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THREE WAYS OF DEFINING OWA OPERATOR ON THE SET OF ALL NORMAL CONVEX FUZZY SETS

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ABSTRACT. We deal with an extension of ordered weighted averaging (OWA, for short) operators to the set of all normal convex fuzzy sets in $[0,1]$. The main obstacle to achieve this goal is the non-existence of a linear order for fuzzy sets. Three ways of dealing with the lack of a linear order on some set and defining OWA operators on the set appeared in the recent literature. We adapt the three approaches for the set of all normal convex fuzzy sets in $[0,1]$ and study their properties. It is shown that each of the three approaches leads to operator with desired algebraic properties, and two of them are also linear.

1. Introduction

Since Yager [11] introduced ordered weighted averaging (OWA, for short) operator, it became one of the most widely used aggregation methods for real numbers. OWA operator is given by the following mapping $\mathrm{OWA}_{\mathbf{w}} : [0,1]^n \to [0,1]$ defined by

$$\mathrm{OWA}_{\mathbf{w}}(a_1,\dots,a_n) = \sum_{i=1}^{n} w_i a_{\sigma(i)}, \tag{1}$$

where $\mathbf{w} = (w_1,\dots,w_n) \in [0,1]^n$, $w_1 + \cdots + w_n = 1$, is called the weighting vector, and σ is a permutation of $\{1,\dots,n\}$ such that $a_{\sigma(1)} \geq \dots \geq a_{\sigma(n)}$. The crucial point of this definition is the existence of a linear order on $[0,1]$, since given inputs have to be reordered before the summation. The wide range of applications of OWA operators led to growing interest of scholars to also concern with OWA operators for some elements different from real numbers. In this paper, we deal with OWA operators on $F([0,1])$ (the set of all fuzzy sets in $[0,1]$).

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However, the extension of definition from real numbers to fuzzy sets is not straightforward, because, as it is well-known, fuzzy sets are not linearly ordered. So, if one desire to define "something like OWA operator" on the set of fuzzy sets, it is necessary to avoid the lack of linear order in some way. There are three various approaches to this task in the recent literature:

- OWA operators on complete lattices, see [5], [6], [8];
- Type-1 OWA operators, see [13], [14], [15]; and
- OWA operator for discrete gradual real intervals, see [7], [9].

All the three approaches were applied and each of them gives appropriate results. But, of course, each effort to avoid the lack of a linear order brings some drawback, too. We discuss this issue and study and compare algebraic properties of the three OWA operators. We also show that two of the three OWA operators are homogeneous and shift-invariant, that is, they are linear.

The structure of the present work is as follows. In Section 2, we recall definition of OWA operators on complete lattices and present its special case, namely OWA operator on lattice of normal convex fuzzy sets in $[0,1]$. In Section 3, we deal with the so-called type-1 OWA operator on the set of all fuzzy sets in X. We focus on OWA operator for discrete gradual real intervals in Section 4. Properties of the three operators are discussed in Section 5. The conclusions are discussed in Section 6.

2. OWA operators on fuzzy sets as special case of OWA operators on complete lattices

The authors of [5] introduced OWA operators on complete lattices (L-OWA, for short). In this section, we present OWA operator on a special case of complete lattice, namely that of all normal convex fuzzy sets in $[0,1]$. See [6], [8] for more details.

We will use the following notation and terminology. Let X be a non-empty set. A *fuzzy set* in X is a mapping from X to $[0,1]$, the class of all fuzzy sets in X will be denoted by $F(X)$. A fuzzy set A in X is *normal* if there exists $x \in X$ such that $A(x) = 1$, and A is *convex* if it satisfies

$$A\big(\lambda x_1 + (1-\lambda)x_2\big) \geq \min\big(A(x_1), A(x_2)\big)$$

for all $\lambda \in [0,1]$ for all $x_1, x_2 \in X$. We denote by $F_{NC}(X)$ the class of all normal convex fuzzy sets in X. We will use operations $\sqcup$, $\sqcap$, relations $\sqsubseteq$, $\preceq$ and special elements $\widetilde{a}$, for $a \in X$, in $F(X)$:

$$(A \sqcup B)(z) = \sup_{x \vee y = z} \big(A(x) \wedge B(y)\big),$$
$$(A \sqcap B)(z) = \sup_{x \wedge y = z} \big(A(x) \wedge B(y)\big), \quad \widetilde{a}(x) = \begin{cases} 1 & \text{if } x = a, \\ 0, & \text{otherwise,} \end{cases} \tag{2}$$

$$A \sqsubseteq B \text{ iff } A \sqcap B = A, \qquad A \preceq B \text{ iff } A \sqcup B = B.$$

The relations $\sqsubseteq$, $\preceq$ are partial (non-linear) orders and they coincide on $F_{NC}(X)$. Note that the algebra $\big(F_{NC}([0,1]), \sqcup, \sqcap, \sqsubseteq, \widetilde{0}, \widetilde{1}\big)$ consisting of all the convex normal functions is a De Morgan algebra [10], i.e., it is, among other things, bounded distributive lattice.

2.1. Definition (the first approach)

The first approach is based on replacement of possibly incomparable input fuzzy sets by an 'appropriate' chain. Let $A_1, \dots, A_n \in F_{NC}([0,1])$, we can get a chain $B_n \sqsubseteq B_{n-1} \sqsubseteq \cdots \sqsubseteq B_1$ with $B_1, \dots, B_n \in F_{NC}([0,1])$ in the following way (see [5, Lemma 3.1]):

$$\begin{aligned} B_1 &= A_1 \sqcup \cdots \sqcup A_n, \\ B_2 &= \big((A_1 \sqcap A_2) \sqcup \cdots \sqcup (A_1 \sqcap A_n)\big) \sqcup \big((A_2 \sqcap A_3) \sqcup \cdots \sqcup (A_2 \sqcap A_n)\big) \\ &\quad \sqcup \cdots \sqcup \big((A_{n-1} \sqcap A_n)\big), \\ &\ \vdots \\ B_n &= A_1 \sqcap \cdots \sqcap A_n. \end{aligned} \tag{3}$$

Definition 2.1. Let $W_1, \dots, W_n \in F([0,1])$. A vector $\mathbf{W} = (W_1, \dots, W_n)$ is said to be a *distributive weighting vector* in $F([0,1])$ if (i) $W_1 \sqcup \cdots \sqcup W_n = \widetilde{1}$; and (ii) $A \sqcap (W_1 \sqcup \cdots \sqcup W_n) = (A \sqcap W_1) \sqcup \cdots \sqcup (A \sqcap W_n)$ for all $A \in F([0,1])$.

Proposition 2.2 ([8, Cor. 3.7]). *A vector $(W_1, \dots, W_n) \in F_{NC}([0,1])^n$ is a distributive weighting vector in $F_{NC}([0,1])$ if and only if there exists $i \in \{1, \dots, n\}$ such that $W_i = \widetilde{1}$.*

Definition 2.3. Let $\mathbf{W} = (W_1, \dots, W_n) \in F_{NC}([0,1])^n$ be a distributive weighting vector in $\big(F_{NC}([0,1]), \sqcap, \sqcup, \sqsubseteq, \widetilde{0}, \widetilde{1}\big)$. The mapping $F_{\mathbf{W}} : F_{NC}([0,1])^n \to F_{NC}([0,1])$ given, for all $(A_1, \dots, A_n) \in F_{NC}([0,1])^n$, by $F_{\mathbf{W}}(A_1, \dots, A_n) = (W_1 \sqcap B_1) \sqcup \cdots \sqcup (W_n \sqcap B_n)$, where $(B_1, \dots, B_n)$ is a linearly ordered vector constructed from $(A_1, \dots, A_n)$ according to the equation (3), is called *an n-ary* OWA *operator* on $F_{NC}([0,1])$.

Remark. This first approach to avoiding the lack of a linear order in definition of OWA operator is based on replacement of input. In the case of standard OWA operator, the chain $a_{\sigma(1)}, \ldots, a_{\sigma(n)}$ consists of the original input elements $a_1, \ldots, a_n$, which are just reordered. In contrast to this, in case of OWA operator $F_{\mathbf{W}}$ given by Definition 2.3, the chain $B_1, \ldots, B_n$ consists of elements (possibly) different from the original input elements $A_1, \ldots, A_n$. So, strictly said, we do not aggregate exactly the given input, just some reasonably chosen substitute input. That is the price we pay for avoiding the non-existence of linear order on the set of fuzzy sets by the first approach.

Example 1. Let the weighting vector be $\mathbf{W} = (W_1, \widetilde{1}) \in F_{NC}([0,1])^2$ and inputs be $A_1, A_2 \in F_{NC}([0,1])$, where A_1, A_2, W_1 are normal convex fuzzy sets in $[0,1]$ given by Figure 1. Then the result of binary OWA operator on $F_{NC}([0,1])$, namely, $F_{\mathbf{W}}(A_1, A_2) = (W_1 \sqcap B_1) \sqcup (W_2 \sqcap B_n)$, is depicted in Figure 1. Recall that $B_1 = A_1 \sqcup A_2$ and $B_2 = A_1 \sqcap A_2$ are computed according to the equation (2), see Figure 2. See [8, Examples 3.9 and 3.11] for more detailed explanation.

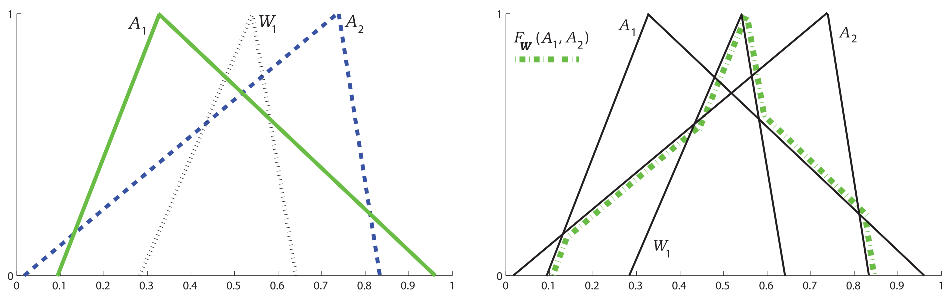

Figure 1. See Example 1.

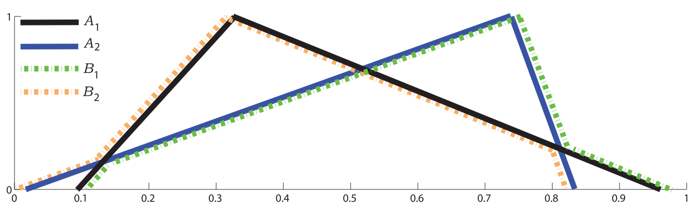

Figure 2. See Example 1.

2.2. Algebraic properties

Some algebraic properties of OWA operator $F_{\mathbf{W}}$ were studied in [5], [6] and [8]. The following theorem deals with other properties that were not under investigation yet.

Theorem 2.4. *Let* $\mathbf{W} = (W_1, \dots, W_n)$ *be a distributive weighting vector in* $\big(F_{NC}([0,1]), \sqcap, \sqcup, \sqsubseteq, \widetilde{0}, \widetilde{1}\big)$ *and* $F_{\mathbf{W}}$ *the corresponding* OWA *operator given by Def. 2.3. Then:*

(i) $F_{\mathbf{W}}(\widetilde{0}, \dots, \widetilde{0}) = \widetilde{0}$.

(ii) $F_{\mathbf{W}}(\widetilde{1}, \dots, \widetilde{1}) = \widetilde{1}$.

(iii) $A_1 \sqsubseteq A_1^*, \dots, A_n \sqsubseteq A_n^*$ *imply* $F_{\mathbf{W}}(A_1, \dots, A_n) \sqsubseteq F_{\mathbf{W}}(A_1^*, \dots, A_n^*)$*, for all* $A_1, A_1^*, \dots, A_n, A_n^* \in F_{NC}([0,1])$.

(iv) $A_1 \sqcap \dots \sqcap A_n \sqsubseteq F_{\mathbf{W}}(A_1, \dots, A_n) \sqsubseteq A_1 \sqcup \dots \sqcup A_n$*, for all* $A_1, \dots, A_n \in F_{NC}([0,1])$.

(v) $F_{\mathbf{W}}$ *is a symmetric operator, i.e.,* $F_{\mathbf{W}}(A_1, \dots, A_n) = F_{\mathbf{W}}(A_{\sigma(1)}, \dots, A_{\sigma(n)})$*, for all* $A_1, \dots, A_n \in F_{NC}([0,1])$*, for each permutation* σ *of* $\{1, \dots, n\}$.

(vi) $F_{\mathbf{W}}$ *is an idempotent operator, i.e.,* $F_{\mathbf{W}}(A, \dots, A) = A$*, for all* $A \in F_{NC}([0,1])$.

(vii) *For weighting vector* $\mathbf{W} = (\widetilde{1}, \widetilde{0}, \dots, \widetilde{0})$ *it holds* $F_{\mathbf{W}}(A_1, \dots, A_n) = A_1 \sqcup \dots \sqcup A_n$*, and for weighting vector* $\mathbf{W} = (\widetilde{0}, \dots, \widetilde{0}, \widetilde{1})$ *it holds* $F_{\mathbf{W}}(A_1, \dots, A_n) = A_1 \sqcap \dots \sqcap A_n$*, for all* $A_1, \dots, A_n \in F_{NC}([0,1])$.

Proof. (i) $F_{\mathbf{W}}(\widetilde{0}, \dots, \widetilde{0}) = (W_1 \sqcap \widetilde{0}) \sqcup \dots \sqcup (W_n \sqcap \widetilde{0}) = \widetilde{0} \sqcup \dots \sqcup \widetilde{0} = \widetilde{0}$. (ii) $F_{\mathbf{W}}(\widetilde{1}, \dots, \widetilde{1}) = (W_1 \sqcap \widetilde{1}) \sqcup \dots \sqcup (W_n \sqcap \widetilde{1}) = W_1 \sqcup \dots \sqcup W_n = \widetilde{1}$. (iii) Let $A_1 \sqsubseteq A_1^*, \dots, A_n \sqsubseteq A_n^*$. Then we have $B_1 \sqsubseteq B_1^*, \dots, B_n \sqsubseteq B_n^*$ and consequently $F_{\mathbf{W}}(A_1, \dots, A_n) = (W_1 \sqcap B_1) \sqcup \dots \sqcup (W_n \sqcap B_n) \sqsubseteq (W_1 \sqcap B_1^*) \sqcup \dots \sqcup (W_n \sqcap B_n^*) = F_{\mathbf{W}}(A_1^*, \dots, A_n^*)$. (iv) The proof immediately follows from [5, Proposition 3.7 (iii)]. (v) Immediately follows from Def. 2.3. (vii) Immediately follows from [5, Proposition 3.8]. (vi) According to Def. 2.1 we have $F_{\mathbf{W}}(A, \dots, A) = (W_1 \sqcap A) \sqcup \dots \sqcup (W_n \sqcap A) = (W_1 \sqcup \dots \sqcup W_n) \sqcap A = \widetilde{1} \sqcap A = A$. $\square$

2.3. Homogeneity and shift-invariance

Definition 2.5. An aggregation function $M : F([0,1])^n \to F([0,1])$ is *homogeneous* if for all $\lambda \in]0, \infty[$ and for all $(A_1, \dots, A_n) \in F_{NC}([0,1])^n$ the following holds: $M(\lambda A_1, \dots, \lambda A_n) = \lambda M(A_1, \dots, A_n)$ whenever $(\lambda A_1, \dots, \lambda A_n) \in F_{NC}([0,1])^n$.

Definition 2.6. An aggregation function $M : F([0,1])^n \to F([0,1])$ is *shift-invariant* (or *stable for translations*) if for all $\lambda \in [0,1]$ and for all $(A_1, \dots, A_n) \in F_{NC}([0,1])^n$ the following holds: $M(A_1 + \lambda, \dots, A_n + \lambda) = M(A_1, \dots, A_n) + \lambda$ whenever $(A_1 + \lambda, \dots, A_n + \lambda) \in F_{NC}([0,1])^n$.

Recall that for $A \in F([0,1])$ and appropriate λ we have $(\lambda A)(x) = A(x/\lambda)$ and $(A + \lambda)(x) = A(x - \lambda)$, moreover λA and $A + \lambda$ are normal convex fuzzy sets if A is normal convex fuzzy set.

Theorem 2.7. OWA *operator* $F_{\mathbf{W}}: F_{NC}([0,1])^n \to F_{NC}([0,1])$ *given by Def. 2.3 is not homogeneous.*

Proof.
We give an counterexample. Let $\lambda = 2$, $A_1 = (0.3, 0.4, 0.5)$, $A_2 = (0.1, 0.2, 0.3)$, $W_1 = (0.2, 0.3, 0.4)$ be triangular fuzzy sets in $[0,1]$, $W_2 = \widetilde{1}$ and $\mathbf{W} = (W_1, W_2)$. Then $\lambda A_1 = (0.6, 0.8, 1)$, $\lambda A_2 = (0.2, 0.4, 0.6)$ and $F_{\mathbf{W}}(A_1, A_2) = (W_1 \sqcap A_1) \sqcup (W_2 \sqcap A_2) = W_1 \sqcup A_2 = W_1$, $F_{\mathbf{W}}(\lambda A_1, \lambda A_2) = (W_1 \sqcap \lambda A_1) \sqcup (W_2 \sqcap \lambda A_2) = W_1 \sqcup \lambda A_2 = \lambda A_2$, hence $F_{\mathbf{W}}(\lambda A_1, \lambda A_2) \neq \lambda F_{\mathbf{W}}(A_1, A_2)$. □

Lemma 2.8. *Let* $A_1, \ldots, A_n \in F(\mathbb{R})$ *and* $\lambda \in]0, \infty[$. *Then:*
(i) $\lambda A_1 \sqcup \cdots \sqcup \lambda A_n = \lambda(A_1 \sqcup \cdots \sqcup A_n)$; and
(ii) $\lambda A_1 \sqcap \ldots \sqcap \lambda A_n = \lambda(A_1 \sqcap \ldots \sqcap A_n)$.

Proof.
(i)
$$\begin{aligned}\big(\lambda A_1 \sqcup \cdots \sqcup \lambda A_n\big)(z) &= \sup_{x_1 \vee \ldots \vee x_n = z} \big((\lambda A_1)(x_1) \wedge \cdots \wedge (\lambda A_n)(x_n)\big)\\ &= \sup_{\frac{x_1}{\lambda} \vee \ldots \vee \frac{x_n}{\lambda} = \frac{z}{\lambda}} \big((A_1)(x_1/\lambda) \wedge \cdots \wedge (A_n)(x_n/\lambda)\big)\\ &= (A_1 \sqcup \cdots \sqcup A_n)(z/\lambda) = \big(\lambda(A_1 \sqcup \cdots \sqcup A_n)\big)(z).\end{aligned}$$

(ii) The proof is analogous. □

Corollary 2.9. *Let* $\lambda \in]0, \infty[$. *If we take* $\lambda A_1, \ldots, \lambda A_n$ *instead of* $A_1, \ldots$ $\ldots, A_n \in F(\mathbb{R})$ *in the equation* (3), *then we get chain* $\lambda B_n \sqsubseteq \ldots \sqsubseteq \lambda B_1$ *instead of* $B_n \sqsubseteq \ldots \sqsubseteq B_1$.

Theorem 2.10. OWA *operator* $F_{\mathbf{W}}: F_{NC}([0,1])^n \to F_{NC}([0,1])$ *given by Def. 2.3 is not shift-invariant.*

Proof. We give an counterexample. Let $\lambda = 0.4$, $A_1 = (0.3, 0.4, 0.5)$, $A_2 = (0.1, 0.2, 0.3)$, $W_1 = (0.2, 0.3, 0.4)$ be triangular fuzzy sets in $[0,1]$, $W_2 = \widetilde{1}$ and $\mathbf{W} = (W_1, W_2)$. Then $A_1 + \lambda = (0.7, 0.8, 0.9)$, $A_2 + \lambda = (0.5, 0.6, 0.7)$ and $F_{\mathbf{W}}(A_1, A_2) = (W_1 \sqcap A_1) \sqcup (W_2 \sqcap A_2) = W_1 \sqcup A_2 = W_1$, $F_{\mathbf{W}}(A_1 + \lambda, A_2 + \lambda) = \big(W_1 \sqcap (A_1 + \lambda)\big) \sqcup \big(W_2 \sqcap (A_2 + \lambda)\big) = W_1 \sqcup (A_2 + \lambda) = A_2 + \lambda$, hence $F_{\mathbf{W}}(A_1 + \lambda, A_2 + \lambda) \neq F_{\mathbf{W}}(A_1, A_2) + \lambda$. □

Lemma 2.11. *Let* $A_1, \ldots, A_n \in F(\mathbb{R})$ *and* $\lambda \in \mathbb{R}$. *Then:*
(i) $(A_1 + \lambda) \sqcup \cdots \sqcup (A_n + \lambda) = (A_1 \sqcup \cdots \sqcup A_n) + \lambda$; and
(ii) $(A_1 + \lambda) \sqcap \ldots \sqcap (A_n + \lambda) = (A_1 \sqcap \ldots \sqcap A_n) + \lambda$.

P r o o f.

(i)
$$\begin{aligned}\big((A_1+\lambda) \sqcup \dots \sqcup (A_n+\lambda)\big)(z) &= \sup_{x_1\vee\dots\vee x_n=z} \\ \big((A_1+\lambda)(x_1) \wedge \dots \wedge (A_n+\lambda)(x_n)\big) &= \sup_{(x_1-\lambda)\vee\dots\vee(x_n-\lambda)=z-\lambda} \\ \big((A_1)(x_1-\lambda) \wedge \dots \wedge (A_n)(x_n-\lambda)\big) &= (A_1\sqcup\dots\sqcup A_n)(z-\lambda) \\ &= \big((A_1\sqcup\dots\sqcup A_n)+\lambda\big)(z).\end{aligned}$$

(ii) The proof is analogous. □

Corollary 2.12. *Let $\lambda \in \mathbb{R}$. If we take $A_1+\lambda, \dots, A_n+\lambda$ instead of $A_1, \dots$ $\dots, A_n \in F(\mathbb{R})$ in the equation* (3)*, then we get chain $B_n+\lambda \sqsubseteq \dots \sqsubseteq B_1+\lambda$ instead of $B_n \sqsubseteq \dots \sqsubseteq B_1$.*

2.4. Operators for numbers and intervals

We show that OWA operator $F_{\mathbf{W}}$ given by Def. 2.3 is an extension of Yager's OWA operator on $[0,1]$, moreover, $F_{\mathbf{W}}$ is closed on the set of all closed subintervals of $[0,1]$.

Theorem 2.13. *Let $w_1, \dots, w_n, a_1, \dots, a_n \in [0,1]$, $\mathbf{w} = (w_1, \dots, w_n)$, $\mathbf{W} = (\widetilde{w}_1, \dots, \widetilde{w}_n)$. Let $F_{\mathbf{W}}$ be the corresponding* OWA *operator given by Def. 2.3 and* $\mathrm{OWA}_{\mathbf{w}} : [0,1]^n \to [0,1]$ *be the standard Yager's* OWA *operator for real numbers given by the equation* (1)*. Then*

$$F_{\mathbf{W}}(\widetilde{a}_1, \dots, \widetilde{a}_n) \equiv \mathrm{OWA}_{\mathbf{w}}(a_1, \dots, a_n), \quad \text{for all} \quad w_1, \dots, w_n$$

such that $F_{\mathbf{W}}$ and $\mathrm{OWA}_{\mathbf{w}}$ are defined.

P r o o f. Operators $F_{\mathbf{W}}$ and $\mathrm{OWA}_{\mathbf{w}}$ are defined simultaneously if and only if weighting vector $(\widetilde{w}_1, \dots, \widetilde{w}_n)$ is distributive and $w_1+\dots+w_n = 1$, i.e., according to Proposition 2.2, $w_k = 1$ for some $k \in \{1, \dots, n\}$ and $w_i = 0$ for all $i \in \{1, \dots, n\} - \{k\}$. Then $F_{\mathbf{W}}(\widetilde{a}_1, \dots, \widetilde{a}_n) = (\widetilde{0} \sqcap B_1) \sqcup \dots \sqcup (\widetilde{0} \sqcap B_{k-1}) \sqcup (\widetilde{1} \sqcap B_k) \sqcup (\widetilde{0} \sqcap B_{k+1}) \sqcup \dots \sqcup (\widetilde{0} \sqcap B_n) = B_k = \widetilde{a_{\sigma(k)}} \equiv 0 \cdot a_{\sigma(1)} + \dots + 0 \cdot a_{\sigma(k-1)} + 1 \cdot a_{\sigma(k)} + 0 \cdot a_{\sigma(k+1)} + \dots + 0 \cdot a_{\sigma(n)} = \mathrm{OWA}_{\mathbf{w}}(a_1, \dots, a_n)$. Recall that $B_i = \widetilde{a_{\sigma(i)}}$ for all $i \in \{1, \dots, n\}$. □

From now on, $\mathbb{I}([0,1]) = \{[a,b] \mid 0 \le a \le b \le 1\}$, $\widetilde{[a,b]}$ stands for the characteristic function of $[a,b]$ and $\mathbb{K}$ stands for the set of all characteristic functions of the closed subintervals of $[0,1]$. Recall that $(\mathbb{K}, \sqcap, \sqcup, \sqsubseteq, \widetilde{0}, \widetilde{1})$ is a subalgebra of F_{NC}, see [10].

Theorem 2.14. *Let $\mathbf{W}$ be a distributive weighting vector in $(\mathbb{K}, \sqcap, \sqcup, \sqsubseteq, \widetilde{0}, \widetilde{1})$ and $F_{\mathbf{W}}$ the* OWA *operator given by Def. 2.3. If $A_1, \dots, A_n \in \mathbb{K}$, then $F_{\mathbf{W}}(A_1, \dots$ $\dots, A_n) \in \mathbb{K}$.*

3. Type-1 OWA operators

In this section, we deal with the type-1 OWA operator (T1-OWA, for short) introduced in [13] and further developed in [14] and [15].

3.1. Definition (the second approach)

A binary operation on $[0,1]$ is called a t-norm if it is commutative, associative, non-decreasing in each component and 1 acts as identity element.

Definition 3.1. Given a weighting vector $\mathbf{W} = (W_1, \ldots, W_n) \in F([0,1])^n$, an associated type-1 OWA operator of dimension n is a mapping $\Phi_{\mathbf{W}} : F(X)^n \to F(X)$ given by

$$\Phi_{\mathbf{W}}(A_1, \ldots, A_n)(y) = \sup_{\sum\limits_{i=1}^{n} \overline{w}_i a_{\sigma(i)} = y} \big(W_1(w_1) * \cdots * W_n(w_n) * A_1(a_1) * \cdots * A_n(a_n)\big),$$

where $*$ is a t-norm, $\overline{w}_i = w_i / \sum_{j=1}^{n} w_j$, and $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$ is a permutation such that $a_{\sigma(i)}$ is the i-th largest element in the set $\{a_1, \ldots, a_n\}$.

Remark. The second approach to avoiding the lack of a linear order in definition of OWA operator is based on a delaying of application of a linear order in the following way: we do not order input elements (fuzzy sets $A_1, \ldots, A_n \in F(X)$) before calculation of $\Phi_{\mathbf{W}}(A_1, \ldots, A_n)(y)$, but we only order elements $a_1, \ldots a_n \in X$ (real numbers) later during the calculation. The reason is clear, there exists a linear order on X, however, there does not exist such an order on $F(X)$. But, this means that overlapping input fuzzy sets are ordered in various ways during the individual partial calculations of the sum $\sum_{k=1}^{n} \overline{w}_i a_{\sigma(i)}$. Thus a fixed weight W_i is not strictly assigned to the particular input $A_{\sigma(i)}$, which is a property of standard OWA operator. That is the price we pay for avoiding the non-existence of linear order on the set of fuzzy sets by the second approach.

Example 2. Let the weighting vector $\mathbf{W} = (W_1, W_2, W_3, W_4) \in F_{NC}([0,1])^4$ and inputs $A_1, A_2, A_3, A_4 \in F_{NC}([0,1])$ be given by Figure 3. Let $*$ stand for minimum. Then the result of type-1 OWA operator $\Phi_{\mathbf{W}}(A_1, A_2, A_3, A_4)$ is depicted in Figure 3 (red dashed line). The example and figure are taken from [13].

3.2. Algebraic properties

Theorem 3.2. *Let $W_1, \ldots, W_n \in F_{NC}([0,1])$, $\Phi_{\mathbf{W}} : F_{NC}([0,1])^n \to F_{NC}([0,1])$ be an associated type-1 OWA operator given by Def. 3.1, where $*$ stands for minimum. Then:*

(i) $\Phi_{\mathbf{W}}(\widetilde{0}, \ldots, \widetilde{0}) = \widetilde{0}$;

(ii) $\Phi_{\mathbf{W}}(\widetilde{1}, \ldots, \widetilde{1}) = \widetilde{1}$;

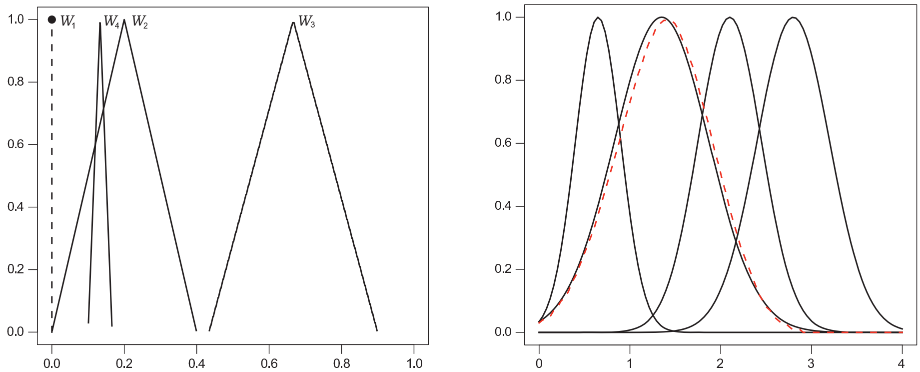

FIGURE 3. See Example 2.

(iii) $A_1 \sqsubseteq B_1, \dots, A_n \sqsubseteq B_n$ *imply* $\Phi_{\mathbf{W}}(A_1, \dots, A_n) \sqsubseteq \Phi_{\mathbf{W}}(B_1, \dots, B_n)$, *for all* $A_1, B_1, \dots, A_n, B_n \in F_{NC}([0,1])$;

(iv) $A_1 \sqcap \ldots \sqcap A_n \sqsubseteq \Phi_{\mathbf{W}}(A_1, \dots, A_n) \sqsubseteq A_1 \sqcup \cdots \sqcup A_n$, *for all* $A_1, \dots, A_n \in F_{NC}([0,1])$;

(v) $\Phi_{\mathbf{W}}$ *is a symmetric operator;*

(vi) $\Phi_{\mathbf{W}}$ *is an idempotent operator;*

(vii) *For weighting vector* $\mathbf{W} = (\widetilde{1}, \widetilde{0}, \dots, \widetilde{0})$ *it holds* $\Phi_{\mathbf{W}}(A_1, \dots, A_n) = A_1 \sqcup \cdots \cdots \sqcup A_n$, *and for weighting vector* $\mathbf{W} = (\widetilde{0}, \dots, \widetilde{0}, \widetilde{1})$ *it holds*

$$\Phi_{\mathbf{W}}(A_1, \dots, A_n) = A_1 \sqcap \ldots \sqcap A_n, \quad \textit{for all } A_1, \dots, A_n \in F_{NC}([0,1]).$$

Proof.
(i) Since the weights $W_1, \dots, W_n$ are normal, there exist $w_1^*, \dots w_n^* \in [0,1]$ such that $W_i(w_i^*) = 1$ for all $i \in \{1, \dots, n\}$. Then

$$\begin{aligned}\Phi_{\mathbf{W}}(\widetilde{0}, \dots, \widetilde{0})(0) &= \sup_{\sum_{i=1}^{n} \overline{w}_i a_{\sigma(i)} = 0} \big(W_1(w_1) * \cdots * W_n(w_n) * \widetilde{0}(a_1) * \cdots * \widetilde{0}(a_n)\big) \\ &\geq W_1(w_1^*) * \cdots * W_n(w_n^*) * \widetilde{0}(0) * \cdots * \widetilde{0}(0) = 1.\end{aligned}$$

And for $y \neq 0$ there exists $a_k \neq 0$, for some $k \in \{1, \dots, n\}$, thus $\widetilde{0}(a_k) = 0$ and consequently $\Phi_{\mathbf{W}}(\widetilde{0}, \dots, \widetilde{0})(y) = 0$. (ii) The proof is similar to that of (i). The proof of (iii), (iv) and (vii) follows from [14, Theorem 1, Theorem 2 and the equations (6)–(7)], and observation that $A \sqsubseteq B$ imply $A_\alpha^- \leq B_\alpha^-$, $A_\alpha^+ \leq B_\alpha^+$, where $[A_\alpha^-, A_\alpha^+]$ denotes α-cut of A. (v) Immediately follows from Def. 3.1. (vi) The proof follows from [14, Theorems 1 and 2 and the equations (6)–(7)]. □

3.3. Homogeneity and shift-invariance

Theorem 3.3. OWA *operator* $\Phi_{\mathbf{W}} : F([0,1])^n \to F([0,1])$ *given by Def. 3.1 is* (i) homogeneous and (ii) shift-invariant.

Proof.

(i) $\Phi_{\mathbf{W}}(\lambda A_1, \dots, \lambda A_n)(y)$

$$= \sup_{\sum_{i=1}^{n} \overline{w}_i a_{\sigma(i)} = y} \big(W_1(w_1) * \dots * W_n(w_n) * (\lambda A_1)(a_1) * \dots * (\lambda A_n)(a_n)\big)$$

$$= \sup_{\sum_{i=1}^{n} \overline{w}_i \frac{a_{\sigma(i)}}{\lambda} = \frac{y}{\lambda}} \big(W_1(w_1) * \dots * W_n(w_n) * A_1(a_1/\lambda) * \dots * A_n(a_n/\lambda)\big)$$

$$= \Phi_{\mathbf{W}}(A_1, \dots, A_n)(y/\lambda) = \lambda \Phi_{\mathbf{W}}(A_1, \dots, A_n)(y).$$

(ii) Observe that $\sum_{i=1}^{n} \overline{w}_i = 1$. Then

$\Phi_{\mathbf{W}}(A_1 + \lambda, \dots, A_n + \lambda)(y)$

$$= \sup_{\sum_{i=1}^{n} \overline{w}_i a_{\sigma(i)} = y} \big(W_1(w_1) * \dots * W_n(w_n) * (A_1 + \lambda)(a_1) * \dots * (A_n + \lambda)(a_n)\big)$$

$$= \sup_{\sum_{i=1}^{n} \overline{w}_i (a_{\sigma(i)} - \lambda) = y - \lambda} \big(W_1(w_1) * \dots * W_n(w_n) * A_1(a_1 - \lambda) * \dots * A_n(a_n - \lambda)\big)$$

$$= \Phi_{\mathbf{W}}(A_1, \dots, A_n)(y - \lambda) = \Phi_{\mathbf{W}}(A_1, \dots, A_n)(y) + \lambda. \qquad \square$$

3.4. Operators for numbers and intervals

We are going to show that OWA operator $\Phi_{\mathbf{W}}$ given by Def. 3.1 is an extension of Yager's OWA operator on $[0,1]$ and is closed on the set of all closed subintervals of $[0,1]$.

Theorem 3.4. *Let* $w_1, \dots, w_n, a_1, \dots, a_n \in [0,1]$, $\mathbf{W} = (\widetilde{w}_1, \dots, \widetilde{w}_n)$ *and* $\overline{\mathbf{w}} = (\overline{w}_1, \dots, \overline{w}_n)$ *with* $\overline{w}_j = w_j / \sum_{i=1}^{n} w_i$, *for all* $j \in \{1, \dots, n\}$. *Let* $\Phi_{\mathbf{W}}$ *be the corresponding type-1* OWA *operator given by Def. 3.1 and* $\mathrm{OWA}_{\overline{\mathbf{w}}}$ *be standard Yager's* OWA *operator for real numbers given by the equation* (1). *Then*

$$\Phi_{\mathbf{W}}(\widetilde{a}_1, \dots, \widetilde{a}_n) \equiv \mathrm{OWA}_{\overline{\mathbf{w}}}(a_1, \dots, a_n).$$

Proof.

For $y = \mathrm{OWA}_{\overline{\mathbf{w}}}(a_1, \dots, a_n) = \sum_{i=1}^{n} \overline{w}_i a_{\sigma(i)}$, we have

$$\Phi_{\mathbf{W}}(\widetilde{a}_1, \dots, \widetilde{a}_n)(y) \geq 1 * \dots * 1 * 1 * \dots * 1 = 1$$

and for $y \neq \sum_{i=1}^{n} \overline{w}_i a_{\sigma(i)}$ it holds

$$\Phi_{\mathbf{W}}(\widetilde{a}_1, \dots, \widetilde{a}_n)(y) = 0.$$

Hence,

$$\Phi_{\mathbf{W}}(\widetilde{a}_1, \dots, \widetilde{a}_n) \equiv \sum_{i=1}^{n} \overline{w}_i a_{\sigma(i)} = \mathrm{OWA}_{\overline{\mathbf{w}}}(a_1, \dots, a_n). \qquad \square$$

Theorem 3.5. *Let* $\mathbf{W}$ *be a distributive weighting vector in* $(\mathbb{K}, \sqcap, \sqcup, \sqsubseteq, \widetilde{0}, \widetilde{1})$ *and* $\Phi_{\mathbf{W}}$ *the* OWA *operator given by Def. 3.1. If* $A_1, \ldots, A_n \in \mathbb{K}$*, then*

$$\Phi_{\mathbf{W}}(A_1, \ldots, A_n) \in \mathbb{K}.$$

4. OWA operator for discrete gradual real intervals

In [7] an OWA operator for discrete gradual real intervals in $[0, 1]$ was proposed. It is based on the concepts introduced in [3] and [4]. Note that gradual interval is more general concept that convex fuzzy set, hence, OWA operators for gradual intervals are applicable to fuzzy sets, too. We will use the following notation and terminology.

Definition 4.1 ([4])**.** A *gradual real number* $\breve{r}$ is defined by an assignment function $A_{\breve{r}} : (0, 1] \to \mathbb{R}$. The set of all gradual real numbers is denoted by $G(\mathbb{R})$.

We will consider *discrete gradual real numbers* $A_{\breve{r}} : \{\alpha_1, \ldots, \alpha_k\} \to \mathbb{R}$, where $0 < \alpha_1 < \alpha_2 < \ldots < \alpha_k = 1$. The set of all discrete gradual real numbers is denoted by $DG_k(\mathbb{R})$. We do not distinguish between gradual number and its assignment function, thus we write $\breve{r}(\alpha)$ instead of $A_{\breve{r}}(\alpha)$. We have a partial order on $DG_k(\mathbb{R})$: $\breve{r} \leq \breve{s}$ iff $\breve{r}(\alpha_i) \leq \breve{s}(\alpha_i)$ for all $i = 1, \ldots, k$.

Gradual interval [4], i.e. interval of gradual real numbers, is defined as follows.

Definition 4.2 ([4])**.** Let $\breve{x}^-, \breve{x}^+$ be gradual real numbers such that $\breve{x}^- \leq \breve{x}^+$. A *gradual interval* $\breve{X}$ is the set

$$\breve{X} = \left\{ \breve{r} \in G(\mathbb{R}) \,|\, \breve{x}^- \leq \breve{r} \leq \breve{x}^+ \right\}.$$

We write $\breve{X} = [\breve{x}^-, \breve{x}^+]$. The set of all gradual intervals will be denoted by $\mathbb{I}(G(\mathbb{R}))$.

If $\breve{x}^-, \breve{x}^+$ are discrete gradual numbers on the same set $\{\alpha_1, \ldots, \alpha_k\}$, then $[\breve{x}^-, \breve{x}^+]$ is called a *discrete gradual interval.* The set of all discrete gradual intervals is denoted by $\mathbb{I}\big(DG_k(\mathbb{R})\big)$. We have a natural partial order on $\mathbb{I}\big(DG_k(\mathbb{R})\big)$, namely $[\breve{x}^-, \breve{x}^+] \preceq_2 [\breve{y}^-, \breve{y}^+]$ iff ($\breve{x}^- \leq \breve{y}^-$ and $\breve{x}^+ \leq \breve{y}^+$), which corresponds to commonly used partial order of intervals: $[a, b] \leq [c, d]$ iff $a \leq c$ and $b \leq d$.

4.1. Definition (the third approach)

We proposed a class of linear orders for discrete gradual real numbers in [7] (this work was generalized in [9]). Based on this class and the concept of admissible order introduced in [1], we also proposed a class of admissible linear orders for discrete gradual intervals in [9]. We skip details, just note that an admissible linear order for discrete gradual intervals will be denoted by $\preceq$ in this paper.

Let $\breve{1} \in DG_k([0,1])$ stand for a discrete gradual real number with $\breve{1}(\alpha_j) = 1$ for all $j = 1, \dots, k$. The following operations will be of use in this paper: $(\breve{x} + \breve{y})(\alpha_j) = \breve{x}(\alpha_j) + \breve{y}(\alpha_j)$, $(\lambda + \breve{y})(\alpha_j) = \lambda + \breve{y}(\alpha_j)$, $(\breve{x} \cdot \breve{y})(\alpha_j) = \breve{x}(\alpha_j) \cdot \breve{y}(\alpha_j)$ and $(\lambda \cdot \breve{y})(\alpha_j) = \lambda \cdot \breve{y}(\alpha_j)$, for all $j = 1, \dots, k$, where $\breve{x}, \breve{y} \in DG_k(\mathbb{R})$ and $\lambda \in \mathbb{R}$.

Definition 4.3. Let $\breve{\mathbf{w}} = (\breve{w}_1, \dots, \breve{w}_n) \in DG_k([0,1])^n$ with $\breve{w}_1 + \dots + \breve{w}_n = \breve{1}$ be a weighting vector of discrete gradual numbers. A discrete gradual intervals OWA operator (DGIOWA, for short) associated with $\breve{\mathbf{w}}$ is a mapping $G^{\preceq}_{\breve{\mathbf{w}}} : \mathbb{I}\big(DG_k([0,1])\big)^n \to \mathbb{I}\big(DG_k([0,1])\big)$ defined by

$$G^{\preceq}_{\breve{\mathbf{w}}}\big([\breve{x}_1^-, \breve{x}_1^+], \dots, [\breve{x}_n^-, \breve{x}_n^+]\big) = \sum_{i=1}^{n} \breve{w}_i \cdot [\breve{x}_{(i)}^-, \breve{x}_{(i)}^+], \tag{4}$$

where $[\breve{x}_{(i)}^-, \breve{x}_{(i)}^+]$, $i = 1, \dots, n$, denotes the ith greatest component of the input $\big([\breve{x}_1^-, \breve{x}_1^+], \dots, [\breve{x}_n^-, \breve{x}_n^+]\big)$ with respect to an admissible order $\preceq$.

Remark. The third approach to avoiding the lack of a linear order in definition of OWA operator is based on a discretization of inputs. The inputs of OWA operator $G^{\preceq}_{\breve{\mathbf{w}}}$ given by Definition 4.3 are discrete gradual intervals for which we have a linear order, so definition of the operator $G^{\preceq}_{\breve{\mathbf{w}}}$ can be done as a straightforward extension of standard Yager's OWA operator for real numbers. Note that the set of all normal convex fuzzy sets in $[0,1]$ is a subset of the set of all gradual intervals in $[0,1]$. Hence, the operator $G^{\preceq}_{\breve{\mathbf{w}}}$ can also be applied to fuzzy sets. The drawback of the approach lays in the fact that the weights are only gradual numbers, not gradual intervals. That is the price we pay for avoiding the non-existence of linear order on the set of fuzzy sets by the third approach.

Example 3. Let $\breve{\mathbf{w}} = (\breve{w}_1, \breve{w}_2, \breve{w}_3) \in DG_k([0,1])^n$ with $\breve{w}_1 + \breve{w}_2 + \breve{w}_3 = \breve{1}$ be a weighting vector of discrete gradual numbers $\breve{w}_1, \breve{w}_2, \breve{w}_3$ given by Figure 4. Let inputs A_1, A_2, A_3 be discrete gradual intervals given by Figure 4. Then the result of DGIOWA operator $G^{\preceq}_{\breve{\mathbf{w}}}(A_1, A_2, A_3)$ is depicted in Figure 4. For more detailed explanation see [9, Example 5].

4.2. Algebraic properties

In the following theorem we will use both the partial order $\preceq_2$ and the admissible (that is linear) order $\preceq$ on $\mathbb{I}\big(DG_k([0,1])\big)$.

Theorem 4.4. *For any operator* $G^{\preceq}_{\breve{\mathbf{w}}}$ *on* $\mathbb{I}\big(DG_k([0,1])\big)$ *given by Def. 4.3 it holds:*

(i) $G^{\preceq}_{\breve{\mathbf{w}}}\big([\breve{0}, \breve{0}], \dots, [\breve{0}, \breve{0}]\big) = [\breve{0}, \breve{0}]$;

(ii) $G^{\preceq}_{\breve{\mathbf{w}}}\big([\breve{1}, \breve{1}], \dots, [\breve{1}, \breve{1}]\big) = [\breve{1}, \breve{1}]$;

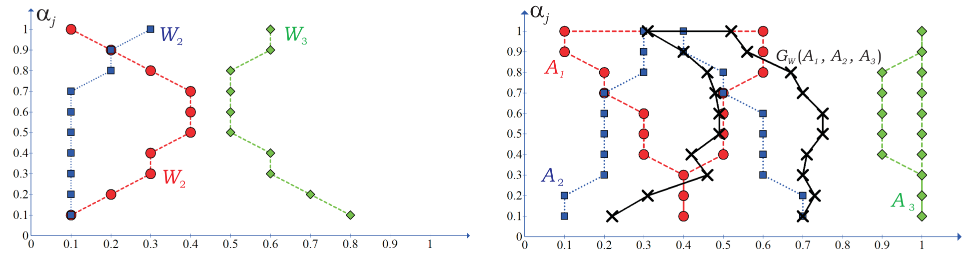

FIGURE 4. See Example 3.

(iii) $[\breve{x}_1^-, \breve{x}_1^+] \preceq_2 [\breve{y}_1^-, \breve{y}_1^+], \dots, [\breve{x}_n^-, \breve{x}_n^+] \preceq_2 [\breve{y}_n^-, \breve{y}_n^+]$ *imply*

$$G_{\breve{\mathbf{w}}}^{\preceq}\big([\breve{x}_1^-, \breve{x}_1^+], \dots, [\breve{x}_n^-, \breve{x}_n^+]\big) \preceq_2 G_{\breve{\mathbf{w}}}^{\preceq}\big([\breve{y}_1^-, \breve{y}_1^+], \dots, [\breve{y}_n^-, \breve{y}_n^+]\big),$$

for all $[\breve{x}_1^-, \breve{x}_1^+], [\breve{y}_1^-, \breve{y}_1^+], \dots, [\breve{y}_n^-, \breve{y}_n^+] \in \mathbb{I}\big(DG_k([0,1])\big)$.

(iv) $\breve{X}_1 \wedge \dots \wedge \breve{X}_n \preceq_2 G_{\breve{\mathbf{w}}}^{\preceq}\left(\breve{X}_1, \dots, \breve{X}_n\right) \preceq_2 \breve{X}_1 \vee \dots \vee \breve{X}_n$, *for all* $\breve{X}_1, \dots, \breve{X}_n \in \mathbb{I}\left(DG_k([0,1])\right)$, *where* $\wedge$ *and* $\vee$ *stand for meet and join with respect to partial order* $\preceq_2$, *respectively.*

(v) $G_{\breve{\mathbf{w}}}^{\preceq}$ *is a symmetric operator.*

(vi) $G_{\breve{\mathbf{w}}}^{\preceq}$ *is an idempotent operator.*

(vii) *For weighting vector* $\breve{\mathbf{w}} = (\breve{1}, \breve{0}, \dots, \breve{0})$ *it holds*

$$G_{\breve{\mathbf{w}}}^{\preceq}(\breve{X}_1, \dots, \breve{X}_n) = \max\nolimits_{\preceq}(\breve{X}_1, \dots, \breve{X}_n),$$

and for weighting vector $\breve{\mathbf{w}} = (\breve{0}, \dots, \breve{0}, \breve{1})$ *it holds*

$$G_{\breve{\mathbf{w}}}^{\preceq}(\breve{X}_1, \dots, \breve{X}_n) = \min\nolimits_{\preceq}(\breve{X}_1, \dots, \breve{X}_n),$$

for all $\breve{X}_1, \dots, \breve{X}_n \in \mathbb{I}\big(DG_k([0,1])\big)$, *where* $\max_{\preceq}$ *and* $\min_{\preceq}$ *stand for maximum and minimum with respect to linear order* $\preceq$, *respectively.*

P r o o f. For proof of items (i)–(iii) see [9, Theorem 3.9] and the rest straightforwardly follows from Def. 4.3 and properties of discrete gradual intervals. □

4.3. Homogeneity and shift-invariance

Observe that $\lambda \breve{X} = \lambda[\breve{x}^-, \breve{x}^+] = [\lambda \breve{x}^-, \lambda \breve{x}^+]$ and $\breve{X} + \lambda = [\breve{x}^-, \breve{x}^+] + \lambda = [\breve{x}^- + \lambda, \breve{x}^+ + \lambda]$.

THEOREM 4.5. OWA *operator* $G_{\breve{\mathbf{w}}}^{\preceq} : \mathbb{I}\big(DG_k([0,1])\big)^n \to \mathbb{I}\ \big(DG_k([0,1])\big)$ *given by Def. 4.3 is homogeneous, i.e., if* $(\lambda \breve{X}_1, \dots, \lambda \breve{X}_n) \in \mathbb{I}\big(DG_k([0,1])\big)^n$, *then* $G_{\breve{\mathbf{w}}}^{\preceq}(\lambda \breve{X}_1, \dots, \lambda \breve{X}_n) = \lambda G_{\breve{\mathbf{w}}}^{\preceq}(\breve{X}_1, \dots, \breve{X}_n)$, *for all* $\lambda \in]0, \infty[$ *and* $(\breve{X}_1, \dots, \breve{X}_n) \in \mathbb{I}\big(DG_k([0,1])\big)^n$.

Proof.

$$G^{\preceq}_{\breve{\mathbf{w}}}\big(\lambda[\breve{x}_1^-,\breve{x}_1^+],\dots,\lambda[\breve{x}_n^-,\breve{x}_n^+]\big)=\sum_{i=1}^{n}\breve{w}_i\lambda[\breve{x}_{(i)}^-,\breve{x}_{(i)}^+]=\lambda G^{\preceq}_{\breve{\mathbf{w}}}\big([\breve{x}_1^-,\breve{x}_1^+],\dots,[\breve{x}_n^-,\breve{x}_n^+]\big).$$

□

Theorem 4.6. OWA *operator* $G^{\preceq}_{\breve{\mathbf{w}}} : \mathbb{I}\big(DG_k([0,1])\big)^n \to \mathbb{I}\big(DG_k([0,1])\big)$ *given by Def. 4.3 is shift-invariant, i.e., if* $(\breve{X}_1+\lambda,\dots,\breve{X}_n+\lambda)\in\mathbb{I}\big(DG_k([0,1])\big)^n$, *then* $G^{\preceq}_{\breve{\mathbf{w}}}(\breve{X}_1+\lambda,\dots,\breve{X}_n+\lambda)=G^{\preceq}_{\breve{\mathbf{w}}}(\breve{X}_1,\dots,\breve{X}_n)+\lambda$, *for all* $\lambda\in[0,1]$ *and* $(\breve{X}_1,\dots,\breve{X}_n)\in\mathbb{I}\big(DG_k([0,1])\big)^n$.

Proof.

$$\begin{aligned}G^{\preceq}_{\breve{\mathbf{w}}}\big([\breve{x}_1^-,\breve{x}_1^+]+\lambda,\dots,[\breve{x}_n^-,\breve{x}_n^+]+\lambda\big)&=\sum_{i=1}^{n}\breve{w}_i([\breve{x}_{(i)}^-,\breve{x}_{(i)}^+]+\lambda)\\&=\sum_{i=1}^{n}(\breve{w}_i[\breve{x}_{(i)}^-,\breve{x}_{(i)}^+]+\breve{w}_i\lambda)\\&=G^{\preceq}_{\breve{\mathbf{w}}}\big([\breve{x}_1^-,\breve{x}_1^+],\ \dots,[\breve{x}_n^-,\breve{x}_n^+]\big)+\lambda.\end{aligned}$$

□

4.4. Operators for numbers and intervals

Theorem 4.7. *Let* $\mathbf{w}=(w_1,\dots,w_n)\in[0,1]^n$ *with* $w_1+\dots+w_n=1$, *and let* $\breve{\mathbf{w}}=(\breve{w}_1,\dots,\breve{w}_n)$ *with* $\breve{w}_i(\alpha_j)=w_i$ *for all* $i=1,\dots,n$, $j=1,\dots,k$. *Let* $G^{\preceq}_{\breve{\mathbf{w}}}$ *be the corresponding* OWA *operator on* $\mathbb{I}\big(DG_k([0,1])\big)$ *given by Def. 4.3 and* $\mathrm{OWA}_{\mathbf{w}}$ *be the standard Yager's* OWA *operator for real numbers given by the equation* (1). *Then*

$$G^{\preceq}_{\breve{\mathbf{w}}}\left([\breve{x}_1,\breve{x}_1],\dots,[\breve{x}_n,\breve{x}_n]\right)\equiv \mathrm{OWA}_{\mathbf{w}}\left(x_1,\dots,x_n\right),$$

where $\preceq$ *stands for an admissible (linear) order on* $\mathbb{I}\big(DG_k([0,1])\big)$ *and* $\breve{x}_i(\alpha_j)=x_i$ *for all* $i=1,\dots,n$, $j=1,\dots,k$.

Proof. See Theorem 3.10 in [9]. □

Theorem 4.8. *Let* $w_1,\dots,w_n\in[0,1]$ *with* $w_1+\dots+w_n=1$, *and let* $\breve{\mathbf{w}}=(\breve{w}_1,\dots,\breve{w}_n)$, *where* $\breve{w}_i(\alpha_j)=w_i$ *for all* $i=1,\dots,n$, $j=1,\dots,k$. *Let* $G^{\preceq}_{\breve{\mathbf{w}}}$ *be the corresponding* OWA *operator on* $\mathbb{I}\big(DG_k([0,1])\big)$ *given by Def. 4.3. Let* x_i^-, $x_i^+\in[0,1]$ *be such that* $x_i^-\le x_i^+$ *for all* $i=1,\dots,n$. *Then there exist* y^-, $y^+\in[0,1]$ *with* $y^-\le y^+$ *such that*

$$G^{\preceq}_{\breve{\mathbf{w}}}\left([\breve{x}_1^-,\breve{x}_1^+],\dots,[\breve{x}_n^-,\breve{x}_n^+]\right)=[\breve{y}^-,\breve{y}^+],$$

where $\breve{x}_i^-(\alpha_j)=x_i^-$, $\breve{x}_i^+(\alpha_j)=x_i^+$ *for all* $i=1,\dots,n$, $j=1,\dots,k$, *and* $\breve{y}^-(\alpha_j)=y^-$, $\breve{y}^+(\alpha_j)=y^+$ *for all* $j=1,\dots,k$.

Remark. Note that, for some appropriate admissible orders of intervals and discrete gradual intervals, the operator $G^{\preceq}_{\check{\mathbf{w}}}$ is an extension of the so-called IVOWA operator [2], which is an operator on the set of all closed subintervals of $[0,1]$. However, we do not have sufficient tools in this paper to enable us to give the exact results in this direction (see [9, Theorem 3.10]).

5. Properties of OWA operators

In the previous sections, we introduced the definitions of three different OWA operators for fuzzy sets in $[0,1]$, namely $F_{\mathbf{W}}$ (L-OWA), $\Phi_{\mathbf{W}}$ (type-1 OWA) and $G^{\preceq}_{\check{\mathbf{w}}}$ (DGIOWA), and proved some their properties that have not been studied in the literature yet. Table 1 summarizes the proved properties and also a few properties proved by other authors in the past.

In the first part of Table 1 (lines 1–3) the inputs, outputs and weights are summarized. Although there are different types of inputs and outputs for the three operators, normal convex fuzzy sets are encompassed in each of them. The only exception is that the weights of DGIOWA operator are not fuzzy sets, they are just gradual numbers.

The second part of Table 1 (lines 4–6) concerns with three basic properties of each aggregation function, that is boundary conditions and monotonicity. According to these properties, the three operators L-OWA, T1-OWA and DGIOWA are rightly called aggregation functions.

In the third part of Table 1 (lines 7–11) frequently used algebraic properties of aggregation functions are included. We can see that all the three operators have desired algebraic properties, namely they are between minimum and maximum, they are exactly minimum or maximum for some special weighting vector, and they are symmetric and idempotent.

The fourth part of Table 1 (lines 12–13) deals with linearity of aggregation functions, that is homogeneity and shift-invariance. The result is that OWA operator defined by the first approach (L-OWA) is neither homogeneous nor shift-invariant, but operators defined by the second (T1-OWA) and the third (DGIOWA) approach are both, hence they are linear aggregation functions.

According to the last part of Table 1 (lines 14–15) all the three defined OWA operators are extensions of standard Yager's operator, more precisely if all the inputs and weights are numbers from $[0,1]$ then all the three operators give the same result as Yager's OWA operator. Moreover, the three operators are closed on the set of all subintervals of $[0,1]$.

Table 1. The summary of properties of the three OWA operators.

OPERATOR / PROPERTY	L-OWA Def. 2.3	T1-OWA Def. 3.1	DGIOWA Def. 4.3
Inputs	$F_{NC}([0,1])$	$F(X)$	$\mathbb{I}\big(DG_k([0,1])\big)$
Weights	$F_{NC}([0,1])$	$F([0,1])$	$DG_k([0,1])$
Output	$F_{NC}([0,1])$	$F(X)$	$\mathbb{I}\big(DG_k([0,1])\big)$
$\mathrm{OWA}(0,\dots,0)=0$	Th. 2.4 (i)	Th. 3.2 (i)	Th. 4.4 (i)
$\mathrm{OWA}(1,\dots,1)=1$	Th. 2.4 (ii)	Th. 3.2 (ii)	Th. 4.4 (ii)
Non-decreasingness	Th. 2.4 (iii)	Th. 3.2 (iii)	Th. 4.4 (iii)
$\min \leq \mathrm{OWA} \leq \max$	Th. 2.4 (iv)	Th. 3.2 (iv)	Th. 4.4 (iv)
Symmetry	Th. 2.4 (v)	Th. 3.2 (v)	Th. 4.4 (v)
Idempotency	Th. 2.4 (vi)	Th. 3.2 (vi)	Th. 4.4 (vi)
$\mathrm{OWA}_{(1,0,\dots,0)}=\max$	Th. 2.4 (vii)	Th. 3.2 (vii)	Th. 4.4 (vii)
$\mathrm{OWA}_{(0,\dots,0,1)}=\min$	Th. 2.4 (vii)	Th. 3.2 (vii)	Th. 4.4 (vii)
Homogeneity	xxx	Th. 3.3 (i)	Th. 4.5
Shift-invariance	xxx	Th. 3.3 (ii)	Th. 4.6
Standard OWA on $[0,1]$	Th. 2.13	Th. 3.4	Th. 4.7
OWA on intervals	Th. 2.14	Th. 3.5	Th. 4.8

6. Conclusion

Various ways of extension of Yager's OWA operator from $[0,1]$ to the set of all normal convex fuzzy sets in $[0,1]$ are presented. The focus of this paper is to describe the methods of dealing with the lack of a linear order on the set of all fuzzy sets and defining OWA operators on the set. The three approaches were investigated, namely L-OWA operator on any complete lattice endowed with a t-norm and a t-conorm [5], type-1 OWA operator for aggregating linguistic information based on the extension principle [13], and DGIOWA operator for discrete gradual real intervals in $[0,1]$ [9]. We have adapted the three approaches to the set of all normal convex fuzzy sets in $[0,1]$, proved some of their algebraic properties, studied their linearity (homogeneity and shift-invariance) and discussed how they perform on $[0,1]$ and the set of interval fuzzy sets in $[0,1]$.

As it can be seen from the summarization in Table 1, all the three approaches to avoiding the lack of a linear order when defining OWA operator on the set of fuzzy sets have led to operator with desired algebraic properties, two of them are also linear, and finally all of them are extensions of standard Yager's operator on $[0, 1]$ and are closed on the set of all subintervals of $[0, 1]$.

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