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ON SOME TRANSFORMATIONS OF FUZZY MEASURES

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ABSTRACT. In this paper, some transformations of fuzzy measures are reviewed. Then, based on them, two new transformations are introduced and their properties are investigated. Also, some examples are provided to illustrate these notions and invariantness under them.

1. Introduction

For probability measure P we have the well-known Bayesian conditional probability, defined for any B such that $P(B) > 0$;

a) local invariantness means $P_{|B}(A) = P(A)$, i.e., A and B are independent (with respect to P);

b) global invariantness means $P_{|B} = P$ and it holds only if $P(B) = 1$;

c) total invariantness means $P_{|B} = P$ for all B such that $P(B) > 0$ which means that P is the Dirac measure.

Inspired by this, here, we try to introduce some transformations for any fuzzy measure. Using these transformations we have new fuzzy measures with new properties. In case we get the same fuzzy measure using the given transformation, the fuzzy measure is called invariant under the transformation.

In this paper, we suggest three transformations. One which is called the conditional transformation has some restriction for the subset B. Drawback is that this approach cannot be applied for any subset B; to avoid this, we introduce the Lukasiewicz and Goguen transformations m_B, m^B and derive m_{B^2}, m^{B^2}. These transformations are studied and the invariantness under them is investigated by some examples.

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2. Preliminaries

We will deal with universe $N = \{1, \dots, n\}$, where $n \geq 2$. A fuzzy measure $m\colon \mathcal{P}(N) \to [0,1]$, in which $\mathcal{P}(N)$ is the power set of N, is an order preserving homomorphism, i.e., m is a monotone set function satisfying $m(\emptyset) = 0$ and $m(N) = 1$. A fuzzy measure is called the Boolean measure if $\mathrm{Range}(m) = \{0,1\}$. Denoting by $\mathcal{M}_n$ and $\mathcal{B}_n$ the set of all fuzzy measures and the set of all Boolean fuzzy measures on N, respectively, it can be shown that $\mathcal{M}_n$ is a convex closure of $\mathcal{B}_n$. The fuzzy measure m_* given by

$$m_*(E) = \begin{cases} 1 & \text{if } E = N, \\ 0, & \text{otherwise} \end{cases}$$

is the smallest and the fuzzy measure m^* given by

$$m^*(E) = \begin{cases} 0 & \text{if } E = \emptyset, \\ 1, & \text{otherwise} \end{cases}$$

is the greatest fuzzy measure. Let i be a fixed element in N. The Dirac measure $\delta_i \in \mathcal{B}_n$ is defined by

$$\delta_i(E) = \begin{cases} 1, & i \in E, \\ 0, & i \notin E. \end{cases}$$

Observe that $m^* = \bigvee_{i=1}^n \delta_i$ and $m_* = \bigwedge_{i=1}^n \delta_i$.

3. Transformations of fuzzy measures

Recently, the following transformation of fuzzy measures on N based on a fixed subset B of N was introduced in [1], compare also [2]. Note that this transformation is based on the Lukasiewicz implication and thus we propose to call it the Lukasiewicz transformation.

Definition 1. Let $m \in \mathcal{M}_n$ and $B \subseteq N$. The Lukasiewicz transformation $L_B\colon \mathcal{M}_n \to \mathcal{M}_n, L_B(m) = m_B$ is given by

$$m_B(E) = 1 - m(B \cup E^c) + m(B \cap E). \tag{1}$$

It is easy to see that for any $B \subseteq N$, m_B given by (1) is a fuzzy measure. Also, $m_N = m$ for any fuzzy measure m. Similarly, $m_\emptyset = m^d$ is the fuzzy measure dual to m, $m^d(E) = 1 - m(E^c)$.

Remark 1. Observe that the Lukasiewicz transformation L_B can be restricted to Boolean measures, i.e., if $m \in \mathcal{B}_n$ then also $L_B(m) = m_B \in \mathcal{B}_n$. Moreover, the transformation L_B commutes with convex combinations, i.e., $L_B(\sum_{i=1}^k c_i.m_i) = \sum_{i=1}^k c_i.L_B(m_i)$, where c_i are non-negative constants such that $\sum_{i=1}^k c_i = 1$. Therefore, it is enough to discuss the Lukasiewicz transformation of Boolean measures only.

Remark 2. It is seen that

$$\begin{aligned}(m^d)_B(E) &= 1 - m^d(B \cup E^c) + m^d(B \cap E)\\ &= m(B^c \cap E) + 1 - m(B^c \cup E^c)\\ &= m_{B^c}(E)\end{aligned}$$

for all $E \subseteq N$. So, $(m^d)_B = m_{B^c}$.

The next result brings the introduction of a new transformation of fuzzy measures, namely double Lukasiewicz transformation, and shows some of its properties.

Theorem 1. *Let $m \in \mathcal{M}_n$ and for $k \in \{2, 3, \dots\}$ define $m_{B^k} = \underbrace{\big(((m_B)_B)\dots\big)_B}_{k-times}$. Then,*

$$m_{B^k} = \begin{cases} m_B, & k \text{ is odd},\\ m_{B^2}, & k \text{ is even}.\end{cases}$$

P r o o f. Observe that from (1) we get

$$\begin{aligned}m_{B^2}(E) &= (m_B)_B(E)\\ &= 1 - m_B(B \cup E^c) + m_B(B \cap E)\\ &= 1 - \Big(1 - m\big(B \cup (B^c \cap E)\big) + m\big(B \cap (B \cup E^c)\big)\Big)\\ &\quad + 1 - m\big(B \cup (B \cap E)^c\big) + m\big(B \cap (B \cap E)\big)\\ &= m(B \cap E) - m(B) + m(B \cup E).\end{aligned} \tag{2}$$

Further,

$$\begin{aligned}m_{B^3} &= \big((m_B)_B\big)_B(E)\\ &= (m_{B^2})_B(E)\\ &= 1 - m_{B^2}(B \cup E^c) + m_{B^2}(B \cap E)\\ &= 1 - m(B \cup E^c) + m(B \cap E)\\ &= m_B.\end{aligned}$$

By induction, the statement follows. □

In formula (2), we have introduced a new transformation of fuzzy measures $L_B^2 \colon \mathcal{M}_n \to \mathcal{M}_n$, $L_B^2(m) = m_{B^2}$ which we propose to call a double Lukasiewicz transformation. Again as in the case of the Lukasiewicz transformation, also double Lukasiewicz transformation can be seen as mappings from $\mathcal{B}_n$ to $\mathcal{B}_n$, and they commute with convex combinations.

In [1], it was shown that $m_B = m$ for all subsets $B \subseteq N$ if and only if m is additive, i.e., a probability measure on N. We have a similar result for double Lukasiewicz transformation.

Corollary 1. *Let $m \in \mathcal{M}_n$. Then $m_{B^2} = m$ for all $B \subseteq N$ if and only if m is additive.*

Proof. The sufficiency follows from the modularity of additive measures. On the other hand, to see the necessity, suppose that $m_{B^2} = m$ for all $B \subseteq N$. Then, due to (2), we have $m(E) = m(B \cap E) - m(B) + m(B \cup E)$ valid for any subsets E, B of N, i.e., m is modular. Trivially, $m(\emptyset) = 0$, and hence m is additive. $\square$

Also, the next result is a corollary of Theorem 1.

Corollary 2. *Let $m \in \mathcal{M}_n$ and $B \subseteq N$. Then both m_B and m_{B^2} are invariant under double Lukasiewicz transformation L_B^2.*

Here, we list some basic properties of double Lukasiewicz transformations:

- for any $m \in \mathcal{M}_n$, $m_{N^2} = m_{\emptyset^2} = m$;
- for any $E \subseteq B \subseteq N$ and $m \in \mathcal{M}_n$, $m_{B^2}(E) = m(E)$;
- for any $B \subseteq E \subseteq N$ and $m \in \mathcal{M}_n$, $m_{B^2}(E) = m(E)$.

In the next parts, we will discuss some further properties of L_B and L_B^2 transformations.

Theorem 2. *Let $m \in \mathcal{M}_n$. The following statements hold for any B, $\emptyset \neq B \subset N$,*

i. $(m_*)_B(E) = \begin{cases} 0, & E \subseteq B, \\ 1, & \textit{otherwise}, \end{cases}$

ii. $(m^*)_B(E) = \begin{cases} 1, & B \cap E \neq \emptyset, \\ 0, & \textit{otherwise}, \end{cases}$

iii. $(\delta_i)_B = \delta_i$.

Proof.

i. We have

$$(m_*)_B(E) = 1 - m_*(B \cup E^c) + m_*(B \cap E) = \begin{cases} 1 - 1 + 0 = 0, & E \subseteq B, \\ 1 - 0 - 0 = 1, & \text{otherwise}. \end{cases}$$

ii. We have

$$(m^*)_B(E) = 1 - m^*(B \cup E^c) + m^*(B \cap E) = \begin{cases} 1 - 1 + 1 = 1, & B \cap E \neq \emptyset, \\ 1 - 1 = 0, & \text{otherwise.} \end{cases}$$

iii. First, let $i \in B$. Then,

$$(\delta_i)_B(E) = 1 - \delta_i(B \cup E^c) + \delta_i(B \cap E) = \begin{cases} 1 - 1 + 1 = \delta_i(E), & i \in E, \\ 1 - 1 + 0 = \delta_i(E), & i \notin E. \end{cases}$$

Now, let $i \notin B$. So,

$$(\delta_i)_B(E) = 1 - \delta_i(B \cup E^c) + \delta_i(B \cap E) = \begin{cases} 1 - 0 + 0 = \delta_i(E), & i \in E, \\ 1 - 1 + 0 = \delta_i(E), & i \notin E. \end{cases}$$

□

Therefore, for any $B \subset N$, we have $(m_*)_B \neq m_*$ and $(m^*)_B \neq m^*$ while, the Dirac measure is invariant under Lukasiewicz transformation.

The next transformation is based on the Goguen implication, see [1], and thus we will call it the Goguen transformation.

Definition 2. Let $B \subseteq N$. The Goguen transformation $G_B \colon \mathcal{M}_n \to \mathcal{M}_n$, $G_B(m) = m^B$, is given by

$$m^B(E) = \frac{m(B \cap E)}{m(B \cup E^c)}, \tag{3}$$

with convention $\frac{0}{0} = 1$. It is easy to see that for any $B \subseteq N$, m^B given by (3) is a fuzzy measure. If $B = N$ then, $m^B = m$ for any fuzzy measure m. Also, if $m \in \mathcal{B}_n$ then, $m^\emptyset = m^d$.

Remark 3. Observe that the Goguen transformation G_B can be restricted to Boolean measures, i.e., if $m \in \mathcal{B}_n$ then also $G_B(m) = m_B \in \mathcal{B}_n$. Moreover, transformation G_B commutes with weighted geometric means, i.e., $G_B(\prod_{i=1}^k m_i^{w_i}) = \prod_{i=1}^k G_B^{w_i}(m_i)$, where w_i are non-negative constants.

The next result brings the introduction of a new transformation of fuzzy measures, namely double Goguen transformation, and shows some of its properties.

Theorem 3. *Let $m \in \mathcal{M}_n$ and for $k \in \{2, 3, \dots\}$ define $m^{B^k} = \underbrace{(((m^B)^B)\dots)^B}_{k-times}$. Then,*

$$m^{B^k} = \begin{cases} m^B, & k \text{ is odd,} \\ m^{B^2}, & k \text{ is even.} \end{cases}$$

Proof. Observe that from (3) we get

$$m^{B^2} = \left(m^B\right)^B(E) = \frac{m^B(B \cap E)}{m^B(B \cup E^c)} = \frac{\dfrac{m(B \cap (B \cap E))}{m(B \cup (B \cap E)^c)}}{\dfrac{m(B \cap (B \cup E^c))}{m(B \cup (B \cup E^c)^c)}} = \frac{m(B \cap E) \cdot m(B \cup E)}{m(B)}. \tag{4}$$

Moreover,

$$m^{B^3} = \left(\left(m^B\right)^B\right)^B(E) = (m^{B^2})^B(E) = \frac{m^{B^2}(B \cap E)}{m^{B^2}(B \cup E^c)} = \frac{m(B \cap E) \cdot m(B)}{m(B) \cdot m(B \cup E^c)} = m^B.$$

By induction, the statement follows. □

In formula (4), we have introduced a new transformation of fuzzy measures $G_B^2 \colon \mathcal{M}_n \to \mathcal{M}_n$, $G_B^2(m) = m^{B^2}$, with convention 0 if $m(B \cup E) = 0$, and 1 if $m(B \cup E) > 0$ but $m(B) = 0$. Again as in the case of Goguen transformations, also double Goguen transformations can be seen as mappings from $\mathcal{B}_n$ to $\mathcal{B}_n$, and they commute with weighted geometric means.

Theorem 4. *Let $m \in \mathcal{M}_n$. The following statements hold for any B, $\emptyset \neq B \subset N$*

i. $(m_*)^B(E) = \begin{cases} 0, & E \subseteq B, \\ 1, & \textit{otherwise}, \end{cases}$

ii. $(m^*)^B(E) = \begin{cases} 1, & B \cap E \neq \emptyset, \\ 0, & \textit{otherwise}, \end{cases}$

iii. $\delta_i^B = \delta_i$.

Proof.

i. By formula (3), we have

$$(m_*)^B(E) = \frac{m_*(B \cap E)}{m_*(B \cup E^c)} = \begin{cases} \frac{0}{1} = 1, & E \subseteq B, \\ \frac{0}{0} = 1, & \text{otherwise}. \end{cases}$$

ii. Using definition of Goguen transformation, we get

$$(m^*)^B(E) = \frac{m^*(B \cap E)}{m^*(B \cup E^c)} = \begin{cases} \frac{1}{1} = 1, & B \cap E \neq \emptyset, \\ \frac{0}{1} = 0, & \text{otherwise}. \end{cases}$$

iii. First, let $i \in B$. We have

$$\delta_i^B(E) = \frac{\delta_i(B \cap E)}{\delta_i(B \cup E^c)} = \begin{cases} 1 = \delta_i(E), & i \in E, \\ \frac{0}{1} = 0 = \delta_i(E), & i \notin E. \end{cases}$$

Now, let $i \notin B$. Then,

$$\delta_i^B(E) = \frac{\delta_i(B \cap E)}{\delta_i(B \cup E^c)} = \begin{cases} \frac{0}{0} = 1 = \delta_i(E), & i \in E, \\ \frac{0}{1} = 0 = \delta_i(E), & i \notin E. \end{cases}$$

□

Therefore, for any $B \subset N$, $(m_*)^B \neq m_*$ and $(m^*)^B \neq m^*$ while, the Dirac measure is invariant under the Goguen transformation.

The combination of Lukasiewicz and Goguen transformations can be considered. Let $B, C \subseteq N$. Then for any $m \in \mathcal{M}_n$

$$\begin{aligned} (m_B)^C(E) &= \frac{m_B(C \cap E)}{m_B(C \cup E^c)} \\ &= \frac{1 - m(B \cup ((C \cap E)^c) + m(B \cap (C \cap E))}{1 - m(B \cup ((C \cup E^c)^c) + m(B \cap (C \cup E^c))}. \end{aligned}$$

Now, one can investigate the followings:

– $B = C$; we get

$$\begin{aligned} (m_B)^B(E) &= \frac{1 - m(N) + m(B \cap E)}{1 - m(B \cup E) + m(B)} \\ &= \frac{m(B \cap E)}{1 - m(B \cup E) + m(B)}. \end{aligned}$$

– $B \subseteq C$; then

$$\begin{aligned} (m_B)^C(E) &= \frac{1 - m(B \cup (C \cap E)^c) + m(B \cap E)}{1 - m(B \cup (C \cup E^c)^c) + m(B)} \\ &= \frac{m^d(B^c \cap (C \cap E)) + m(B \cap E)}{m^d(B^c \cap (C \cup E^c)) + m(B)}. \end{aligned}$$

– $C \subseteq B$; we have

$$\begin{aligned} (m_B)^C(E) &= \frac{1 - m(N) + m(C \cap E)}{1 - m(B \cup (C^c \cap E)) + m(B \cap (C \cup E^c))} \\ &= \frac{m(C \cap E)}{1 - m(B \cup (C^c \cap E)) + m(B \cap (C \cup E^c))} \\ &= \frac{m(C \cap E)}{m^d(B^c \cap (C \cup E^c)) + m(B \cap (C \cup E^c))}. \end{aligned}$$

– $B \cap C = \emptyset$; then

$$\begin{aligned}(m_B)^C(E) &= \frac{1 - m((C \cap E)^c)}{1 - m(B \cup (C^c \cap E)) + m(B \cap E^c)} \\ &= \frac{m^d(C \cap E)}{m(B \cup (C^c \cap E)) + m^d(B^c \cup E)}.\end{aligned}$$

Observe that

$$(m_\emptyset)^\emptyset(E) = \begin{cases} 1, & m(E) = 1, \\ 0, & m(E) \neq 1, \end{cases}$$

for all $E \subseteq N$. So, $(m_\emptyset)^\emptyset \in \mathcal{B}_n$. We have these two special cases

– $(m_\emptyset)^\emptyset(E) = m_*(E)$ if and only if $m(E) \neq 1$ for all $E \subset N$.

– $(m_\emptyset)^\emptyset = m^*$ if and only if $m = m^*$.

Also, we may apply the transformations with opposite order, i.e., first Goguen and then Lukasiewicz transformation. Regarding this idea, we get

$$\begin{aligned}(m^B)_C(E) &= 1 - m^B(C \cup E^c) + m^B(C \cap E) \\ &= 1 - \frac{m(B \cap (C \cup E^c))}{m(B \cup (C \cup E^c)^c)} + \frac{m(B \cap (C \cap E))}{m(B \cup (C \cap E)^c)}.\end{aligned}$$

Now, one can investigate the followings:

– $B = C$; we get

$$\begin{aligned}(m^B)_B(E) &= 1 - \frac{m(B)}{m(B \cup E)} + \frac{m(B \cap E)}{m(N)} \\ &= 1 - \frac{m(B)}{m(B \cup E)} + m(B \cap E).\end{aligned}$$

– $B \subseteq C$; we have

$$(m^B)_C(E) = 1 - \frac{m(B)}{m(B \cup (C \cup E^c)^c)} + \frac{m(B \cap E)}{m(B \cup (C \cap E)^c)}.$$

– $C \subseteq B$; then

$$(m^B)_C(E) = 1 - \frac{m(B \cap (C \cup E^c))}{m(B \cup (C \cup E^c)^c)} + \frac{m(C \cap E)}{m(B \cup E)}.$$

– $B \cap C = \emptyset$; so

$$(m^B)_C(E) = 1 - \frac{m(B \cap E^c)}{m(B \cup (C \cup E^c)^c)} + \frac{0}{m(B \cup (C \cap E)^c)}.$$

Observe that

$$(m^\emptyset)_\emptyset(E) = \begin{cases} 1, & m(E) \neq 0, \\ 0, & m(E) = 0 \end{cases} \quad \text{for all } E \subseteq N. \quad \text{So, } (m^\emptyset)_\emptyset \in \mathcal{B}_n.$$

We have these two special cases

- $(m^{\emptyset})_{\emptyset}(E) = m^*(E)$ if and only if $m(E) \neq 0$ for all $\emptyset \neq E \subseteq N$.
- $(m^{\emptyset})_{\emptyset} = m_*$ if and only if $m = m_*$.

As it is seen the order of transformations matters and in general case $(m^B)_C \neq (m_B)^C$. Applying two Lukasiewicz transformations is another way to obtain new fuzzy measure. This method consists of double Lukasiewicz transformation as special case. In fact we have

$$\begin{aligned}(m_B)_C(E) &= 1 - m_B(C \cup E^c) + m_B(C \cap E)\\ &= 1 - \Big(1 - m\big(B \cup (C \cup E^c)^c\big) + m\big(B \cap (C \cup E^c)\big)\Big)\\ &\quad + 1 - m(B \cup (C \cap E)^c) + m(B \cap (C \cap E))\\ &= 1 + m\big(B \cup (C^c \cap E)\big) - m\big(B \cap (C \cup E^c)\big) - m\big(B \cup (C \cap E)^c\big)\\ &\quad + m\big(B \cap (C \cap E)\big).\end{aligned}$$

Now, one can investigate the followings:

- $B = C$; we get double Lukasiewicz transformation.
- $B \subseteq C$; we have

$$(m_B)_C(E) = 1 + m\big(B \cup (C^c \cap E)\big) - m(B) - m\big(B \cup (C \cap E)^c\big) + m(B \cap E).$$

- $C \subseteq B$; then

$$\begin{aligned}(m_B)_C(E) &= 1 + m(B \cup E) - m\big(C \cup (B \cap E^c)\big) - m(N) + m(C \cap E)\\ &= m(B \cup E) - m\big(C \cup (B \cap E^c)\big) + m(C \cap E).\end{aligned}$$

- $B \cap C = \emptyset$; we have

$$\begin{aligned}(m_B)_C(E) &= 1 + m\big(C^c \cap (B \cup E)\big) - m(B \cap E^c) - m\big((C \cap E)^c\big) + m(\emptyset)\\ &= 1 + m\big(C^c \cap (B \cup E)\big) - m(B \cap E^c) - m\big((C \cap E)^c\big).\end{aligned}$$

The other combination which we obtain by the Goguen transformation, can be defined as

$$(m^B)^C(E) = \frac{m^B(C \cap E)}{m^B(C \cup E^c)} = \frac{\dfrac{m(B \cap (C \cap E))}{m(B \cup (C \cap E)^c)}}{\dfrac{m(B \cap (C \cup E^c))}{m(B \cup (C \cup E^c)^c)}}.$$

Now, one can investigate the followings:

- $B = C$; we get double Goguen transformation.
- $B \subseteq C$; we have

$$(m^B)^C(E) = \frac{m(B \cap E) \cdot (B \cup (C^c \cap E))}{m(B \cup (C \cap E)^c).m(B)}.$$

– $C \subseteq B$; then

$$(m^B)^C(E) = \frac{\dfrac{m(C \cap E)}{m(N)}}{\dfrac{m(C \cup (B \cap E^c))}{m(B \cup E)}} = \frac{m(C \cap E) \cdot m(B \cup E)}{m(C \cup (B \cap E^c))}.$$

– $B \cap C = \emptyset$; so

$$(m^B)^C(E) = \frac{\dfrac{m(\emptyset)}{m((C \cap E)^c)}}{\dfrac{m(B \cap E^c)}{m(B \cup (C^c \cap E))}} = \frac{m(\emptyset) \cdot m(B \cup (C^c \cap E))}{m((C \cap E)^c) \cdot m(B \cap E^c)}.$$

Therefore, as it is shown, using the mentioned transformations successively gives new fuzzy measures with different properties.

Definition 3. For any fuzzy measure $m \in \mathcal{M}_n$ and $B \subseteq N$ with $m(B) > 0$ one can define

$$m_{|B}(E) = \frac{m(B \cap E)}{m(B)}, \tag{5}$$

with convention $\frac{0}{0} = 1$. This transformation is called the conditional transformation.

It is easy to see that for any $B \subseteq N$, $m_{|B}$ given by (5) is a fuzzy measure. Also, $m_{|N} = m$ for any $m \in \mathcal{M}_n$.

Theorem 5. *Let $m \in \mathcal{M}_n$. Then,*

i. *for any B, $\emptyset \neq B \subset N$, we have*

$$(m^*)_{|B}(E) = \begin{cases} 0, & B \cap E = \emptyset, \\ 1, & B \cap E \neq \emptyset. \end{cases}$$

ii. *for any $B \subseteq N$ such that $i \in B$, we have $(\delta_i)_{|B} = \delta_i$.*

Proof.

i. We have

$$(m^*)_{|B}(E) = \frac{m^*(B \cap E)}{m^*(B)} = \begin{cases} \frac{0}{1} = 0, & B \cap E = \emptyset, \\ \frac{1}{1} = 1, & B \cap E \neq \emptyset. \end{cases}$$

ii. We have

$$(\delta_i)_{|B}(E) = \frac{\delta_i(B \cap E)}{\delta_i(B)} = \begin{cases} \frac{1}{1} = 1 = \delta_i(E), & i \in E, \\ \frac{0}{1} = 0 = \delta_i(E), & i \notin E. \end{cases}$$

□

Therefore, for any $B \subset N$, $(m^*)_{|B} \neq m^*$ while the Dirac measure is invariant under transformation given by (5) if $i \in B$. Note that if $m = m_*$ then, considering condition $m(B) > 0$, we just have the case $B = N$ which by what has been mentioned before, $(m_*)_{|N} = m_*$.

Applying the conditional transformation successively gives the new fuzzy measure. To do so, for any B and C, $\emptyset \neq B, C \subseteq N$ consider

$$\begin{aligned}(m_{|B})_{|C}(E) &= \frac{m_{|B}(C \cap E)}{m_{|B}(C)} \\ &= \frac{\dfrac{m(B \cap C \cap E)}{m(B)}}{\dfrac{m(B \cap C)}{m(B)}} \\ &= \frac{m(B \cap C \cap E)}{m(B \cap C)}.\end{aligned}$$

As it can be seen, this expression is commutative, i.e., $(m_{|B})_{|C} = (m_{|C})_{|B}$. Also, observe that if $B \subseteq C$, then

$$\begin{aligned}(m_{|B})_{|C}(E) &= \frac{m(B \cap C \cap E)}{m(B \cap C)} \\ &= \frac{m(B \cap E)}{m(B)} \\ &= m_{|B}(E)\end{aligned}$$

and specially

$$(m_{|B})_{|B}(E) = m_{|B}(E).$$

Furthermore,

$$m_{|B^k} = m_{|B} \quad \text{for all } k \in \{2, 3, \dots\}.$$

4. Conclusion

To obtain new fuzzy measures, some transformations have been introduced in literature. Here, using Lukasiewicz and Goguen transformations, two new ones namely double Lukasiewicz and double Goguen transformations have been introduced and discussed. Then, the invariantness of some Boolean fuzzy measures under these new transformations has been studied. Also, as it has been shown, applying the mentioned transformations successively gives new fuzzy measures with different properties. Furthermore, the order of transformations matters.

As the application of fuzzy measures in different fields such as decision making and specially aggregating is inevitable, generating new fuzzy measures seems necessary. Study on more general transformations can be considered for the future research.

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