathbb{R}$ the boundary position and $u_0$, $v_0\colon \Omega \mapsto \mathbb{R}$ be the initial displacement and velocity of the middle surface $\Omega$. We set

$$a = \frac{1}{6\rho}, \quad a_{ijk\ell} = \frac{2}{\rho} A_{ijk\ell}, \quad b_{ijk\ell} = \frac{2}{\rho} B_{ijk\ell}, \quad f = \frac{2F}{\rho}$$

the new mechanical and material characteristics. The rigid obstacle is represented by a function $\Phi\colon \Omega \to \mathbb{R}$ and the unknown contact force between the plate and the obstacle by an unknown function $g$.

The vertical displacement $u\colon (0,T] \times \Omega \mapsto \mathbb{R}$ is then a solution of the following hyperbolic initial-boundary value problem

$$\begin{aligned} e(x)u_{tt} - a \operatorname{div}\left(e^3 \operatorname{grad} u_{tt}\right) + \left[e^3(x)(a_{ijk\ell}u_{t,x_ix_j} + b_{ijk\ell}u_{x_ix_j})\right]_{x_kx_\ell} & \\ = f + g \qquad \text{in } (0,T] \times \Omega, & \qquad (1) \end{aligned}$$

$$0 \le g \perp \left(u - \frac{1}{2}e - \Phi\right) \ge 0 \qquad \text{in } (0,T] \times \Omega, \qquad (2)$$

$$u(t,x) = w(t,x),\ M(u)(t,x) = 0 \qquad \text{in } (0,T] \times \partial\Omega \qquad (3)$$

$$u(0,x) = u_0(x),\ u_t(0,x) = v_0(x) \qquad \text{in } \Omega, \qquad (4)$$

with the bending moment

$$M(u) = e^3(x)(a_{ijkl}u_{t,x_ix_j} + b_{ijkl}u_{x_ix_j})n_k n_\ell\,.$$

The Einstein summation convention is employed above and further.

We introduce the Hilbert spaces

$$H \equiv L_2(\Omega), \quad H^k(\Omega) = \big\{y \in H\colon D^\alpha y \in H,\ |\alpha| \le k\big\}, \quad k \in \mathbb{N}$$

with the standard inner products $(\cdot,\cdot)$, $(\cdot,\cdot)_k$ and the norms $|\cdot|_0$, $\|\cdot\|_k$. Further we set

$$H_0^1(\Omega) = \big\{y \in H^1(\Omega)\colon y(\xi) = 0,\ \xi \in \partial\Omega \text{ (in the sense of traces)}\big\}$$

and the Hilbert space

$$V = H^2(\Omega) \cap H_0^1(\Omega)$$

with the inner product and the norm

$$((y,z)) = \int_\Omega y_{x_i x_j}(x) z_{x_i x_j}(x)\,\mathrm{d}x,\ \|y\| = ((y,y))^{1/2}, \qquad y,z \in V.$$

We denote by $V^*$ the dual space of linear bounded functionals over $V$ with duality pairing $\langle F, y\rangle_* = F(y)$, $F \in V^*$, $y \in V$. It is a Banach space with a norm $\|\cdot\|_*$. The spaces $V$, $H$, $V^*$, $H_0^1(\Omega)$ fulfil the compact embedings

$$V \hookrightarrow\hookrightarrow H \hookrightarrow\hookrightarrow V^*,\ V \hookrightarrow\hookrightarrow H_0^1(\Omega).$$

We set $I = (0,T)$, $Q = I \times \Omega$. For a Banach space $X$ we denote by $L_p(I;X)$ the Banach space of all functions $y\colon I \mapsto X$ such that $\|y(\cdot)\|_X \in L_p(0,T)$, $p \ge 1$, by $L_\infty(I;X)$ the space of essentially bounded functions with values in $X$, by $C(\bar I;X)$ the space of continuous functions $y\colon \bar I \mapsto X$. For $k \in \mathbb{N}$ we denote by $C^k(\bar I;X)$ the spaces of $k$-times continuously differentiable functions defined on $\bar I$ with values in $X$. If $X$ is a Hilbert space, we set

$$H^k(I;X) = \left\{v \in C^{k-1}(\bar I;X)\colon \frac{\mathrm{d}^k v}{\mathrm{d}t^k} \in L_2(I;X)\right\}$$

the Hilbert spaces with the inner products

$$(u,v)_{H^k(I,X)} = \int_I \left[(u,v)_X + \sum_{j=1}^k (u^j, v^j)_X\right]\mathrm{d}t, \qquad k \in \mathbb{N}.$$

We denote by $\dot v$, $\ddot v$ the first and the second order time derivative of a function $v\colon I \to X$ and continue with the following assumptions.

The symmetric and positively definite fourth-order tensors $a_{ijk\ell}$, $b_{ijk\ell}$ fulfil

$$a_{ijk\ell} = a_{k\ell ij} = a_{jik\ell}, \quad b_{ijk\ell} = b_{k\ell ij} = b_{jik\ell},$$

$$\alpha_0 > 0,\ \alpha_0\,\varepsilon_{ij}\varepsilon_{ij} \le a_{ijk\ell}\,\varepsilon_{ij}\varepsilon_{k\ell} \le \alpha_1\,\varepsilon_{ij}\varepsilon_{ij} \qquad \text{for all} \quad \{\varepsilon_{ij}\} \in \mathbb{R}^{2\times 2}_{sym},$$

$$\beta_0 > 0,\ \beta_0\,\varepsilon_{ij}\varepsilon_{ij} \le b_{ijk\ell}\,\varepsilon_{ij}\varepsilon_{k\ell} \le \beta_1\,\varepsilon_{ij}\varepsilon_{ij} \qquad \text{for all} \quad \{\varepsilon_{ij}\} \in \mathbb{R}^{2\times 2}_{sym}, \tag{5}$$

where the Einstein summation convention is employed and $\mathbb{R}^{2\times 2}_{sym}$ is the set of all second-order symmetric tensors.

The plate has the variable thickness

$$e\in E_{ad}:=\bigl\{e\in H^2(\Omega):0<e_{\min}\le e(x)\le e_{\max}\ \text{ for all }\ x\in\bar\Omega,\ \|e\|_{H^2(\Omega)}\le\hat e\bigr\}. \tag{6}$$

We assume the right-hand side $f\in L_2(Q)$, the obstacle $\Phi\in C(\bar\Omega)$ and boundary and initial functions fulfilling:

$$\begin{aligned} &w\in H^2\bigl(I;H^1(\Omega)\bigr)\cap H^1\bigl(I;H^2(\Omega)\bigr),\\ &w(t,x)>\frac12 e_{\max}+\Phi(x)\quad\text{for all}\quad x\in\bar Q, \end{aligned}\tag{7}$$

$$u_0(x)=w(0,x),\quad x\in\Omega;\quad v_0\in H^1(\Omega). \tag{8}$$

For $u,y\in H^2(\Omega)$ we define the following bilinear forms

$$A\colon (u,y)\mapsto a_{ijk\ell}u_{ij}y_{k\ell},\quad B\colon (u,y)\mapsto b_{ijk\ell}u_{ij}y_{k\ell} \tag{9}$$

and introduce a shifted cone

$$\begin{aligned}\mathscr{K}(e):=\Bigl\{&y\in L_2(I;V)+w:\\ &\dot y\in L_2\bigl(I;H^1(\Omega)\bigr),\ y(t,x)\ge\frac12 e(x)+\Phi(x)\ \text{ for all }\ (t,x)\in Q\Bigr\}.\end{aligned}\tag{10}$$

Applying the integration by parts both with respect to the time and plane variables we obtain the following weak (variational) formulation of the Problem (1)–(4). We remark that the acceleration term $\ddot u$ does not appear there.

**Problem** $(\mathscr{P}(e))$**.** To find a function $u\in\mathscr{K}(e)$ such that $\dot u\in L_2\bigl(I;H^2(\Omega)\bigr)$, $u(0,x)=u_0(x)$, $x\in\Omega$; and the variational inequality

$$\begin{aligned}&\int_Q\Bigl(e^3\bigl(A(\dot u,y-u)+B(u,y-u)-a\nabla\dot u\cdot\nabla(\dot y-\dot u)\bigr)-e\dot u(\dot y-\dot u)\Bigr)\,\mathrm{d}x\ \mathrm{d}t\\ &\quad+\int_\Omega\bigl(ae^3\nabla\dot u\cdot\nabla(y-u)+e\dot u(y-u)\bigr)(T,.)\ \mathrm{d}x\\ &\ge\int_\Omega\Bigl(ae^3\nabla v_0\cdot\nabla\bigl(y(0,\cdot)-u_0\bigr)+ev_0\bigl(y(0,\cdot)-u_0\bigr)\Bigr)\ \mathrm{d}x\\ &\quad+\int_Q f(y-u)\ \mathrm{d}x\ \mathrm{d}t\end{aligned}\tag{11}$$

is satisfied for any $y\in\mathscr{K}(e)$.

### 2.2. Formulation and solving of the penalized problem

For any $\eta > 0$ we define the penalized problem approximating the original problem $\mathscr{P}(e)$.

**Penalized Problem** ($\mathscr{P}_\eta(e)$)**.** To find $u \in L_2(I;V) + w$ such that

$$\dot{u} \in L_2\big(I; H^2(\Omega)\big), \quad \ddot{u} \in L_2\big(I; H^1(\Omega)\big)$$

and

$$\int\limits_Q \Big(e\ddot{u}z + e^3\big(a\,\nabla\ddot{u}\cdot\nabla z + A(\dot{u},z) + B(u,z)\big)\Big)\,\mathrm{d}x\,\mathrm{d}t =$$

$$\int\limits_Q \left[\eta^{-1}\left(u - \frac{1}{2}e - \varPhi\right)^{-} + f\right] z\,\mathrm{d}x\,\mathrm{d}t \quad \text{for all} \quad z \in L_2(I;V), \tag{12}$$

$$u(0,x) = u_0(x), \quad \dot{u}(0,x) = v_0(x) \qquad x \in \Omega \tag{13}$$

with $\omega^- = \max\{0, -\omega\}$, $\omega\colon \Omega \mapsto \mathbb{R}$.

We verify the existence and the uniqueness of a solution of the penalized problem $\mathscr{P}_\eta(e)$ together with *a priori* estimates inevitable for solving the original state Problem $\mathscr{P}(e)$.

**Theorem 2.1.** *For any $e \in E_{ad}$ there exists a unique solution $u \equiv u_\eta$ of the problem* (12), (13)*.*

Proof. Let $w_i \in V$; $i \in \mathbb{N}$ be a basis of $V$. We construct the Galerkin approximation $u_m$ of a solution in a form

$$u_m(t) = \sum_{i=1}^{m} \alpha_i(t) w_i + w, \quad \alpha_i(t) \in \mathbb{R}, \qquad i = 1,\ldots,m,\ m \in \mathbb{N} \tag{14}$$

given by a solution of the approximated initial value problem for a system of second-order ordinary differential equations

$$\int\limits_\Omega \Big[e(x)\ddot{u}_m w_i + e^3(x)\big(a\nabla\ddot{u}_m\cdot\nabla w_i + A(\dot{u}_m, w_i) + B(u_m, w_i)\big)\Big]\,\mathrm{d}x =$$

$$\int\limits_\Omega \left[\eta^{-1}\left(u_m - \frac{1}{2}e(x) - \varPhi(x)\right)^{-} + f(t)\right] w_i\,\mathrm{d}x, \qquad i = 1,\ldots,m; \tag{15}$$

$$u_m(0) = u_{0m},\ \dot{u}_m(0) = v_{0m};\ u_{0m} \to u_0 \text{ in } H^2(\Omega),\ v_{0m} \to v_0 \text{ in } H^1(\Omega). \tag{16}$$

A solution originally existing only locally can be prolonged to the whole time interval $I$ with the $\eta$-independent *a priori* estimates

$$\|u_m\|_{C(\bar{I},H^2(\Omega))} + \|\dot{u}_m\|_{L_2(I,H^2(\Omega))} + \|\dot{u}_m\|_{C(\bar{I},H^1(\Omega))} \le$$
$$C_1(\alpha_0, \alpha_1, \beta_0, \beta_1, e_{\min}, e_{\max}, \hat{e}, u_0, v_0, w, f). \tag{17}$$

We have directly from the Galerkin system the $\eta$-dependent estimate

$$\|\ddot{u}_m\|_{L_2(I;H^1(\Omega))} \le C_2(\alpha_0,\alpha_1,\beta_0,\beta_1,e_{\min},e_{\max},\hat{e},u_0,v_0,w,f,\eta). \tag{18}$$

We proceed with the convergence of the Galerkin approximation. Applying the estimates (17), (18), the Aubin-Lions compact imbedding theorem [10], Sobolev imbedding theorems and the interpolation theorems in Sobolev spaces [7] we obtain for a subsequence of $\{u_m\}$ (denoted again by $\{u_m\}$) a function

$$u \in C(\bar{I}, V+w) \quad \text{with} \quad \dot{u} \in L_2\big(I, H^2(\Omega)\big), \quad \ddot{u} \in L_2\big(I, H^1(\Omega)\big)$$

and the convergences

$$\begin{aligned}
\ddot{u}_m &\rightharpoonup \ddot{u} && \text{in } L_2\big(I, H^1(\Omega)\big),\\
\dot{u}_m &\rightharpoonup \dot{u} && \text{in } L_2(I;V),\\
\dot{u}_m &\to \dot{u} && \text{in } C\big(\bar{I}; H^1(\Omega)\big),\\
u_m &\to u && \text{in } C\big(\bar{I}; H^2(\Omega)\big).
\end{aligned} \tag{19}$$

Let $\mu \in \mathbb{N}$, $y_\mu = \sum_{i=1}^{\mu} \phi_i(t) w_i$, $\phi_i \in C_0^\infty(0,T)$, $i = 1,\dots,\mu$. The convergence process (19) implies

$$\int_Q \Big[e\,\ddot{u} y_\mu + e^3\big(a\nabla\ddot{u}\cdot\nabla y_\mu + A(\dot{u}, y_\mu) + B(u, y_\mu)\big)\Big]\,\mathrm{d}x\,\mathrm{d}t =$$
$$\int_Q \left[\eta^{-1}\left(u - \frac{1}{2}e(x) - \Phi(x)\right)^{-} + f(t)\right] y_\mu\,\mathrm{d}x\,\mathrm{d}t.$$

Functions $\{y_\mu\}$ form a dense subset of the set $L_2(I;V)$ and hence a function $u$ fulfils the identity (12).

The approximated Galerkin initial conditions (16) imply the initial conditions (13).

The proof of the uniqueness can be performed in a standard way using the Gronwall lemma. □

### 2.3. Solving the state problem

In order to solve the original state problem $\mathscr{P}(e)$ we perform the convergence of $\{u_\eta\}$ as $\eta \to 0+$. We need the $\eta$-independent estimates of $u_\eta$ and its derivatives.

The estimates (17), (18) and the convergences (19) imply the estimates

$$\begin{aligned}
&\|\dot{u}_\eta\|_{C(\bar{I},H^1(\Omega))} + \|\dot{u}_\eta\|_{L_2(I,H^2(\Omega))} + \|u_\eta\|_{C(\bar{I},H^2(\Omega))}\\
&\le C_1(\alpha_0,\alpha_1,\beta_0,\beta_1,e_{\min},e_{\max},\hat{e},u_0,v_0,w,f) \qquad \text{for all } e \in E_{ad},
\end{aligned} \tag{20}$$

$$\begin{aligned}
&\|\ddot{u}_\eta\|_{L_2(I;H^1(\Omega))}\\
&\le C_2(\alpha_0,\alpha_1,\beta_0,\beta_1,e_{\min},e_{\max},\hat{e},u_0,v_0,w,f,\eta) \qquad \text{for all } e \in E_{ad}.
\end{aligned} \tag{21}$$

We express a penalized variational formulation in a form

$$\begin{aligned}
&\int_Q \Big(e^3\big(A(\dot u_\eta, y) + B(u_\eta, y) - a\nabla \dot u_\eta \cdot \nabla \dot y\big) - e\dot u_\eta \dot y\Big)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \int_\Omega \big(ae^3\nabla \dot u_\eta \cdot \nabla y + e\dot u_\eta y\big)(T,\cdot)\,\mathrm{d}x \\
&= \int_\Omega \Big(ae^3\nabla v_0 \cdot \nabla y(0,\cdot) + ev_0 y(0,\cdot)\Big)\,\mathrm{d}x \\
&\quad + \int_Q \left(\eta^{-1}\left(u_\eta - \frac{1}{2}e - \varPhi\right)^{-} + f\right) y\,\mathrm{d}x\,\mathrm{d}t \qquad \text{for all} \quad y \in \mathscr{Y},
\end{aligned} \tag{22}$$

$$\mathscr{Y} \equiv \Big\{y \in L_2(I;V);\ \dot y \in L_2\big(I;H_0^1(\Omega)\big)\Big\}.$$

Using the assumptions (2.1) we obtain after inserting $y = w - u_\eta$ in (22) the crucial $L_1$ estimate of the penalty term

$$\begin{aligned}
&\left\|\eta^{-1}\left(u_\eta - \frac{1}{2}e - \varphi\right)^{-}\right\|_{L_1(Q)} \\
&\le C_3(\alpha_0, \alpha_1, \beta_0, \beta_1, e_{\min}, e_{\max}, \hat e, u_0, v_0, w, f).
\end{aligned} \tag{23}$$

After coming back to a weak penalty formulation with $\ddot u$ we have a dual estimate of accelerations

$$\begin{aligned}
&\| - a\operatorname{div}\big(e^3\nabla \ddot u_\eta\big) + e\ddot u_\eta\|_{L_1(I;V^*)} \\
&\le C_4(\alpha_0, \alpha_1, \beta_0, \beta_1, e_{\min}, e_{\max}, \hat e, u_0, v_0, w, f).
\end{aligned} \tag{24}$$

We use further the following generalization due to [9] of Aubin's-Lion's compactness theorem mentioned and used above:

**Lemma 2.2.** *Let $X_0 \hookrightarrow\hookrightarrow X \hookrightarrow X_1$ be Banach spaces, the first reflexive and separable, $1 < p < \infty$, $1 \le q < \infty$. Then*

$$W \equiv \big\{v \colon v \in L_p(I;X_0),\ \dot v \in L_q(I;X_1)\big\} \hookrightarrow\hookrightarrow L_p(I;X).$$

We apply this compactness result with the spaces $X_0 = L_2(\varOmega)$, $X = H^{-1}(\varOmega)$, $X_1 = V^*$. The sequence $\big\{-a\operatorname{div}(e^3\nabla \dot u_\eta) + e\dot u_\eta\big\}$ is then compact in $L_2\big(I;H^{-1}(\varOmega)\big)$ and we obtain the important strong convergence of the sequence $\{\dot u_k\}$ to $\dot u$, $u \in \mathscr{K}(e)$; for a subsequence

$$\{u_k\} \quad \text{of the sequence} \quad \{u_{\eta_k}\}, \quad \eta_k \to 0+.$$

Applying, simultaneously, the estimate (20) we have the convergences

$$\begin{aligned} \dot{u}_k &\to \dot{u} \quad \text{in } L_2\big(I; H^1(\Omega)\big), \\ \dot{u}_k &\rightharpoonup \dot{u} \quad \text{in } L_2\big(I; H^2(\Omega)\big), \\ u_k &\rightharpoonup^* u \quad \text{in } L_\infty\big(I; H^2(\Omega)\big), \\ u_k &\to u \quad \text{in } C\ \big(\bar{I}; H^1(\Omega)\big). \end{aligned} \tag{25}$$

After inserting the test function $z = y - u_k,\ y \in \mathscr{K}(e)$ in the penalized equation (11) we perform the convergence (25) in the same way as in [1] implying that the limit $u \in \mathscr{K}(e)$ is a solution of the State problem together with the estimates not dependent on the variable thickness $e$ and the existence theorem follows.

**Theorem 2.3.** *For any $e \in E_{ad}$ there exists a solution $u \equiv u(e) \in \mathscr{K}(e)$ of the Problem $\mathscr{P}(e)$ fulfilling the estimate*

$$\begin{aligned} &\|\dot{u}(e)\|_{L_\infty(I,H^1(\Omega))} + \|\dot{u}(e)\|_{L_2(I,H^2(\Omega))} + \|u(e)\|_{C(\bar{I},H^2(\Omega))} \\ &\le C_1(\alpha_0, \alpha_1, \beta_0, \beta_1, e_{\min}, e_{\max}, \hat{e}, u_0, v_0, w, f) \qquad \textit{for all} \quad e \in E_{ad}. \end{aligned} \tag{26}$$

**Remark 2.4.** Applying the estimate (24) and the Alaoglu $w*$-compatness theorem [6] we obtain the weak existence of the acceleration term

$$\big[-a \operatorname{div}(e^3 \nabla u) + eu\big]^{\cdot\cdot} \in \big(L_\infty(I; V)\big)^*$$

fulfilling the estimate

$$\begin{aligned} &\big\| \big[-a \operatorname{div}(e^3 \nabla u) + eu\big]^{\cdot\cdot} \big\|_{(L_\infty(I;V))^*} \\ &\le C_4(\alpha_0, \alpha_1, \beta_0, \beta_1, e_{\min}, e_{\max}, \hat{e}, u_0, v_0, w, f) \qquad \text{for all} \quad e \in E_{ad}. \end{aligned} \tag{27}$$

## 3. Optimal control problem

The state problem $\mathscr{P}(e),\ e \in E_{ad}$ is in general not uniquely solved, moreover there is a lack of compactness in any bounded set of solutions due to not enough regular acceleration term (see Remark 2.4). Hence, we formulate the optimal design problem with states restricted of solutions of $\mathscr{P}(e)$ achieved as limits of solutions of penalized problems $\mathscr{P}_\eta(e),\ \eta > 0$. We obtain a solution of the corresponding optimal control as the limit of the sequence of solutions of penalized control problems.

We consider a continuous cost functional

$$(e, u) \mapsto J(e, u) \in \mathbb{R}^+, \quad (e, u) \in C(\bar{\Omega}) \times C\big([-1ex]I; H^1(\Omega)\big).$$

We recall

$$E_{ad} = \big\{ e \in H^2(\Omega) \colon 0 < e_{\min} \le e(x) \le e_{\max} \ \text{ for all } \ x \in \bar{\Omega},\ \|e\|_{H^2(\Omega)} \le \hat{e} \big\}$$

the set of admissible thicknesses compact in the space $C(\bar{\Omega})$.

We start with a penalized control problem for any $\eta > 0$. Let us set

$$\mathscr{J}_\eta(e) = J\big(e, u_\eta(e)\big), \qquad e \in E_{ad}\,, \tag{28}$$

where $u_\eta(e)$ is a unique solution of the penalized problem $\mathscr{P}_\eta(e)$.

**Optimal Control Problem** ($\mathscr{P}^\eta_{opt}$ **).** To find $e^\eta_* \in E_{ad}$ such that

$$\mathscr{J}_\eta(e^\eta_*) \le \mathscr{J}_\eta(e) \qquad \text{for all} \quad e \in E_{ad}\,. \tag{29}$$

**Theorem 3.1.** *There exists a solution of the Optimal Control Problem $\mathscr{P}^\eta_{opt}$.*

Proof. Let $\{e_n\}$, $n \in \mathbb{N}$; be a minimizing sequence for the cost functional $\mathscr{J}_\eta$, i.e.,

$$\lim_{n\to\infty} \mathscr{J}_\eta\{e_n\} = \inf_{e\in E_{ad}} \mathscr{J}_\eta(e).$$

There exists $e^\eta_* \in E_{ad}$ and a subsequence denoted again by $(e_n)$ such that

$$e_n \rightharpoonup e^\eta_* \text{ in } H^2(\Omega), \quad e_n \to e^\eta_* \text{ in } C(\bar\Omega). \tag{30}$$

The corresponding sequence $\big\{u_\eta(e_n)\big\} \equiv \{u^n_\eta\}$ contains due to the estimates (20), (21) the subsequence again denoted by $\{u^n_\eta\}$ and fulfilling

$$\begin{aligned} u^n_\eta &\rightharpoonup u^\eta_* \quad \text{in } H^1\big(I; H^2(\Omega)\big),\\ \ddot u^n_\eta &\rightharpoonup \ddot u^\eta_* \quad \text{in } L_2\big(I; H^1(\Omega)\big),\\ u^n_\eta &\to u^\eta_* \quad \text{in } C^1\big(\bar I, H^1(\Omega)\big). \end{aligned} \tag{31}$$

The previous convergences and the formulation of the Penalized Problems $\mathscr{P}_\eta(e_n)$ imply that $u^\eta_* = u_\eta(e^\eta_*)$. The continuity property of the cost functional $J$ and the formula (28) then imply

$$\mathscr{J}_\eta(e^\eta_*) = \min_{e\in E_{ad}} \mathscr{J}_\eta(e) \tag{32}$$

and the thickness $e^\eta_*$ is a solution of the Optimal Control Problem $\mathscr{P}^\eta_{opt}$. $\square$

We have verified in the Theorem 2.3 the existence of a solution of the state problem $\mathscr{P}(e)$ using the penalization method. In order to apply this method also in solving the optimal design problem with solutions of $\mathscr{P}(e)$ in the role of the states we consider the sequence of problems $\mathscr{P}_{\eta_k}(e_k)$ with $\eta_k \to 0+$, $e_k \rightharpoonup e$ in $H^2(\Omega)$. It can be verified in the same way as in the previous section with fixed $e$ the existence of a solution $u \equiv u(e)$ of the Problem $\mathscr{P}(e)$ fulfilling the convergence

$$\begin{aligned} \dot u_{\eta_k}(e_k) &\to \dot u \quad \text{in } L_2\big(I; H^1(\Omega)\big),\\ \dot u_{\eta_k}(e_k) &\rightharpoonup \dot u \quad \text{in } L_2\big(I; H^2(\Omega)\big),\\ u_{\eta_k}(e_k) &\rightharpoonup^* u \quad \text{in } L_\infty\big(I; H^2(\Omega)\big),\\ u_{\eta_k}(e_k) &\to u \quad \text{in } C\big(\bar I; H^1(\Omega)\big). \end{aligned} \tag{33}$$

We set

$$U_{ad}(e) = \big\{u \in \mathscr{K}(e) : \ u \text{ is a solution of } \mathscr{P}(e) \\ \text{fulfilling the convergence (33) with } e_k \rightharpoonup e \text{ in } H^2(\Omega)\big\}$$

and formulate

**Optimal Control Problem** ($\mathscr{P}_{opt}$)**.** To find a couple $(e_*, u_*)$ such that

$$J(e_*, u_*) \le J(e, u) \qquad \text{for all} \quad (e, u) \in E_{ad} \times U_{ad}(e).$$

**Theorem 3.2.** *There exists a solution of the Optimal Control Problem $\mathscr{P}_{opt}$.*

P r o o f. Using solutions of the penalized optimal control problems $\mathscr{P}^{\eta}_{opt}$ we obtain applying the approach from [8] the existence of the optimal thickness function $e_*$ .

Let $e \in E_{ad}$ be arbitrary and $u \in U_{ad}(e)$. There exist sequences $\eta_k \to 0+$, $e_k \to e$ and $u_{\eta_k}(e)$ such that the convergence (33) holds. Simultaneously, there exists a subsequence of $\{e^{\eta_k}_*\}$ (denoted again by $\{e^{\eta_k}_*\}$) of solutions of the penalized problems $\mathscr{P}^{\eta_k}_{opt}$ and the element $e_*$ such that

$$e^{\eta_k}_* \rightharpoonup e_* \text{ in } H^2(\Omega), \quad e^{\eta_k}_* \to e_* \text{ in } C(\bar{\Omega}).$$

Applying the minimum result (29), the continuity property of the functional $J$ we obtain for $u_* \in U_{ad}(e_*)$ the relations

$$\begin{aligned} J(e_*, u_*) &= \lim_{k\to\infty} J\big(e^{\eta_k}_*, u_{\eta_k}(e^{\eta_k}_*)\big) \\ &\le \lim_{k\to\infty} J\big(e, u_{\eta_k}(e)\big) \\ &= J(e, u) \qquad \text{for all} \quad (e, u) \in E_{ad} \times U_{ad}(e) \end{aligned}$$

and hence the couple $(e_*, u_*)$ is a solution of the Optimal Control Problem $\mathscr{P}_{opt}$.

□

**Remark 3.3.** There is an open question to verify the existence of an optimal thickness without any additional conditions with respect to the set of solutions of state problems. We mentioned in Remark 2.4 the existence of the acceleration term from the dual space $\big(L_\infty(I; V)\big)^*$. In contrast to the proof of the existence of the state $u \in \mathscr{K}(e)$ in Theorem 2.3 we cannot apply Lemma 2.2 using *a priori* estimates in the space $L_1(I; V^*)$.

REFERENCES

[1] BOCK, I.—JARUŠEK, J.: *Unilateral dynamic contact of viscoelastic von Kármán plates,* Adv. Math. Sci. Appl. **16** (2006), 175–187.

[2] ______ *Dynamic contact problem for a bridge modeled by a viscoelastic von Kármán system,* Z. Angew. Math. Phys. **61** (2010), 865–876.

[3] BOCK, I.—KEČKEMÉTYOVÁ, M.: *An optimal design with respect to a variable thickness of a viscoelastic beam in a dynamic boundary contact,* Tatra Mt. Math. Publ. **48** (2011), 15–24.
[4] ______ *Regularized optimal control problem for a beam vibrating against an elastic foundation,* Tatra Mt. Math. Publ. **63** (2015), 53–71.
[5] CHRISTENSEN, R. M.: *Theory of Viscoelasticity.* Academic Press, New York, 1982.
[6] CONWAY, J. B.: *A Course in Functional Analysis.* Springer-Verlag, New York, 1990.
[7] ECK C.—JARUŠEK, J.—KRBEC, M.: *Unilateral Contact Problems in Mechanics. Variational Methods and Existence Theorems,* in: Pure Appl. Math. (Boca Raton), Vol. 270, Boca Raton, FL, Chapman & Hall/CRC, 2005.
[8] HASLINGER, J.—MÄKINEN, A. E.—STEBEL, J.: *Shape optimization for Stokes problem with threshhold slip boundary conditions,* Discrete Cont. Dyn. Syst. Ser. S. **10** (2017), 1281–1301.
[9] JARUŠEK, J.—MÁLEK, J.—NEČAS, J.—ŠVERÁK, V.: *Variational inequality for a viscous drum vibrating in the presence of an obstacle,* Rend. Mat. Ser. VII. **12** (1992), 943–958.
[10] LIONS, J. L.: *Quelques Méthodes de Résolutions des Problémes aux Limites non Linéaires.* Dunod, Paris, 1969.

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