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DISTRIBUTION FUNCTIONS OF RATIO SEQUENCES. AN EXPOSITORY PAPER

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ABSTRACT. This expository paper presents known results on distribution functions $g(x)$ of the sequence of blocks $X_n = \left(\frac{x_1}{x_n}, \frac{x_2}{x_n}, \dots, \frac{x_n}{x_n}\right)$, $n = 1, 2, \dots$, where x_n is an increasing sequence of positive integers. Also presents results of the set $G(X_n)$ of all distribution functions $g(x)$. Specially:

- continuity of $g(x)$;
- connectivity of $G(X_n)$;
- singleton of $G(X_n)$;
- one-step $g(x)$;
- uniform distribution of X_n, $n = 1, 2, \dots$;
- lower and upper bounds of $g(x)$;
- applications to bounds of $\frac{1}{n}\sum_{i=1}^{n} \frac{x_i}{x_n}$;
- many examples, e.g., $X_n = \left(\frac{2}{p_n}, \frac{3}{p_n}, \dots, \frac{p_{n-1}}{p_n}, \frac{p_n}{p_n}\right)$, where p_n is the nth prime, is uniformly distributed.

The present results have been published by 25 papers of several authors between 2001–2013.

1. Introduction

Let x_n, $n=1,2,\dots$, be an increasing sequence of positive integers (by "increasing" we mean strictly increasing). The double sequence x_m/x_n, $m, n = 1, 2, \dots$ is called *the ratio sequence* of x_n. It was introduced by T. Šalát [16]. He studied its everywhere density. For further study of the ratio sequences, O. Strauch and J. T. Tóth [24] introduced a sequence X_n of blocks

$$X_n = \left(\frac{x_1}{x_n}, \frac{x_2}{x_n}, \dots, \frac{x_n}{x_n}\right), \qquad n = 1, 2, \dots$$

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and they studied the set $G(X_n)$ of its distribution functions. The motivation is that the existence of strictly increasing $g(x) \in G(X_n)$ implies everywhere density of x_m/x_n, the basic problem studied by Šalát [16]. Further motivation is that the block sequences are a tool for study of distribution functions of sequences, see [20, p. 12, 1.9]. Organization of the paper:

In Section 2 we follow the notations and basic properties of distribution functions used in [5], [12] and [21, p. 1–28, 1.8.23].

In Section 3 we list main properties of $g(x)$ and $G(X_n)$ without proofs.

In Section 4 we add proofs of some properties in Section 3. Specially:

4.1 Basic properties;
4.2 Continuity of $g(x) \in G(X_n)$;
4.3 Singleton $G(X_n) = \{g(x)\}$;
4.4 U.d. of X_n;
4.5 One-step d.f.s $c_\alpha(x)$;
4.6 Connectivity of $G(X_n)$;
4.7 Boundaries of $g(x) \in G(X_n)$;
4.8 Lower and upper d.f.s in $G(X_n)$;
4.9 Construction $H \subset G(X_n)$;
4.10 $g(x) \in G(X_n)$ with constant intervals;
4.11 Transformation of X_n by $1/x \bmod 1$.

Many examples with x_n and $G(X_n)$ are given in Section 5. The paper is completed in Section 6 with comments on another block sequences.

2. Definitions

- From now on $1 \le x_1 < x_2 < \cdots$ denotes the sequence of positive integers and $x \in [0,1)$.
- Denote by $F(X_n, x)$ the step distribution function

$$F(X_n, x) = \frac{\#\{i \le n; \frac{x_i}{x_n} < x\}}{n},$$

  for $x \in [0,1)$ and for $x = 1$ we define $F(X_n, 1) = 1$.
- Denote by $A(t)$ the counting function

$$A(t) = \#\{n \in \mathbb{N}; x_n < t\}.$$

Directly from the definition we obtain

$$F(X_m, x) = \frac{n}{m} F\left(X_n, x\frac{x_m}{x_n}\right)$$

for each $m \le n$ and

$$\frac{nF(X_n, x)}{xx_n} = \frac{A(xx_n)}{xx_n}$$

for every $x \in [0, 1)$.

- The lower asymptotic density $\underline{d}$ and the upper asymptotic density $\overline{d}$ of x_n, $n = 1, 2, \dots,$[1] are defined as

$$\underline{d} = \liminf_{t\to\infty} \frac{A(t)}{t} = \liminf_{n\to\infty} \frac{n}{x_n}, \quad \overline{d} = \limsup_{t\to\infty} \frac{A(t)}{t} = \limsup_{n\to\infty} \frac{n}{x_n}.$$

- A non-decreasing function $g\colon [0, 1] \to [0, 1]$, $g(0) = 0$, $g(1) = 1$ is called distribution function (abbreviated d.f.). We shall identify any two d.f.s coinciding at common points of continuity.
- Similarly, the inequality $g_1(x) \le g_2(x)$ we consider only in the common points of continuity.
- A d.f. $g(x)$ is a d.f. of the sequence of blocks X_n, $n = 1, 2, \dots,$ if there exists an increasing sequence $n_1 < n_2 < \cdots$ of positive integers such that

$$\lim_{k\to\infty} F(X_{n_k}, x) = g(x)$$

a.e. on $[0, 1]$. This is equivalent to the weak convergence, i.e., the preceding limit holds for every point $x \in [0, 1]$ of continuity of $g(x)$.
- Denote by $G(X_n)$ the set of all d.f.s of X_n, $n = 1, 2, \dots$ If $G(X_n) = \{g(x)\}$ is a singleton, the d.f. $g(x)$ is also called the asymptotic d.f. (abbreviated a.d.f.) of X_n.
- Also for a sequence $y_n \in [0, 1)$, $n = 1, 2, \dots,$ we have defined in [21, 1.3] the step d.f.

$$F_N(x) = \frac{\#\{n \le N; y_n \in [0, x)\}}{N}$$

and $G(y_n)$ is the set of all possible weak limits $F_{N_k}(x) \to g(x)$.
- The lower d.f. $\underline{g}(x)$ and the upper d.f. $\overline{g}(x)$ of a sequence X_n, $n = 1, 2, \dots$ are defined as

$$\underline{g}(x) = \inf_{g\in G(X_n)} g(x), \quad \overline{g}(x) = \sup_{g\in G(X_n)} g(x).$$

[1] $\underline{d} = \underline{d}(x_n)$, $\overline{d} = \overline{d}(x_n)$.

- If $\lim_{k\to\infty} F(X_{n_k},x)=g(x)$ and $\lim_{k\to\infty}\frac{n_k}{x_{n_k}}=d_g$ we shall call d_g as a *local asymptotic density* for d.f. $g(x)$.

  In this paper we frequently use the following two theorems of Helly (see the First and Second Helly theorem [21, Th. 4.1.0.10 and Th. 4.1.0.11, p. 4–5]).
- *Helly's selection principle*: For any sequence $g_n(x)$, $n=1,2,\dots$, of d.f.s in $[0,1]$ there exists a subsequence $g_{n_k}(x)$, $k=1,2,\dots$, and a d.f. $g(x)$ such that $\lim_{k\to\infty} g_{n_k}(x)=g(x)$ a.e.
- *Second Helly theorem*: If we have $\lim_{n\to\infty} g_n(x)=g(x)$ a.e. in $[0,1]$, then for every continuous function $f:[0,1]\to\mathbb{R}$ we have $\lim_{n\to\infty}\int_0^1 f(x)\mathrm{d}g_n(x)=\int_0^1 f(x)\,\mathrm{d}g(x)$.
- Note that applying Helly's selection principle, from the sequence $F(X_n,x)$, $n=1,2,\dots$, one can select a subsequence $F(X_{n_k},x)$, $k=1,2,\dots$, such that $\lim_{k\to\infty} F(X_{n_k},x)=g(x)$ holds not only for the continuity points x of $g(x)$, but also for all $x\in[0,1]$.
- We will use the one-step d.f. $c_\alpha(x)$ with the step 1 at α defined on $[0,1]$ via
$$c_\alpha(x)=\begin{cases}0, & \text{if } x\le\alpha;\\ 1, & \text{if } x>\alpha,\end{cases}$$
  while always $c_\alpha(0)=0$ and $c_\alpha(1)=1$.

3. Overview of basic results

$G(X_n)$ has the following properties:

1. If $g(x)\in G(X_n)$ increases and is continuous at $x=\beta$ and $g(\beta)>0$, then there exists $1\le\alpha<\infty$ such that $\alpha g(x\beta)\in G(X_n)$. If every d.f. of $G(X_n)$ is continuous at 1, then $\alpha=1/g(\beta)$, [24, Prop. 3.1, Th. 3.2].
2. Assume that all d.f.s in $G(X_n)$ are continuous at 0 and $c_1(x)\notin G(X_n)$. Then for every $\tilde{g}(x)\in G(X_n)$ and every $1\le\alpha<\infty$ there exists $g(x)\in G(X_n)$ and $0<\beta\le 1$ such that $\tilde{g}(x)=\alpha g(x\beta)$ a.e. [24, Th. 3.3].
3. Assume that all d.f.s in $G(X_n)$ are continuous at 1. Then all d.f.s in $G(X_n)$ are continuous on $(0,1]$, i.e., only possible discontinuity is in 0 [24, Th. 4.1].
4. If $\underline{d}(x_n)>0$, then every $g(x)\in G(X_n)$ is continuous on $[0,1]$, [24, Th. 6.2(iv)].
5. If $\underline{d}(x_n)>0$, then there exists $g(x)\in G(X_n)$ such that $g(x)\ge x$ for every $x\in[0,1]$, [24, Th. 6.2(ii)]. Generally, [3, Th. 6)], every $G(X_n)$ contains $g(x)\ge x$ for every $x\in[0,1]$.

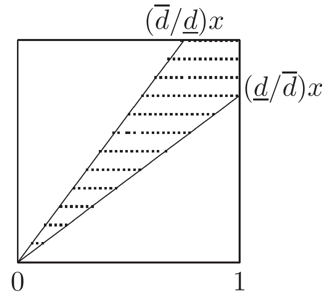
6. If $\overline{d}(x_n) > 0$, then there exists $g(x) \in G(X_n)$ such that $g(x) \leq x$ for every $x \in [0,1]$, [24, Th. 6.2].
7. Assume that $G(X_n)$ is singleton, i.e., $G(X_n) = \{g(x)\}$. Then either $g(x) = c_0(x)$ for $x \in [0,1]$; or $g(x) = x^\lambda$ for some $0 < \lambda \leq 1$ and $x \in [0,1]$. Moreover, if $\overline{d}(x_n) > 0$, then $g(x) = x$, [24, Th. 8.2].
8. $\max_{g \in G(X_n)} \int_0^1 g(x)\,\mathrm{d}x \geq \frac{1}{2}$, [24, Th. 7.1] (c.f. 5.).
9. Assume that every d.f. $g(x) \in G(X_n)$ has a constant value on the fixed interval $(u,v) \subset [0,1]$ (maybe different). If $\underline{d}(x_n) > 0$ then all d.f.s in $G(X_n)$ has infinitely many intervals with constant values, [22].
10. There exists an increasing sequence x_n, $n = 1,2,\dots$, of positive integers such that $G(X_n) = \{h_\alpha(x); \alpha \in [0,1]\}$, where $h_\alpha(x) = \alpha$, $x \in (0,1)$ is the constant d.f. [9, Ex. 1].
11. There exists an increasing sequence x_n, $n = 1,2,\dots$, of positive integers such that $c_1(x) \in G(X_n)$ but $c_0(x) \notin G(X_n)$, where $c_0(x)$ and $c_1(x)$ are one–jump d.f.s with the jump of height 1 at $x = 0$ and $x = 1$, respectively.
12. There exists an increasing sequence x_n, $n = 1,2,\dots$, of positive integers such that $G(X_n)$ is non-connected [9, Ex. 2].
13. We have (see [24, Prop. 3.1, Th. 3.2]):
    Let $g(x) \in G(X_n)$, $\beta \in (0,1)$, and assuming that
    (i) $g(x)$ is continuous at β,
    (ii) $g(x)$ increases at β,[2]
    (iii) $g(\beta) > 0$,
    (iv) all d.f. in $G(X_n)$ are continuous at 1.
    Then
$$\frac{g(x\beta)}{g(\beta)} \in G(X_n).$$
14. Taking the following limits (i)–(iii) for a sequence of indices n_k, $k = 1,2,\dots$
    (i) $\lim_{k\to\infty} F(X_{n_k}, x) = g(x)$,
    (ii) $\lim_{k\to\infty} \frac{n_k}{x_{n_k}} = d_g$,
    then (see [24, Prop. 6.1]) there exists
    (iii) $\lim_{k\to\infty} \frac{A(xx_{n_k})}{xx_{n_k}} = d_g(x)$ and
$$\frac{g(x)}{x} d_g = d_g(x)$$
    for $x \in [0,1]$. Here the limits (i) and (iii) can be considered for all $x \in (0,1]$ or all continuity points $x \in (0,1]$ of $g(x)$ and the constant d_g in (ii) we call *local density*.

[2]The assumption (ii) can be replaced by a requirement that β is a limit point of $\frac{x_i}{x_{n_k}}$, $i = 1,2,\dots,n_k$, $k = 1,2,\dots$, where weakly $F(X_{n_k}, x) \to g(x)$.

15. Specially (see [24, Th. 6.2 (iii), (iv)]), if $\underline{d} > 0$ then

$$x\frac{\underline{d}}{\overline{d}} \le g(x) \le x\frac{\overline{d}}{\underline{d}}$$

for every $x \in [0,1]$ and furthermore $g(x)$ is everywhere continuous. Thus $\underline{d} = \overline{d} > 0$ implies u.d. of the block sequence X_n, $n = 1,2,\dots$



16. $G(X_n) = \{x^\lambda\}$ if and only if $\lim_{n\to\infty}(x_{k.n}/x_n) = k^{1/\lambda}$ for every $k = 1,2,\dots$ Here as in 7. we have $0 < \lambda \le 1$, [7].
17. If $\underline{d}(x_n) > 0$, then all d.f.s $g(x) \in G(X_n)$ are continuous, nonsingular and bounded by $h_1(x) \le g(x) \le h_2(x)$, where

$$h_1(x) = \begin{cases} x\frac{\underline{d}}{\overline{d}} & \text{if } x \in \left[0, \frac{1-\overline{d}}{1-\underline{d}}\right], \\ \frac{\underline{d}}{\frac{1}{x}-(1-\underline{d})} & \text{otherwise,} \end{cases} \qquad h_2(x) = \min\left(x\frac{\overline{d}}{\underline{d}}, 1\right).$$

Furthermore, there exists x_n, $n = 1,2,\dots$, such that $h_2(x) \in G(X_n)$ and for every x_n we have $h_1(x) \notin G(X_n)$, [3, Th. 7] and moreover
18. for a given fixed $g(x) \in G(X_n)$, $x \in [0,1]$ we have $h_{1,g}(x) \le g(x) \le h_{2,g}(x)$, where

$$h_{1,g}(x) = \begin{cases} x\frac{\underline{d}}{d_g} & \text{if } x < y_0 = \frac{1-d_g}{1-\underline{d}}, \\ x\frac{1}{d_g} + 1 - \frac{1}{d_g} & \text{if } y_0 \le x \le 1, \end{cases}$$

$$h_{2,g}(x) = \min\left(x\frac{\overline{d}}{d_g}, 1\right)$$

[3, Th. 6].
19. These boundaries are established by observing that for every $g(x) \in G(X_n)$

$$0 \le \frac{g(y) - g(x)}{y - x} \le \frac{1}{d_g}$$

for $x < y$, $x, y \in [0,1]$.

4. Overview of proofs

In this section we give proofs of some properties described in Section 3.

4.1. Basic properties

Using

$$x_i < xx_m \Longleftrightarrow x_i < \left(x\frac{x_m}{x_n}\right)x_n$$

and that these inequalities imply $i < m$, it directly follows from definition $F(X_n, x)$ that

$$F(X_m, x) = \frac{n}{m}F\left(X_n, x\frac{x_m}{x_n}\right), \tag{1}$$

for every $m \leq n$ and $x \in [0, 1)$. Also for any increasing sequence of positive integers x_n, $n = 1, 2, \dots$, we define a counting function $A(t)$ as

$$A(t) = \#\{n \in \mathbb{N};\ x_n < t\}.$$

Then for every $x \in (0, 1]$ we have the equality

$$\frac{nF(X_n, x)}{xx_n} = \frac{A(xx_n)}{xx_n}, \tag{2}$$

which we shall use to compute the asymptotic density of x_n. We have the lower asymptotic density $\underline{d}$, and the upper asymptotic density $\overline{d}$ of x_n, $n = 1, 2, \dots$ as

$$\underline{d} = \liminf_{t\to\infty} \frac{A(t)}{t} = \liminf_{n\to\infty} \frac{n}{x_n}, \quad \overline{d} = \limsup_{t\to\infty} \frac{A(t)}{t} = \limsup_{n\to\infty} \frac{n}{x_n}.$$

Using Helly's selection principle from the sequence (m, n) we can select a subsequence (m_k, n_k) such that $F(X_{n_k}) \to g(x)$, $F(X_{m_k}) \to \tilde{g}(x)$ as $k \to \infty$, furthermore $x_{m_k}/x_{n_k} \to \beta$ and $m_k/n_k \to \alpha$, but α may be infinity. These limits have the following connection.

Theorem 1 ([24, Prop. 3.1]). *Let m_k and n_k be two increasing integer sequences satisfying $m_k \leq n_k$, for $k = 1, 2, \dots$ and assume that*

(i) $\lim_{k\to\infty} F(X_{n_k}, x) = g(x)$ *a.e.,*
(ii) $\lim_{k\to\infty} F(X_{m_k}, x) = \tilde{g}(x)$ *a.e.,*
(iii) $\lim_{k\to\infty} \frac{x_{m_k}}{x_{n_k}} = \beta > 0$,
(iv) $g(\beta - 0) > 0$.

Then there exists $\lim_{k\to\infty} \frac{n_k}{m_k} = \alpha < \infty$ *such that*

$$\tilde{g}(x) = \alpha g(x\beta) \quad \text{a.e. on } [0, 1], \quad \text{and} \quad \alpha = \frac{\tilde{g}(1 - 0)}{g(\beta - 0)}. \tag{3}$$

P r o o f. Firstly we prove

$$\lim_{k\to\infty} F\left(X_{n_k}, x\frac{x_{m_k}}{x_{n_k}}\right) = g(x\beta). \tag{4}$$

Denoting $\beta_k = x_{m_k}/x_{n_k}$ and substituting $u = x\beta_k$, we find

$$0 \le \int_0^1 \big(F(X_{n_k}, x\beta_k) - g(x\beta_k)\big)^2 \,\mathrm{d}x = \frac{1}{\beta_k}\int_0^{\beta_k} \big(F(X_{n_k}, u) - g(u)\big)^2 \,\mathrm{d}u$$
$$\le \frac{1}{\beta_k}\int_0^1 \big(F(X_{n_k}, u) - g(u)\big)^2 \,\mathrm{d}u \to 0,$$

which leads to $\big(F(X_{n_k}, x\beta_k) - g(x\beta_k)\big) \to 0$ a.e. as $k \to \infty$ (here necessarily $\beta > 0$). Furthermore,

$$\int_0^1 \big(F(X_{n_k}, x\beta_k) - g(x\beta)\big)^2 \,\mathrm{d}x$$
$$= \int_0^1 \big(F(X_{n_k}, x\beta_k) - g(x\beta_k) + g(x\beta_k) - g(x\beta)\big)^2 \,\mathrm{d}x$$
$$\le 2\left(\int_0^1 \big(F(X_{n_k}, x\beta_k) - g(x\beta_k)\big)^2 \,\mathrm{d}x + \int_0^1 \big(g(x\beta_k) - g(x\beta)\big)^2 \,\mathrm{d}x\right).$$

Since $g(x)$ is continuous a.e. on $[0,1]$ then $\big(g(x\beta_k) - g(x\beta)\big) \to 0$ a.e. and applying the Lebesgue theorem of dominant convergence we find $\int_0^1 \big(g(x\beta_k) - g(x\beta)\big)^2 \mathrm{d}x \to 0$. This gives (4). The existence of the limit $\lim_{k\to\infty} \frac{n_k}{m_k} = \alpha < \infty$ follows from (1) and (iv). Now, let $t_n \in [0,1)$ increases to 1 and $\tilde{g}(x)$ be continuous in t_n. Then $g(x\beta)$ is also continuous in t_n and $\tilde{g}(t_n) = \alpha g(t_n\beta)$ for $n = 1, 2, \dots$. The limit of this equation gives the desired form of α. □

The equality (2) gives

Theorem 2 ([24, Prop. 6.1]). *Assume for a sequence n_k, $k = 1, 2, \dots$ that*

(i) $\lim_{k\to\infty} F(X_{n_k}, x) = g(x)$,

(ii) $\lim_{k\to\infty} \frac{n_k}{x_{n_k}} = d_g$.

Then there exists

(iii) $\lim_{k\to\infty} \frac{A(xx_{n_k})}{xx_{n_k}} = d_g(x)$ *and*

$$g(x) = \frac{x}{d_g} d_g(x). \tag{5}$$

Here the limits (i) *and* (iii) *can be considered for all $x \in (0,1]$ or all continuity points $x \in (0,1]$ of $g(x)$.*

4.2. Continuity of $g \in G(X_n)$

If all $g \in G(X_n)$ are everywhere continuous on $[0,1]$, then relation (3) is of the form

$$\frac{g(x\beta)}{g(\beta)} \in G(X_n). \tag{6}$$

As a criterion for continuity of all $g \in G(X_n)$ we can adapt the Wiener-Schoenberg theorem (cf. [12, 6, p. 55]), but here we give the following simple sufficient condition.

Theorem 3 ([24, Th. 4.1]). *Assume that all d.f.s in $G(X_n)$ are continuous at 1. Then all d.f.s in $G(X_n)$ are continuous on $(0,1]$, i.e., the only discontinuity point can be 0.*

Proof. Assume that $x_{m_k}/x_{n_k} \to \beta$ and $F(X_{n_k}, x) \to g(x)$ as $k \to \infty$. If from (m_k, n_k) we can select two sequences (m'_k, n'_k) and (m''_k, n''_k) such that $n'_k/m'_k \to \alpha_1$ and $n''_k/m''_k \to \alpha_2$ with a finite $\alpha_1 \neq \alpha_2$, then $\alpha_1 g(x\beta)$, $\alpha_2 g(x\beta) \in G(X_n)$ and thus one of such d.f. $\tilde{g}(x)$ must be discontinuous at 1 (it holds also for g continuous at β). Thus, assuming that $G(X_n)$ has only continuous d.f.s at 1, the limits $x_{m_k}/x_{n_k} \to \beta > 0$ and $F(X_{n_k}, x) \to g(x)$ imply the convergence of n_k/m_k. Now by [24, Th. 3.2]: If β is a point of discontinuity of $g(x)$ with $g(\beta + 0) - g(\beta - 0) = h > 0$, then there exists a closed interval $I \subset [0,1]$, with length $|I| \geq h$ such that for every $\frac{1}{\alpha} \in I$ we have $\alpha g(x\beta) \in G(X_n)$. Thus $g(x)$ cannot have a discontinuity point in $(0,1]$. $\square$

Theorem 4 ([24, Th. 6.2]).

(i) *If $\overline{d} > 0$, then there exits $g \in G(X_n)$ such that $g(x) \leq x$ for every $x \in [0,1]$.*

(ii) *If $\underline{d} > 0$, then there exits $g \in G(X_n)$ such that $g(x) \geq x$ for every $x \in [0,1]$.*

(iii) *If $\underline{d} > 0$, then for every $g \in G(X_n)$ we have*

$$(\underline{d}/\overline{d})x \leq g(x) \leq (\overline{d}/\underline{d})x \tag{7}$$

for every $x \in [0,1]$.

(iv) *If $\underline{d} > 0$, then every $g \in G(X_n)$ is everywhere continuous in $[0,1]$.*

(v) *If $\underline{d} > 0$, then for every limit point $\beta > 0$ of x_m/x_n there exist $g \in G(X_n)$ and $0 \leq \alpha < \infty$ such that $\alpha g(x\beta) \in G(X_n)$.*

Proof. (i). Assume that $n_k/x_{n_k} \to \overline{d}$ as $k \to \infty$. Select a subsequence n'_k of n_k such that $F(X_{n'_k}, x) \to g(x)$ a.e. on $[0,1]$. Since $d_g(x) \leq \overline{d}$ a.e. in (5) gives $\big(g(x)/x\big)\overline{d} \leq \overline{d}$ a.e., which leads to $g(x) \leq x$ a.e. and implies $g(x) \leq x$ for every $x \in [0,1]$.

(ii). Similarly to (i), let $n_k/x_{n_k} \to \underline{d}$ as $k \to \infty$. Select a subsequence n'_k of n_k such that $F(X_{n'_k}, x) \to g(x)$ a.e. on $[0,1]$. Since $d_2(x) \geq \underline{d}$ a.e., (5) implies $\big(g(x)/x\big)\underline{d} \geq \underline{d}$ a.e. again, which gives $g(x) \geq x$ a.e., whence, $g(x) \geq x$ everywhere on $x \in [0,1]$.

(iii). For any $g \in G(X_n)$ there exists n_k such that $F(X_{n_k}, x) \to g(x)$ a.e. From n_k we can choose a subsequence n'_k such that $n'_k/x_{n'_k} \to d_1$. Using (5) and the fact that $\underline{d} \leq d_1 \leq \overline{d}$ and $\underline{d} \leq d_2 \leq \overline{d}$ we have $\big(g(x)/x\big)\underline{d} \leq \overline{d}$ and $\big(g(x)/x\big)\overline{d} \geq \underline{d}$ a.e. If $\underline{d} > 0$, these inequalities are valid for every $x \in (0,1]$.

(iv). Continuity of $g \in G(X_n)$ at 1 follows from [24, Prop. 4.2]: Denote

$$\overline{d}(\varepsilon) = \limsup_{n\to\infty} \frac{\#\{i \leq n; (1-\varepsilon)x_n < x_i < x_n\}}{n}.$$

Every $g \in G(X_n)$ is continuous at 1 if and only if $\lim_{\varepsilon\to 0} \overline{d}(\varepsilon) = 0$. Since

$$\overline{d}(\varepsilon) \leq \limsup_{n\to\infty} \varepsilon \frac{x_n}{n} = \frac{\varepsilon}{\underline{d}},$$

applying [24, Th. 4.1] = Theorem 3, we have continuity of g in $(0,1]$. Continuity at 0 follows from (7).

(v). It follows from the fact that if $\underline{d} > 0$ and $\lim_{k\to\infty} x_{m_k}/x_{n_k} = \beta > 0$ for $m_k < n_k$, then $\limsup_{k\to\infty} n_k/m_k < \infty$. More precisely, if we pick (m'_k, n'_k) from (m_k, n_k) such that $n'_k/m'_k \to \alpha$, then

$$\frac{\underline{d}}{\overline{d}\beta} \leq \alpha \leq \frac{\overline{d}}{\underline{d}\beta}. \tag{8}$$

This is so because if we select (m''_k, n''_k) from (m'_k, n'_k) such that $n''_k/x_{n''_k} \to d_1$ and $m''_k/x_{m''_k} \to d_2$, then, by

$$\frac{n''_k}{m''_k} = \frac{\frac{n''_k}{x_{n''_k}} x_{n''_k}}{\frac{m''_k}{x_{m''_k}} x_{m''_k}},$$

we see $\alpha = d_1/(d_2\beta)$. □

4.3. Singleton $G(X_n) = \{g\}$

For general $G(X_n)$, the connection between $G(X_n)$ and $G(x_m/x_n \bmod 1)$ is open, but for singleton $G(X_n)$ we have

Theorem 5 ([24, Th. 8.1])**.** *If $G(X_n) = \{g\}$, then $G(x_m/x_n \bmod 1) = \{g\}$.*

Proof. A proof of the theorem is the same as the proof of [19, Prop. 1, (ii)], since

$$\lim_{n\to\infty} \frac{|X_n|}{|X_1| + \cdots + |X_n|} = \lim_{n\to\infty} \frac{n}{n(n+1)/2} = 0.$$

□

Theorem 6 ([24, Th. 8.2]). *Assume that $G(X_n) = \{g\}$. Then either*

(i) *$g(x) = c_0(x)$ for $x \in [0,1]$ or*

(ii) *$g(x) = x^\lambda$ for some $0 < \lambda \le 1$ and $x \in [0,1]$. Moreover,*

(iii) *if $\overline{d} > 0$ then $g(x) = x$.*

Proof. Let $G(X_n) = \{g\}$. We divide the proof into the following six steps.

(I). By [24, Th. 7.1], we have $\int_0^1 g(x)\mathrm{d}x \ge \frac{1}{2}$ which implies $g(x) \ne c_1(x)$.

(II). g must be continuous on $(0,1)$, since otherwise [24, Th. 3.2], for a discontinuity point $\beta \in (0,1)$, guarantees the existence of $\alpha_1 \ne \alpha_2$ such that $\alpha_1 g(x\beta) = \alpha_2 g(x\beta) = g(x)$ a.e. which is a contradiction.

(III). Assume that $g(x)$ increases in every point $\beta \in (0,1)$. In this case relation (5) gives the well-known Cauchy equation $g(x)g(\beta) = g(x\beta)$ for a.e. $x, \beta \in [0,1]$ For a monotonic $g(x)$ the Cauchy equation has solutions only of the type $g(x) = x^\lambda$.

(IV). Assume that $g(x)$ has a constant value on the interval $(\gamma,\delta) \subset [0,1]$. For $\beta \in (0,1]$ $g(x)$ satisfies two conditions: (j) $g(x)$ increases in β and (jj) $g(\beta) > 0$. Then the basic relation (3) gives $g(x) = \alpha g(x\beta)$ which implies that $g(x)$ has a constant value also on $\beta(\gamma,\delta)$ and if $\delta \le \beta$ then also on $\beta^{-1}(\gamma,\delta)$. Thus, if (γ_i,δ_i), $i \in \mathcal{I}$ is a system of all intervals (maximal under inclusion) in which $g(x)$ possesses constant values, then for every $i \in \mathcal{I}$ there exists $j \in \mathcal{I}$ such that $\beta(\gamma_i,\delta_i) = (\gamma_j,\delta_j)$ and vice-versa for every $j \in \mathcal{I}$, $\delta_j \le \beta$, there exists $i \in \mathcal{I}$ such that $\beta^{-1}(\gamma_j,\delta_j) = (\gamma_i,\delta_i)$. This is true also for $\beta = \beta_1^{n_1}\beta_2^{n_2}\dots$, where $\beta_1,\beta_2,\dots$ satisfy (j) and (jj) and $n_1,n_2,\dots \in \mathbb{Z}$. Thus, there exists $0 < \theta < 1$ such that every such β has the form θ^n, $n \in \mathbb{N}$. The end points γ_i, δ_i (without $\gamma_i = 0$) satisfy (j) and (jj) and thus the intervals (γ_i,δ_i) is of the form (θ^n,θ^{n-1}), $n = 1,2,\dots$ and all discontinuity points of $g(x)$ are θ^n, $n = 1,2,\dots$, a contradiction with (II). For $g(x) = c_0(x)$ there exists no $\beta \in (0,1]$ satisfying (j) and (jj).

(V). We have the possibilities $g(x) = c_0(x)$ and $g(x) = x^\lambda$ for some $\lambda > 0$. Applying [24, Th. 7.1] we have $\int_0^1 g(x)\,\mathrm{d}x \ge 1/2$ which reduces λ to $\lambda \le 1$.

(VI). If $\overline{d} > 0$, then by [24, Th. 6.2, (i)] = Theorem 4 must be $g(x) \le x$ which is contrary to $x^\lambda > x$ for $\lambda < 1$. □

The possibilities (i), (ii) are achievable. Trivially, for $x_n = [n^\lambda]$, $G(X_n) = \{x^{1/\lambda}\}$ and for x_n satisfying $\lim_{n\to\infty} x_n/x_{n+1} = 0$ we have $G(X_n) = \{c_0(x)\}$. Less trivially, every lacunary x_n, i.e., $x_n/x_{n+1} \le \lambda < 1$, gives $G(X_n) = \{c_0(x)\}$.

The following limit covers all of $G(X_n) = \{g\}$.

Theorem 7 ([24, Th. 8.3]). *The set $G(X_n)$ is a singleton if and only if*

$$\lim_{m,n\to\infty}\left(\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}\left|\frac{x_i}{x_m}-\frac{x_j}{x_n}\right| - \frac{1}{2m^2}\sum_{i,j=1}^{m}\left|\frac{x_i}{x_m}-\frac{x_j}{x_m}\right| - \frac{1}{2n^2}\sum_{i,j=1}^{n}\left|\frac{x_i}{x_n}-\frac{x_j}{x_n}\right|\right)=0. \tag{9}$$

P r o o f. It follows directly from the limit (9) in the form

$$\lim_{m,n\to\infty}\int_0^1\bigl(F(X_m,x)-F(X_n,x)\bigr)^2\,\mathrm{d}x=0,$$

after applying

$$\int_0^1\bigl(g(x)-\tilde{g}(x)\bigr)^2\,\mathrm{d}x = \int_0^1\!\!\int_0^1|x-y|\,\mathrm{d}g(x)\,\mathrm{d}\tilde{g}(y) - \frac{1}{2}\int_0^1\!\!\int_0^1|x-y|\,\mathrm{d}g(x)\,\mathrm{d}g(y) - \frac{1}{2}\int_0^1\!\!\int_0^1|x-y|\,\mathrm{d}\tilde{g}(x)\,\mathrm{d}\tilde{g}(y) \tag{10}$$

for $g(x)=F(X_m,x)$ and $\tilde{g}(x)=F(X_n,x)$. □

4.4. U.d. of X_n

By Theorem 5, u.d. of the single block sequence X_n implies the u.d. of the ratio sequence x_m/x_n. Applying [24, Th. 6.3, (i)] $(\underline{d}/\overline{d})x \le g(x) \le (\overline{d}/\underline{d})x$ for every $x\in[0,1]$, we have

Theorem 8. *If the increasing sequence x_n of positive integers has a positive asymptotic density, i.e., $\underline{d}=\overline{d}>0$, then the associated ratio sequence x_m/x_n, $m=1,2,\dots,n$, $n=1,2,\dots$ is u.d. in $[0,1]$.*

Positive asymptotic density is not necessary. According to T. Š a l á t [16] we can use also a sequence x_n with $\underline{d}=0$.

Theorem 9 ([24, Th. 9.2]). *Let x_n be an increasing sequence of positive integers and $h\colon[0,\infty)\to[0,\infty)$ be a function satisfying*

(i) *$A(x)\sim h(x)$ as $x\to\infty$, where*

(ii) *$h(xy)\sim xh(y)$ as $y\to\infty$ and for every $x\in[0,1]$, and*

(iii) *$\lim_{n\to\infty}\frac{n}{h(x_n)}=1$.*

Then X_n (and consequently x_m/x_n) is u.d. in $[0,1]$.

Proof. Starting with (2) $F(X_n, x)n = A(xx_n)$ it follows from (i) that

$$\frac{F(X_n, x)n}{h(xx_n)} \to 1$$

as $n \to \infty$, then by (ii)

$$\frac{F(X_n, x)n}{xh(x_n)} \to 1$$

which gives by (iii) the limit

$$F(X_n, x)\frac{n}{h(x_n)} \to x$$

as $n \to \infty$. □

Assuming only (i) and (ii), we have $\liminf_{n\to\infty} n/h(x_n) \geq 1$, since otherwise $n_k/h(x_{n_k}) \to \alpha < 1$ implies $F(X_{n_k}, x) \to x/\alpha$ for every $x \in [0, 1]$ which is a contradiction. Also, $G(X_n) \subset \{x\lambda; \lambda \in [0, 1]\}$.

Another criterion can be found by using the so called L^2 discrepancy of the block X_n defined by

$$D^{(2)}(X_n) = \int_0^1 \big(F(X_n, x) - x\big)^2 \,\mathrm{d}x,$$

which can be expressed (cf. [19, IV. Appl.]) as

$$D^{(2)}(X_n) = \frac{1}{n^2}\sum_{i,j=1}^{n} F\left(\frac{x_i}{x_n}, \frac{x_j}{x_n}\right),$$

where

$$F(x, y) = \frac{1}{3} + \frac{x^2 + y^2}{2} - \frac{x + y}{2} - \frac{|x - y|}{2}.$$

Thus

$$D^{(2)}(X_n) = \frac{1}{3} + \frac{1}{nx_n^2}\sum_{i=1}^{n} x_i^2 - \frac{1}{nx_n}\sum_{i=1}^{n} x_i - \frac{1}{2n^2x_n}\sum_{i,j=1}^{n} |x_i - x_j|,$$

which gives (cf. [19]).

Theorem 10. *For every increasing sequence x_n of positive integers we have*

$$\lim_{n\to\infty} D^{(2)}(X_n) = 0 \iff \lim_{n\to\infty} F(X_n, x) = x.$$

The left hand-side can be divided into three limits (cf. [18, Th. 1])

$$\lim_{n\to\infty} D^{(2)}(X_n) = 0 \iff \begin{cases} (i)\ \lim_{n\to\infty} \frac{1}{nx_n}\sum_{i=1}^{n} x_i = \frac{1}{2}, \\ (ii)\ \lim_{n\to\infty} \frac{1}{nx_n^2}\sum_{i=1}^{n} x_i^2 = \frac{1}{3}, \\ (iii)\ \lim_{n\to\infty} \frac{1}{n^2x_n}\sum_{i,j=1}^{n} |x_i - x_j| = \frac{1}{3}. \end{cases}$$

Weyl's criterion for u.d. of X_n is not well applicable in our case. It says (cf. [17, (7)]).

Theorem 11. *X_n is u.d. if and only if*

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} e^{2\pi i h \frac{x_k}{x_n}} = 0$$

for all positive integers h.

4.5. One-step d.f. $c_\alpha(x)$

In [24] there is proved that singleton $G(X_n) = \{c_1(x)\}$ does not exist, since (by [24, Th. 7.1]) for every increasing sequence x_n of positive integers we have

$$\max_{g(x)\in G(X_n)} \int_0^1 g(x)\,\mathrm{d}x \ge \frac{1}{2}. \tag{11}$$

In [24] is also proved (see Th. 8.4, 8.5) that

Theorem 12.

$$G(X_n) = \{c_0(x)\} \Longleftrightarrow \lim_{n\to\infty}\frac{1}{nx_n}\sum_{i=1}^{n} x_i = 0, \tag{12}$$

$$G(X_n) = \{c_0(x)\} \Longleftrightarrow \lim_{n\to\infty}\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}\left|\frac{x_i}{x_m} - \frac{x_j}{x_n}\right| = 0, \tag{13}$$

$$G(X_n) \subset \{c_\alpha(x); \alpha\in[0,1]\} \Longleftrightarrow \lim_{n\to\infty}\frac{1}{n^2 x_n}\sum_{i,j=1}^{n}|x_i - x_j| = 0. \tag{14}$$

P r o o f.

(12). $\int_0^1 x\,\mathrm{d}g(x) = 1 - \int_0^1 g(x)\,\mathrm{d}x = 0$ only if $g(x) = c_0(x)$.

(13). Assume that $F(X_{m_k}, x) \to \tilde{g}(x)$ and $F(X_{n_k}, x) \to g(x)$ a.e. as $k \to \infty$. Riemann-Stieltjes integration yields

$$\frac{1}{m_k n_k}\sum_{i=1}^{m_k}\sum_{j=1}^{n_k}\left|\frac{x_i}{x_{m_k}} - \frac{x_j}{x_{n_k}}\right| = \int_0^1\int_0^1 |x-y|\,\mathrm{d}F(X_{m_k}, x)\,\mathrm{d}F(X_{n_k}, y) \tag{15}$$

which, after using Helly's theorem, tends to

$$\int_0^1\int_0^1 |x-y|\,\mathrm{d}\tilde{g}(x)\,\mathrm{d}g(y) \tag{16}$$

as $k \to \infty$. Then (16) is equal to 0 if and only if $\tilde{g}(x) = g(x) = c_\alpha(x)$ for some fixed $\alpha \in [0,1]$. By Theorem 6, α must be 0 ($\overline{d} = 0$ follows from Theorem 4, part (i)).

(14). Again $\int_0^1\int_0^1 |x-y|\,\mathrm{d}g(x)\,\mathrm{d}g(y)=0$ if and only if $g(x)=c_\alpha(x)$ for $\alpha\in[0,1]$ and thus

$$\lim_{k\to\infty}\frac{1}{n_kn_k}\sum_{i=1}^{n_k}\sum_{j=1}^{n_k}\left|\frac{x_i}{x_{n_k}}-\frac{x_j}{x_{n_k}}\right| = 0$$

for every $n_k\to\infty$. □

Furthermore, if $G(X_n)\subset\{c_\alpha(x);\alpha\in[0,1]\}$, then $\underline{d}(x_n)=0$. Here we prove that

Theorem 13 ([9, Th. 6]). *Let x_n, $n=1,2,\dots$, be an increasing sequence of positive integers. Assume that $G(X_n)\subset\{c_\alpha(x);\alpha\in[0,1]\}$. Then $c_0(x)\in G(X_n)$ and if $G(X_n)$ contains two different d.f.s, then also $c_1(x)\in G(X_n)$.*

Proof. We start from the equation (2) (see [24, p. 756, (1)])

$$F(X_m,x)=\frac{n}{m}F\left(X_n,x\frac{x_m}{x_n}\right),$$

which is valid for every $m\le n$ and $x\in[0,1]$. Assuming, for two increasing sequences of indices $m_k\le n_k$, that, as $k\to\infty$

(i) $F(X_{m_k},x)\to c_{\alpha_1}(x)$ a.e.,

(ii) $F(X_{n_k},x)\to c_{\alpha_2}(x)$ a.e.,

(iii) $\frac{n_k}{m_k}\to\gamma$,

(iv) $\frac{x_{m_k}}{x_{n_k}}\to\beta$,

(such sequences $m_k\le n_k$ exist by Helly theorem) then we have:

a) If $\beta>0$ and $\gamma<\infty$ (see (3) in [24]), then

$$c_{\alpha_1}(x)=\gamma c_{\alpha_2}(x\beta) \tag{13}$$

for almost all $x\in[0,1]$.

b) If $\beta=0$ and $\gamma<\infty$, then by Helly theorem there exists subsequence (m'_k,n'_k) of (m_k,n_k) such that $F\Big(X_{n'_k},x\frac{x_{m'_k}}{x_{n'_k}}\Big)\to h(x)$ a.e. and since

$$F\left(X_{n_k},x\frac{x_{m'_k}}{x_{n'_k}}\right)\le F(X_{n_k},x\beta')$$

for every $\beta'>0$ and sufficiently large k, we get $h(x)\le c_{\alpha_2}(x\beta')$. Summarizing, we have

$$c_{\alpha_1}(x)\le\gamma c_{\alpha_2}(x\beta') \tag{14}$$

for every $\beta'>0$ a.e. on $[0,1]$.

We distinguish the following steps (notions (i)–(iv), a) and b) are preserve):

1^0. Let $c_{\alpha_1}(x)\in G(X_n)$, $0\le\alpha_1<1$, and let m_k, $k=1,2,\dots$, be an increasing sequence of positive integers for which

(i) $F(X_{m_k}, x) \to c_{\alpha_1}(x)$.
Relatively to the m_k, we choose an arbitrary sequence n_k, $m_k \le n_k$, such that

(iii) $\frac{n_k}{m_k} \to \gamma$, $1 < \gamma < \infty$.
From (m_k, n_k) we select a subsequence (m'_k, n'_k) such that

(ii) $F(X_{n'_k}, x) \to c_{\alpha_2}(x)$ a.e. on $[0, 1]$,

(iv) $\frac{x_{m'_k}}{x_{n'_k}} \to \beta$ for some $\beta \in [0, 1]$.

a) If $\beta > 0$, then (13) $c_{\alpha_1}(x) = \gamma c_{\alpha_2}(x\beta)$ a.e. is impossible, because $\gamma > 1$ and for $x > \alpha_1$ we have $c_{\alpha_1}(x) = 1$. Thus $\beta = 0$.

b) The condition $\beta = 0$ implies (14) $c_{\alpha_1}(x) \le \gamma c_{\alpha_2}(x\beta')$ for every $\beta' > 0$ and a.e. on $x \in [0, 1]$. If $\alpha_2 > 0$, then $c_{\alpha_2}(x\beta') = 0$ for all $x < \frac{\alpha_2}{\beta'}$, which implies, using $\beta' \le \alpha_2$, that $c_{\alpha_1}(x) = 0$ for $x \in (0, 1)$, and this is contrary to the assumption $\alpha_1 < 1$.

Thus $\alpha_2 = 0$ and we have: If $0 \le \alpha_1 < 1$ and $c_{\alpha_1}(x) \in G(X_n)$ then $c_0(x) \in G(X_n)$. Now, applying [24, Th. 7.1] we have $\max_{c_\alpha(x)\in G(X_n)} \int_0^1 c_\alpha(x)\,\mathrm{d}x = 1 - \alpha \ge \frac{1}{2}$. Then the assumption $c_{\alpha_1}(x) \in G(X_n)$, $0 \le \alpha_1 < 1$ is true, thus $c_0(x) \in G(X_n)$ holds.

2^0 In this case we start with the sequence n_k and we assume that $c_{\alpha_2}(x) \in G(X_n)$, $0 < \alpha_2 \le 1$, and

(ii) $F(X_{n_k}, x) \to c_{\alpha_2}(x)$ a.e. on $[0, 1]$.
Then we choose arbitrary m_k such that $m_k \le n_k$ and

(iii) $\frac{n_k}{m_k} \to \gamma$, $1 < \gamma < \infty$.
From (m_k, n_k) we select a subsequence (m'_k, n'_k) such that

(ii) $F(X_{m'_k}, x) \to c_{\alpha_1}(x)$ a.e. on $[0, 1]$,

(iv) $\frac{x_{m'_k}}{x_{n'_k}} \to \beta$ for some $\beta \in [0, 1]$.

a) If $\beta > 0$, then by (13) $c_{\alpha_1}(x) = \gamma c_{\alpha_2}(x\beta)$ a.e. If $\alpha_1 < 1$, then $\gamma > 1$ implies $c_{\alpha_1}(x) > 1$ for some $x \in (0, 1)$, a contradiction. Thus $\alpha_1 = 1$ (in this case $\beta \le \alpha_2$).

b) Now, $\beta = 0$ implies (14) $c_{\alpha_1}(x) \le \gamma c_{\alpha_2}(x\beta')$ for every $\beta' > 0$ and a.e. on $x \in [0, 1]$ and the assumption $\alpha_2 > 0$ implies $c_{\alpha_2}(x\beta') = 0$ for all $x < \frac{\alpha_2}{\beta'}$, which gives $\alpha_1 = 1$. Summarizing, if $G(X_n)$ contains two different d.f.s, then it contains $c_0(x)$ and $c_1(x)$ simultaneously. □

4.6. Connectivity of $G(X_n)$

As we have mentioned in the introduction, for a usual sequence y_n the set $G(y_n)$ of all d.f. of y_n is nonempty, closed and connected in the weak topology, and consists either of one or infinitely many functions. The closedness of $G(X_n)$ is clear, but connectivity of $G(X_n)$ is open. A general block sequence Y_n with non-connected $G(Y_n)$ can be found trivially. For our special X_n we have only the following sufficient condition.

Theorem 14 ([24, Th. 5.1]). *If*

$$\lim_{n\to\infty}\left(\frac{1}{n(n+1)}\sum_{i=1}^{n+1}\sum_{j=1}^{n}\left|\frac{x_i}{x_{n+1}}-\frac{x_j}{x_n}\right| - \frac{1}{2(n+1)^2}\sum_{i,j=1}^{n+1}\left|\frac{x_i}{x_{n+1}}-\frac{x_j}{x_{n+1}}\right| - \frac{1}{2n^2}\sum_{i,j=1}^{n}\left|\frac{x_i}{x_n}-\frac{x_j}{x_n}\right|\right) = 0, \tag{17}$$

then $G(X_n)$ is connected in the weak topology.

Proof. The connection follows from the limit

$$\lim_{n\to\infty}\int_0^1 \left(F(X_{n+1},x)-F(X_n,x)\right)^2 \,\mathrm{d}x = 0,$$

since by a theorem of H. G. Barone [2] if t_n is a sequence in a metric space (X,ρ) satisfying

(i) any subsequence of t_n contains a convergent subsequence and

(ii) $\lim_{n\to\infty}\rho(t_n,t_{n+1})=0$,

then the set of all limit points of t_n is connected. Next we use the expression

$$\int_0^1 \big(g(x)-\tilde{g}(x)\big)^2\,\mathrm{d}x = \int_0^1\int_0^1 |x-y|\,\mathrm{d}g(x)\,\mathrm{d}\tilde{g}(y) - \frac{1}{2}\int_0^1\int_0^1 |x-y|\,\mathrm{d}g(x)\,\mathrm{d}g(y) - \frac{1}{2}\int_0^1\int_0^1 |x-y|\,\mathrm{d}\tilde{g}(x)\,\mathrm{d}\tilde{g}(y).$$

Putting $g(x)=F(X_{n+1},x)$ and $\tilde{g}(x)=F(X_n,x)$ we get the desired limit.[3] □

As a consequence we have:

Theorem 15. *If $\lim_{n\to\infty}\frac{x_n}{x_{n+1}}=1$, then $G(X_n)$ is connected.*

[3] $\rho^2(g,\tilde{g})=\int_0^1\big(g(x)-\tilde{g}(x)\big)^2\,\mathrm{d}x$.
In Example 4 is given X_n such that $G(X_n)$ is connected but $\limsup_{n\to\infty}\rho(t_{n+1},t_n)=1$.

P r o o f. After some manipulation (17) it follows from

$$\lim_{n\to\infty}\left(\frac{1}{nx_n}\sum_{i=1}^{n}x_i\right)\left(1-\frac{x_n}{x_{n+1}}\right)=0.$$

□

Note that by [24, Th. 4.1] all d.f.'s in $G(X_n)$ are continuous everywhere on $[0,1]$ if they are continuous at 0 and 1.

In [24, Th. 3.2] is proved that if $g(x)\in G(X_n)$, $g(x)$ increases at $\beta\in[0,1)$, $g(\beta)>0$, then there exists $\alpha\in[1,\infty)$ such that $\alpha g(x\beta)\in G(X_n)$. Using this fact, we can define on $G(X_n)$ the relation $\tilde{g}(x)\prec g(x)$ if there exist α,β such that $\tilde{g}(x)=\alpha g(x\beta)$. For every element $g(x)\in G(X_n)$ we define $\big[g(x)\big]$ as the set of all $\tilde{g}(x)\in G(X_n)$ for which $\tilde{g}(x)\prec g(x)$. Assuming that all d.f.s in $G(X_n)$ are continuous and strictly increasing, then we have

$$\big[g(x)\big]=\big\{g(x\beta)/g(\beta);\beta\in(0,1]\big\}.$$

Denote as $G\big(g(x)\big)$ the set of all possible limits $\lim_{k\to\infty}g(x\beta_k)/g(\beta_k)$, where $\beta_k\to 0$ and put

$$\big[g(x)\big]^*=\big[g(x)\big]\cup G\big(g(x)\big).$$

Theorem 16. *Assume that all d.f.s in $G(X_n)$ are continuous and strictly increasing. If $G(X_n)=\cup_{i=1}^{k}\big[g_i(x)\big]^*$, then $G(X_n)$ is connected if and only if $g_i(x)$, $i=1,2,\dots,k$ can be reordered into $g_{i_n}(x)$, $n=1,2,\dots,k$ such that*

(i) $\big[g_{i_n}(x)\big]^*\cap\big[g_{i_{n+1}}(x)\big]^*\neq\emptyset$, $n=1,2,\dots,k-1$.

P r o o f. 1^0 Firstly we prove that $\big[g(x)\big]^*$ is nonempty, closed and connected, for every $g(x)\in G(X_n)$. Note that, in the following we say that we can go connectively $g_1(x)\to g_2(x)$ through the set H if for every $\varepsilon>0$ there exists a chain $g_{i_n}(x)\in H$, $n=1,2,\dots,m$ such that $\rho(g_1,g_{i_1})<\varepsilon$, $\rho(g_{i_2},g_{i_3})<\varepsilon,\dots$ $\dots$, $\rho(g_{i_m},g_2)<\varepsilon$.

Connectivity: If $g_1(x)=g(x\beta_1)/g(\beta_1)$ and $g_2(x)=g(x\beta_2)/g(\beta_2)$ then we can go connectively $g_1(x)\to g_2(x)$ through $g(x\beta)/g(\beta)$, where β is between β_1 and β_2, since

$$\frac{g(x\beta)}{g(\beta)}-\frac{g(x\beta')}{g(\beta')}=\left(\frac{g(x\beta)-g(x\beta')}{g(\beta)}+g(x\beta')\frac{g(\beta')-g(\beta)}{g(\beta)g(\beta')}\right)\to 0$$

as $(\beta'-\beta)\to 0$, where $\beta,\beta'\geq\varepsilon>0$.

If $g_1(x)=\lim_{k\to\infty}g(x\beta_k)/g(\beta_k)$ and $g_2(x)=\lim_{k\to\infty}g(x\beta_k')/g(\beta_k')$, then we can go connectively

$$g_1(x)\to g(x\beta_k)/g(\beta_k)\to g(x\beta_k')/g(\beta_k')\to g_2(x)$$

through $\big[g(x)\big]$. Similarly for the rest

$$g_1(x)=g(x\beta_1)/g(\beta_1)\quad\text{and}\quad g_2(x)=\lim_{k\to\infty}g(x\beta_k)/g(\beta_k).$$

Closedness: If $\lim_{k\to\infty} g(x\beta_k)/g(\beta_k) = g_1(x)$, we can select β_k such that $\beta_k \to \beta$. If $\beta > 0$, then from continuity $g(x)$ we have $g_1(x) = g(x\beta)/g(\beta)$. The closedness of $G\big(g(x)\big)$ follows from definition of $G\big(g(x)\big)$.

2^0. Assume that (i) holds and select $g_n^*(x) \in \big[g_{i_n}(x)\big]^* \cap \big[g_{i_{n+1}}(x)\big]^*$, $i = 1, 2, \dots$ $\dots, k-1$. Let $g_1(x) \in \big[g_{i_1}(x)\big]^*$ and $g_2(x) \in \big[g_{i_3}(x)\big]^*$. Then we can go connectively

$$g_1(x) \to \frac{g_{i_1}(x\beta_1)}{g_{i_1}(\beta_1)} \to g_1^*(x) \to \frac{g_{i_2}(x\beta_2)}{g_{i_2}(\beta_2)} \to g_2^*(x) \to \frac{g_{i_3}(x\beta_3)}{g_{i_3}(\beta_3)} \to g_2(x),$$

similarly in a general case.

3^0. Assume that (i) does not hold. Then $\big[g_i(x)\big]^*$, $i = 1, 2, \dots, k$, can be divided into two parts such that

$$\Big(\cup_{i\in A}\big[g_i(x)\big]^*\Big) \cap \Big(\cup_{i\in B}\big[g_i(x)\big]^*\Big) = \emptyset,$$

where $A \cup B = \{1, 2, \dots, k\}$. From closedness of such sets follows $\rho(g, \tilde{g}) \geq \delta > 0$ for some δ and every $g(x) \in \cup_{i\in A}\big[g_i(x)\big]^*$ and $\tilde{g}(x) \in \cup_{i\in B}\big[g_i(x)\big]^*$, which contradicts the connectivity of $G(X_n)$. $\square$

4.7. Boundaries of $g(x) \in G(X_n)$

Theorem 17 ([3, Th. 5]). *For every increasing sequence of positive integers x_n, $n = 1, 2, \dots,$ there exists $g(x) \in G(X_n)$ such that $g(x) \geq x$ for all $x \in [0, 1]$.*

Proof. If $\underline{d} > 0$, select n_k so that $\frac{n_k}{x_{n_k}} \to \underline{d} > 0$, and $F(X_{n_k}, x) \to g(x)$. For such $g(x)$, (5) implies

$$\frac{g(x)}{x}\underline{d} \geq \underline{d}\,.$$

Now, let $\underline{d} = 0$. Select n_k such that

$$\frac{n_k}{x_{n_k}} = \min_{i\leq n_k} \frac{i}{x_i},$$

and $F(X_{n_k}, x) \to g(x)$. Then for every $x \in (0, 1]$,

$$\frac{A(xx_{n_k})}{xx_{n_k}} \geq \frac{n_k - 1}{x_{n_k}}.$$

Applying (2) yields

$$\frac{F(X_{n_k}, x)}{x}\frac{n_k}{x_{n_k}} \geq \frac{n_k - 1}{x_{n_k}},$$

and taking the limit, as $k \to \infty$, we obtain $g(x) \geq x$ for all $x \in [0, 1]$.[4] $\square$

[4] L. Mišík.

Theorem 18 ([3, Th. 6]). *Let $x_1 < x_2 < \dots$ be a sequence of positive integers with positive lower asymptotic density $\underline{d} > 0$, and upper asymptotic density $\overline{d}$. Then all d.f.s $g(x) \in G(X_n)$ are continuous, non-singular, and bounded by $h_1(x) \le g(x) \le h_2(x)$, where*

$$h_1(x) = \begin{cases} x\frac{\underline{d}}{\overline{d}}, & \text{if } x \in \left[0, \frac{1-\overline{d}}{1-\underline{d}}\right]; \\ \frac{\underline{d}}{\frac{1}{x}-(1-\underline{d})}, & \text{otherwise}, \end{cases} \tag{18}$$

$$h_2(x) = \min\left(x\frac{\overline{d}}{\underline{d}}, 1\right). \tag{19}$$

Moreover, $h_1(x)$ and $h_2(x)$ are the best possible in the following sense: for given $0 < \underline{d} \le \overline{d}$, there exists $x_1 < x_2 < \cdots$ with lower and upper asymptotic densities $\underline{d}$, $\overline{d}$, such that $\underline{g}(x) = h_1(x)$ for $x \in \left[\frac{1-\overline{d}}{1-\underline{d}}, 1\right]$; also, there exists $x_1 < x_2 < \cdots$ with given $0 < \underline{d} \le \overline{d}$ such that $\overline{g}(x) = h_2(x) \in G(X_n)$.

Proof. For $g(x) \in G(X_n)$, let n_k, $k = 1, 2, \dots$, be an increasing sequence of indices such that $F(X_{n_k}, x) \to g(x)$. From n_k we can select a subsequence (for simplicity written as the original n_k)[5] such that

$$\frac{n_k}{x_{n_k}} \to d_g > 0. \tag{20}$$

Then, by (5), we have

$$g(x) = x\frac{d_g(x)}{d_g}, \quad \text{where} \quad \frac{A(xx_{n_k})}{xx_{n_k}} \to d_g(x) \tag{21}$$

for arbitrary $x \in (0, 1]$.

We will continue in six steps 1^0–6^0.

1^0. We prove the continuity of $g(x)$ at $x = 1$ (improving (iv) in [24, Th. 6.2]) for each $g(x) \in G(X_n)$.

In view of the definition of the counting function $A(t)$

$$0 \le A(x_{n_k}) - A(xx_{n_k}) \le x_{n_k} - xx_{n_k};$$

thus,

$$0 \le \frac{A(x_{n_k})}{x_{n_k}} - \frac{A(xx_{n_k})}{x_{n_k}} = \frac{n_k - 1}{x_{n_k}} - \frac{A(xx_{n_k})}{xx_{n_k}}x \le 1 - x,$$

and, as $k \to \infty$, we have $0 \le d_g - d_g(x)x \le 1 - x$, which implies

$$0 \le d_g - d_g(x) + d_g(x)(1 - x) \le 1 - x.$$

Consequently, $\lim_{x\to 1} d_g(x) = d_g$, and so $\lim_{x\to 1} g(x) = \lim_{x\to 1} x\frac{d_g(x)}{d_g} = 1$. Since $g(x) \in G(X_n)$ is arbitrary, [24, Th. 4.1, Th. 6.2] gives continuity of $g(x)$ in the whole unit interval $[0, 1]$.

[5] We call d_g a *local asymptotic density* related to $g(x)$.

2^0. We prove that $g(x)$ has a bounded right derivative for every $x \in (0,1)$, and for each $g(x) \in G(X_n)$.

For $0 < x < y < 1$ again

$$0 \le A(yx_{n_k}) - A(xx_{n_k}) \le (y-x)x_{n_k},$$

which implies

$$0 \le \frac{A(yx_{n_k})}{yx_{n_k}}y - \frac{A(xx_{n_k})}{xx_{n_k}}x \le y - x.$$

Letting $k \to \infty$, we get

$$0 \le d_g(y)y - d_g(x)x \le y - x,$$

hence

$$0 \le g(y) - g(x) = \frac{d_g(y)y - d_g(x)x}{d_g} \le \frac{y-x}{d_g}.$$

Consequently,

$$0 \le \frac{g(y)-g(x)}{y-x} \le \frac{1}{d_g} \tag{22}$$

for all $x, y \in (0,1)$, $x < y$, which gives the upper bound of the right derivatives of $g(x)$ for every $x \in (0,1)$. Note that a singular d.f. (continuous, strictly increasing, having zero derivative a.e.) has infinite right Dini derivatives in a dense subset of $(0,1)$.

3^0. We prove a local form of Theorem 17.

As $\underline{d} \le d_g \le \overline{d}$, (21) implies

$$x\frac{\underline{d}}{d_g} \le g(x) \le x\frac{\overline{d}}{d_g} \tag{23}$$

for every $x \in [0,1]$. It follows from (22), that there exists an extreme point $A_g = (x_g, y_g)$ on the line $y = x\frac{\underline{d}}{d_g}$ such that $g(x)$ has no common point with this line for $x > x_g$. This point A_g is the intersection of the lines

$$y = x\frac{\underline{d}}{d_g} \quad \text{and,} \quad y = x\frac{1}{d_g} + 1 - \frac{1}{d_g} \tag{24}$$

therefore,

$$A_g = (x_g, y_g) = \left(\frac{1-d_g}{1-\underline{d}}, \frac{\underline{d}}{d_g}\frac{1-d_g}{1-\underline{d}}\right). \tag{25}$$

It means that for a given $g(x) \in G(X_n)$, $h_{1,g}(x) \le g(x) \le h_{2,g}(x)$, where

$$h_{1,g}(x) = \begin{cases} x\frac{\underline{d}}{d_g}, & \text{if } x < y_0 = \frac{1-d_g}{1-\underline{d}}; \\ x\frac{1}{d_g} + 1 - \frac{1}{d_g}, & \text{if } y_0 \le x \le 1, \end{cases} \tag{26}$$

$$h_{2,g}(x) = \min\left(x\frac{\overline{d}}{d_g}, 1\right). \tag{27}$$

4^0. Now we find $h_1(x)$, and $h_2(x)$ such that

$$h_1(x) \le h_{1,g}(x) \le h_{2,g}(x) \le h_2(x)$$

for every $g \in G(X_n)$.

In the parametric expression (25) of A_g, the local asymptotic density d_g defined by (20) belongs to the interval $[\underline{d}, \overline{d}]$. The well-known Darboux property of the asymptotic density implies that for an arbitrary $d \in [\underline{d}, \overline{d}]$ there exists an increasing n_k, $k = 1, 2, \dots$, such that $\frac{n_k}{x_{n_k}} \to d$[6], and then the Helly selection principle implies the existence of a subsequence of n_k such that $F(X_{n_k}, x) \to g(x)$ for some $g(x) \in G(X_n)$. Thus, if $g(x)$ runs over $G(X_n)$, then d_g runs over the entire interval $[\underline{d}, \overline{d}]$. Substituting $d_g = 1 - x_g(1 - \underline{d})$ in $A_g = (x_g, y_g)$ we get

$$y_g = y_g(x_g) = \frac{\underline{d}}{\frac{1}{x_g} - (1 - \underline{d})},$$

where $x_g = \frac{1-d_g}{1-\underline{d}}$ runs through the interval $I = \left[\frac{1-\overline{d}}{1-\underline{d}}, 1\right]$ for $d_g \in [\underline{d}, \overline{d}]$. By putting $x_g = x$, and $y_g = h_1$ we find a part of $h_1(x)$ for $x \in I$ in (18). The remaining part of $h_1(x)$, and also the whole $h_2(x)$, follow from the basic inequality (23), see [3, Fig. 1.]. The optimality of $h_1(x)$ and $h_2(x)$ are proved in 5^0 and 6^0 pages 518–522 of [3]. [7] $\square$

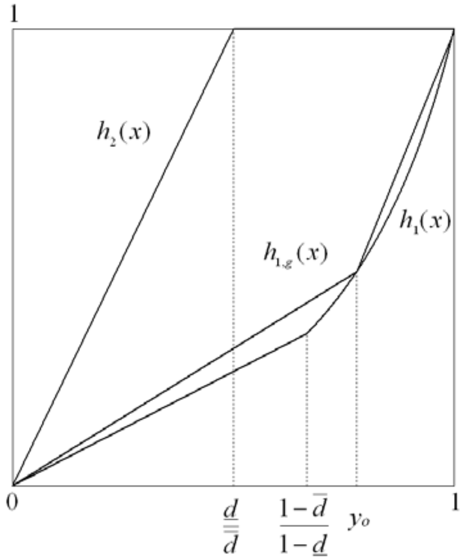


Figure: Boundaries of $g(x) \in G(X_n)$

[6] A simple proof follows from the fact that for every $d \in (\underline{d}, \overline{d})$ there exist infinitely many $n \in \mathbb{N}$ such that $A(n)/n \le d \le A(n+1)/(n+1)$. These n we denote as n_k.

[7] L. Mišík for the idea of (22).

Application

An application of d.f.s in Theorem 18 to elementary number theory:

Theorem 19 ([3, Th. 7]). *For every increasing sequence $x_1 < x_2 < \cdots$ of positive integers with lower and upper asymptotic densities $0 < \underline{d} \le \overline{d}$ we have*

$$\frac{1}{2}\frac{\underline{d}}{\overline{d}} \le \liminf_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \frac{x_i}{x_n}, \tag{28}$$

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \frac{x_i}{x_n} \le \frac{1}{2} + \frac{1}{2}\left(\frac{1-\min(\sqrt{\underline{d}},\overline{d})}{1-\underline{d}}\right)\left(1-\frac{\underline{d}}{\min(\sqrt{\underline{d}},\overline{d})}\right). \tag{29}$$

Here the equations in (28) *and* (29) *can be attained.*

Proof. By Helly theorem, if $F(X_{n_k},x) \to g(x)$, then

$$\int_0^1 x \mathrm{d}F(X_{n_k},x) = \frac{1}{n_k}\sum_{i=1}^{n_k} \frac{x_i}{x_{n_k}} \to \int_0^1 x\,\mathrm{d}g(x) = 1 - \int_0^1 g(x)\,\mathrm{d}x.$$

If $\underline{d} > 0$, then $h_1(x) \le g(x) \le h_2(x)$ which implies

$$1 - \int_0^1 h_2(x)\,\mathrm{d}x \le 1 - \int_0^1 g(x)\,\mathrm{d}x \le 1 - \int_0^1 h_1(x)\,\mathrm{d}x. \tag{30}$$

For $x_1 < x_2 < \cdots$ for which $h_2(x) \in G(X_n)$ in the left of (30) we have equation, but in every case $h_1(x) \notin G(X_n)$ for $0 < \underline{d} < \overline{d}$, which implies strong inequality in the right, i.e.,

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \frac{x_i}{x_n} < 1 - \frac{1}{2}\frac{\underline{d}}{\overline{d}}\left(\frac{1-\overline{d}}{1-\underline{d}}\right)^2 - \frac{\underline{d}}{(1-\underline{d})^2}\left(\log\frac{\underline{d}}{\overline{d}} - (\overline{d}-\underline{d})\right). \tag{31}$$

Since for every $g(x) \in G(X_n)$ in 3^0 we have $h_{1,g}(x) \le g(x) \le h_{2,g}(x)$, then

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \frac{x_i}{x_n} \le \max_{g(x)\in G(X_n)} \left(1 - \int_0^1 h_{1,g}(x)\,\mathrm{d}x\right). \tag{32}$$

If the maximum in (32) is attained in $g_0(x) \in G(X_n)$ and $h_{1,g_0}(x) \in G(X_n)$, then $g_0(x) = h_{1,g_0}(x)$ and we have

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \frac{x_i}{x_n} = 1 - \int_0^1 h_{1,g_0}(x)\,\mathrm{d}x. \tag{33}$$

Using (26) we find

$$\int_0^1 h_{1,g}(x)\,\mathrm{d}x = \frac{1}{2}\left(1 + \frac{1-d_g}{1-\underline{d}}\left(\frac{\underline{d}}{d_g} - 1\right)\right)$$

for $d_g \in [\underline{d}, \overline{d}]$ with derivative $\left(\int_0^1 h_{1,g}(x)\,\mathrm{d}x\right)' = \frac{1}{2(1-\underline{d})}\left(1 - \frac{\underline{d}}{(d_g)^2}\right)$ and which gives that $\min \int_0^1 h_{1,g}(x)\mathrm{d}x$ is attained in $d_{g_0} = \min(\sqrt{\underline{d}}, \overline{d})$.

Now, to prove (33) we can construct integer $x_1 < x_2 < \cdots$ with $0 < \underline{d} \le \overline{d}$ such that $h_{1,g_0}(x) \in G(X_n)$.

We starting with the sequence of indices n_k, and then by (26) we must find indices $m'_k < m_k < n_k$ and integers $x_{m'_k} < x_{m_k} < x_{n_k}$ such that

(i) $\frac{n_k}{x_{n_k}} \to d_{g_0}$,

(ii) $\frac{m_k}{n_k} \to \frac{\underline{d}}{d_{g_0}} \frac{1-d_{g_0}}{1-\underline{d}}$,

(iii) $\frac{x_{m_k}}{x_{n_k}} \to \frac{1-d_{g_0}}{1-\underline{d}}$,

(iv) $\frac{x_{m'_k}}{x_{n_k}} \to 0$,

(v) $\frac{m'_k}{n'_k} \to 0$,

(vi) $\frac{m'_k}{x_{m'_k}} \to \overline{d}$.

Then from (i), (ii) and (iii) follows $\frac{m_k}{x_{m_k}} \to \underline{d}$. Furthermore we must again assumed

(v) $x_{m_k} - x_{m'_k} \ge m_k - m'_k$,

(vi) $x_{n_k} - x_{m_k} \ge n_k - m_k$,

(vii) $x_{m'_{k+1}} - x_{n_k} \ge m'_{k+1} - n_k$,

(viii) $n_k < m'_{k+1}$,

(ix) $m'_1 \le x_{m'_1}$.

It can be solved naturally and complement values x_n are defined linearly. $\square$

Algorithm [4, p. 5]

Let $1 \le x_1 < x_2 < \cdots$ be an increasing sequence of positive integers. Put $x_0 = 0$ and

$$t_n = x_n - x_{n-1}, \qquad n = 1, 2, \ldots$$

For every $n = 1, 2, \ldots$ we compute the finite integer sequence

$$t_1^{(n)}, t_2^{(n)}, \ldots, t_n^{(n)}$$

from $t_1, t_2, \ldots$ by the following procedure:

1^0. For $n = 1$, $t_1^{(1)} = t_1 = x_1$;

2^0. For $n = 2$, $t_1^{(2)} = t_1 + t_2 - 1 = x_2 - 1$ and $t_2^{(2)} = 1$;

3^0. Assume that for $n-1 \geq 2$ we have $t_i^{(n-1)}$, $i = 1, 2, \dots, n-1$. For n we first define the initial auxiliary sequence $t'_1, t'_2, \dots, t'_n$ such that $t'_i = t_i^{(n-1)}$, $i = 1, 2, \dots, n-1$, and $t'_n = t_n$. Then we repeatedly modify this sequence using following steps (a) and (b).

(a) If there exists k, $1 < k < n$, such that $t'_1 = t'_2 = \dots = t'_{k-1} > t'_k$ and $t'_n > 1$, then we put $t'_k := t'_k + 1$, $t'_n := t'_n - 1$ and $t'_i := t'_i$ in all other cases.

(b) If such k does not exist and $t'_n > 1$, then we put $t'_1 := t'_1 + 1$, $t'_n := t'_n - 1$ and $t'_i := t'_i$ in all other cases.

Repeated application of (a) and (b) shows that the step 3^0 terminates if $t'_n = 1$ and outputs the sequence $t_1^{(n)} := t'_1, \dots, t_n^{(n)} := t'_n$.

4^0. Put $n - 1 := n$ and use the output $t_1^{(n)}, \dots, t_n^{(n)}$ as the new input in 3^0.

Thus the final output of Algorithm is the infinite sequence of finite integers block $t_1^{(n)}, t_2^{(n)}, \dots, t_n^{(n)}$ for $n = 1, 2, \dots$

Lemma 1 ([4, Lemma 1]). *Assuming that $t_n \neq 1$ for infinitely many n, then the output $t_1^{(n)}, t_2^{(n)}, \dots, t_n^{(n)}$ of the Algorithm can be of the following two possible forms:*

(A) $t_1^{(n)} = \dots = t_m^{(n)} = D_n > t_{m+1}^{(n)} \geq t_{m+2}^{(n)} = t_{m+3}^{(n)} = \dots t_n^{(n)} = 1$,

(B) $t_1^{(n)} = \dots = t_m^{(n)} = D_n > t_{m+1}^{(n)} = \dots = t_{m+s}^{(n)} = D_n - 1 \geq t_{m+s+1}^{(n)} = \dots = t_n^{(n)} = 1$,

for some $m = m(n)$, $s = s(n)$ and for $D_n := t_1^{(n)}$.

Lemma 2 ([4, Lemma 2]). *For D_n defined in Lemma 1 there are two possibilities:*

(I) *D_n is bounded;*

(II) *$D_n \to \infty$.*

In the case (I) *we have only the form* (A) *and $D_n =$ const. $= c \geq 2$ for all sufficiently large n.*
In the case (II) *both cases* (A) *and* (B) *are possible.*

Construction [4, p. 8]

Assume that, for every $n = 1, 2, \dots$, we have given n-terms sequence

$$t_1^{(n)}, t_2^{(n)}, \dots, t_n^{(n)}$$

such that for every $n = 1, 2, \dots$

$$t_1^{(n)} \leq t_1^{(n+1)}, t_2^{(n)} \leq t_2^{(n+1)}, \dots, t_n^{(n)} \leq t_n^{(n+1)}. \tag{34}$$

Then, we define x_n, $x_j^{(n)}$ and $X_n^{(n)}$ as

$$x_n = \sum_{i=1}^{n} t_i^{(n)}, \quad n = 1, 2, \dots; \tag{35}$$

$$x_j^{(n)} = \sum_{i=1}^{j} t_i^{(n)}, \quad j = 1, 2, \dots, n; \tag{36}$$

$$X_n^{(n)} = \left(\frac{x_1^{(n)}}{x_n^{(n)}}, \frac{x_2^{(n)}}{x_n^{(n)}}, \dots, \frac{x_n^{(n)}}{x_n^{(n)}} \right), \qquad n = 1, 2, \dots \tag{37}$$

Clearly $x_n^{(n)} = x_n$ and using (34) we see that

$$x_j = \sum_{i=1}^{j} t_i^{(j)} \le \sum_{i=1}^{j} t_i^{(n)} = x_j^{(n)}, \qquad j = 1, 2, \dots, n$$

which implies

$$F(X_n^{(n)}, x) \le F(X_n, x) \qquad \text{for all} \quad x \in [0,1], \quad n = 1, 2, \dots \tag{38}$$

Selecting a sequence of indices n_k, $k = 1, 2, \dots$, such that $F(X_{n_k}, x) \to g(x)$ and $F(X_{n_k}^{(n_k)}, x) \to \tilde{g}(x)$ for all $x \in [0,1]$, we have

$$\tilde{g}(x) \le g(x) \qquad \text{for all} \quad x \in [0,1]. \tag{39}$$

The case $\underline{d} = 0$ [4, p. 12]

In the case $\underline{d} = 0$ the Algorithm implies $\lim_{n\to\infty} D_n = \infty$ since if $D_n =$ const. $= c$, then $t_1^{(n)}, t_2^{(n)}, \dots, t_n^{(n)}$ satisfy (A) and $d_g = \frac{1}{\alpha(c-1)+1} \ge \frac{1}{c} > 0$. Note that, in the opposite direction, $\lim_{n\to\infty} D_n = \infty$ need not imply $\underline{d} = 0$, see the Construction.

The following theorem we shall formulate for the case (B), since the case (A) gives the same result, putting $\gamma = 0$ and $s_k = 0$.

Theorem 20 ([4, Th. 3]). *Let x_n, $n = 1, 2, \dots$, be an increasing sequence of positive integers such that $\underline{d} = 0$ and let $t_1^{(n)}, t_2^{(n)}, \dots, t_n^{(n)}$ be a sequence produced by Algorithm. For a selected sequence of indices n_k, $k = 1, 2, \dots$, assume that*

(i) *$F(X_{n_k}, x) \to g(x)$ and $F(X_{n_k}^{(n_k)}, x) \to \tilde{g}(x)$ for all $x \in [0,1]$;*

(ii) *$t_1^{(n_k)} = \dots = t_{m_k}^{(n_k)} = D_{n_k} > t_{m_k+1}^{(n_k)} = \dots = t_{m_k+s_k}^{(n_k)} = D_{n_k} - 1$
$\ge t_{m_k+s_k+1}^{(n_k)} = \dots = t_{n_k}^{(n_k)} = 1$;*

(iii) *$\frac{m_k}{n_k} \to \alpha$;*

(iv) *$\frac{s_k}{n_k} \to \gamma$.*

Then we have $\tilde{g}(x) \le g(x)$ for all $x \in [0,1]$, where

(a) *If* $\alpha+\gamma>0$ *then* $d_g=0$ *and* $\tilde{g}(x)=x(\alpha+\gamma)$ *for all* $x\in[0,1]$.

(b) *If* $\alpha+\gamma=0$ *and* $\frac{m_k+s_k}{n_k}D_{n_k}\to\infty$ *then* $d_g=0$ *and* $\tilde{g}(x)=0$ *for all* $x\in(0,1)$.

(c) *If* $\alpha+\gamma=0$ *and* $\frac{m_k+s_k}{n_k}D_{n_k}\to\delta$, $0<\delta<\infty$, *then* $d_g=\frac{1}{\delta+1}$ *and*

$$\tilde{g}(x)=\begin{cases}0 & \text{if } x<y_2=\frac{\delta}{\delta+1},\\ x(\delta+1)-\delta & \text{if } y_2\le x\le 1.\end{cases}$$

(d) *If* $\alpha+\gamma=0$ *and* $\frac{m_k+s_k}{n_k}D_{n_k}\to\delta=0$, *then* $d_g=1$ *and* $\tilde{g}(x)=x$.

4.8. Lower and upper d.f.s

In Theorem 17 we gave the result [3, Th. 6] that for every integer sequence $1\le x_1<x_2<\cdots$ with $\underline{d}>0$ and every d.f. $g(x)\in G(X_n)$ we have $h_1(x)\le g(x)\le h_2(x)$, where $h_1(x)$ and $h_2(x)$ are defined in (18) and (19), respectively. Furthermore, by [3, Th. 6, 6^0 of Proof], there exists an integer sequence $1\le x_1<x_2<\cdots$ with $\underline{d}>0$ such that $h_2(x)\in G(X_n)$. In this case $h_2(x)=\overline{g}(x)$ and $G(X_n)$ has the following additional properties.

Theorem 21 ([4, Th. 5]). *Let* $1\le x_1<x_2<\cdots$ *be an integer sequence with* $\underline{d}>0$ *such that* $h_2(x)\in G(X_n)$. *Then the set* $G(X_n)$ *contains uncountable many different d.f.s* $g_\alpha(x)$, $\alpha\in[1,\infty)$, *of the form*

$$g_\alpha(x)=\begin{cases}x\frac{1}{\alpha\beta}\frac{\overline{d}}{\underline{d}} & \text{if } x\in\left[0,\frac{\underline{d}}{\overline{d}}\beta\right],\\ \frac{1}{\alpha} & \text{if } x\in\left[\frac{\underline{d}}{\overline{d}}\beta,\beta\right],\\ \text{nondecreasing} & \text{if } x\in\left[\beta,1\right],\end{cases}\tag{40}$$

where for $\beta=\beta(\alpha)$ *we have* $1\le\alpha\beta\le\frac{\overline{d}}{\underline{d}}$. *Furthermore,* $g(x)=x$ *is also in* $G(X_n)$.

P r o o f. We use two steps.

1^0. Assume that $F(X_{n_k},x)\to h_2(x)$ as $k\to\infty$ for $x\in[0,1]$. For every $\alpha\in[1,\infty)$ we can choose $n'_k>n_k$ so that

(i) $\frac{n'_k}{n_k}\to\alpha$.

From the sequence (n'_k,n_k), $k=1,2,\dots$, we can select a subsequence (with the same notation) such that

(ii) $\frac{x_{n_k}}{x_{n'_k}}\to\beta$,

where $\beta=\beta(\alpha)$ but it is not given uniquely. We have only $\frac{1}{\alpha}\frac{\underline{d}}{\overline{d}}\le\beta\le\frac{1}{\alpha}\frac{\overline{d}}{\underline{d}}$ because

$$\frac{n'_k}{n_k}\frac{x_{n_k}}{x_{n'_k}}=\frac{\frac{n'_k}{x_{n'_k}}}{\frac{n_k}{x_{n_k}}}\to\alpha\beta$$

and which gives $\alpha<\infty\Leftrightarrow\beta>0$. Now, from (n'_k,n_k) we again select a subsequence such that

(iii) $F(X_{n'_k}, x) \to g(x)$

for all $x \in [0,1]$. Applying the identity (1)

$$F(X_{n_k}, x) = \frac{n'_k}{n_k} F\left(X_{n'_k}, x\frac{x_{n_k}}{x_{n'_k}}\right) \tag{41}$$

and assuming that $\underline{d} > 0$, which implies everywhere continuity of $g(x)$ (see [24, Th. 6.2]) and $g(x) > 0$ for $0 < x \le 1$, then we can take limit in (41) to obtain

$$h_2(x) = \alpha g_\alpha(x\beta) \tag{42}$$

for $x \in [0,1]$. Now, using $h_2(x) = 1$ for $x \in \left[\frac{\underline{d}}{\overline{d}}, 1\right]$, (42) implies $g_\alpha(x) = \frac{1}{\alpha}$ for $x \in \left[\frac{\underline{d}}{\overline{d}}\beta, \beta\right]$ and $h'_2(x) = \frac{\overline{d}}{\underline{d}}$ for $x \in \left[0, \frac{\underline{d}}{\overline{d}}\right]$ implies $g'_\alpha(x) = \frac{\overline{d}}{\underline{d}}\frac{1}{\alpha\beta}$ for $x \in \left[0, \frac{\underline{d}}{\overline{d}}\beta\right]$. Then we obtain (40) and since $g_\alpha(x) \le h_2(x)$, then $1 \le \alpha\beta$.

2^0. Again, let $F(X_{n_k}, x) \to h_2(x)$ for $x \in [0,1]$. For every limit point[8] $\beta > 0$ of $\frac{x_i}{x_{n_k}}$, $i = 1, 2, \dots, n_k$, $k = 1, 2, \dots$, we can select $m_k < n_k$ such that

(i) $\frac{x_{m_k}}{x_{n_k}} \to \beta$,

(ii) $\frac{n_k}{m_k} \to \alpha$

(iii) $F(X_{m_k}, x) \to g(x)$.

The identity (1) in the form $F(X_{m_k}, x) = \frac{n_k}{m_k} F\left(X_{n_k}, x\frac{x_{m_k}}{x_{n_k}}\right)$ implies

$$g(x) = \alpha h_2(x\beta) = \frac{h_2(x\beta)}{h_2(\beta)} \tag{43}$$

for $x \in [0,1]$. From the form of $h_2(x)$ we have guaranteed that $\beta \in \left[0, \frac{\underline{d}}{\overline{d}}\right]$ is a limit point of $\frac{x_i}{x_{n_k}}$ and in this case (43) gives

$$g(x) = \frac{x\beta\frac{\overline{d}}{\underline{d}}}{\beta\frac{\overline{d}}{\underline{d}}} = x.$$

For $\beta > \frac{\underline{d}}{\overline{d}}$, if exists, we have $g(x) = h_2(x\beta)$ for $x \in [0,1]$, i.e.,

$$g(x) = \begin{cases} x\beta\frac{\overline{d}}{\underline{d}} & \text{if } x \in \left[0, \frac{\underline{d}}{\overline{d}}\frac{1}{\beta}\right], \\ 1 & \text{if } x \in \left[\frac{\underline{d}}{\overline{d}}\frac{1}{\beta}, 1\right]. \end{cases} \tag{44}$$

□

Finally, for $h_2(x)$ defined in (19) for which $h_2(x) = \overline{g}(x)$ for special $1 \le x_1 < x_2 < \cdots$, we see directly that

$$h_2(xy) \le h_2(x)h_2(y) \tag{45}$$

[8] In the following α and β have another meaning as in 1^0.

for every $x, y \in [0,1]$. Also for $h_1(x)$ defined in (18), in the case $x \geq \sqrt{\frac{1-\overline{d}}{1-\underline{d}}}$, for which there exists a special sequence x_n (see [24, pp. 774–777, Ex. 11.2]) such that the lower d.f. $\underline{g}(x) = h_1(x)$ we have[9]

$$\left(\frac{\underline{d}}{\frac{1}{x} - (1-\underline{d})}\right)\left(\frac{\underline{d}}{\frac{1}{y} - (1-\underline{d})}\right) \leq \frac{\underline{d}}{\frac{1}{xy} - (1-\underline{d})} \tag{46}$$

for $xy \geq \sqrt{\frac{1-\overline{d}}{1-\underline{d}}}$. In the following theorem we extend (45) and (46) for arbitrary lower $\underline{g}(x)$ and upper $\overline{g}(x)$ d.f.s.

Theorem 22 ([4, Th. 6])**.** *For every increasing sequence of positive integers $1 \leq x_1 < x_2 < \cdots$, with $\underline{d} > 0$, the lower d.f. $\underline{g}(x)$ and the upper d.f. $\overline{g}(x)$ satisfy*

$$\underline{g}(x).\underline{g}(y) \leq \underline{g}(x.y) \leq \overline{g}(x.y) \leq \overline{g}(x).\overline{g}(y) \tag{47}$$

for every $x, y \in (0,1)$.

Proof. $\underline{d} > 0$ implies that arbitrary $g(x) \in G(X_n)$ is everywhere continuous and $g(x) > 0$ for $x > 0$. Let $y \in (0,1)$.

1^0. Firstly we prove the left-hand side of (47).

a) If y is an increasing point[10] of $g(x)$, $n = 1, 2, \dots$ then by (6) we have $\frac{g(xy)}{g(y)} \in G(X_n)$ and thus $\underline{g}(x) \leq \frac{g(xy)}{g(y)}$ which implies

$$\underline{g}(x)\underline{g}(y) \leq \underline{g}(x)g(y) \leq g(xy) \tag{48}$$

for every $x \in (0,1)$.

b) Let $g(x)$ does not increase at y. Since every $g(x) \in G(X_n)$ is continuous and $\frac{\underline{d}}{\overline{d}}x \leq g(x) \leq \frac{\overline{d}}{\underline{d}}x$ for $x \in [0,1]$, there exists the nearest neighboring point $y_1 < y$, $y_1 > 0$ at which $g(x)$ increases. Thus $\frac{g(xy_1)}{g(y_1)} \in G(X_n)$ which implies $\underline{g}(x) \leq \frac{g(xy_1)}{g(y_1)}$. Because $g(y_1) = g(y)$, $g(xy_1) \leq g(xy)$, then again

$$\underline{g}(x)\underline{g}(y) \leq \underline{g}(x)g(y) = \underline{g}(x)g(y_1) \leq g(xy_1) \leq g(xy) \tag{49}$$

for every $x \in (0,1)$.

Since $g \in G(X_n)$ is arbitrary, and for $x, y \in (0,1)$ by (47) and (48) we have $\underline{g}(x)\underline{g}(y) \leq g(xy)$, then the definition of lower d.f. of $G(X_n)$ as

$$\underline{g}(xy) = \inf_{g \in G(X_n)} g(xy) \quad \text{implies} \quad \underline{g}(x)\underline{g}(y) \leq \underline{g}(xy).$$

2^0. Now, we prove the right-hand side of (47).

[9] This holds also for arbitrary $x, y \in (0,1)$, since it is equivalent to $x(1-y) \leq 1-y$.

[10] Either $g(y-\varepsilon) < g(y)$ or $g(y) < g(y+\varepsilon)$, for arbitrary $\varepsilon > 0$.

a) Again, if y is an increasing point of $g(x)$, then $\frac{g(xy)}{g(y)} \in G(X_n)$, thus $\frac{g(xy)}{g(y)} \le \overline{g}(x)$ which implies

$$g(xy) \le g(y)\overline{g}(x) \le \overline{g}(y)\overline{g}(x) \tag{50}$$

for $x \in (0,1)$.

b) Let $g(x)$ be non increasing at y and let y_2 be the nearest point to the right at which $g(x)$ is increasing. Again, by $\underline{\frac{d}{d}}x \le g(x) \le \overline{\frac{d}{d}}x$, this point exists and thus for given $g(x) \in G(X_n)$ we have $\frac{g(xy_2)}{g(y_2)} \in G(X_n)$, $\frac{g(xy_2)}{g(y_2)} \le \overline{g}(x)$ which implies

$$g(xy) \le g(xy_2) \le g(y_2)\overline{g}(x) \le g(y)\overline{g}(x) \le \overline{g}(y)\overline{g}(x) \tag{51}$$

for $x \in (0,1)$. Then

$$\overline{g}(xy) = \sup_{g \in G(X_n)} g(xy) \quad \text{implies} \quad \overline{g}(x.y) \le \overline{g}(x).\overline{g}(y)$$

for $x, y \in (0,1)$. □

Note that by J. A c z é l [1, p. 144–145, Th. 4] every continuous d.f. $g(xy) = g(x)g(y)$ has the form $g(x) = x^c$ for a constant c and $x \in [0,1]$.

4.9. Construction $H \subset G(X_n)$

Basic open problem is that characterize a nonempty set H of d.f.s for which there exists an increasing sequence of positive integers x_n such that $G(X_n) = H$. In [3] we found integer sequence $1 \le x_1 < x_2 < \cdots$ such that the piecewise linear function $h_2(x)$ defined in (19) belongs to $G(X_n)$. In [4] is the following extension of this construction:

Theorem 23. *Let H be a nonempty set of d.f.s defined on $[0,1]$. Then there exists an integer sequence $1 \le x_1 < x_2 < \cdots$ such that $H \subset G(X_n)$.*

P r o o f.
1^0. To the set H it can be constructed a sequence of continuous strictly increasing piecewise linear functions $h_n(x)$, $n = 1, 2, \dots$, such that every $f(x) \in H$ is a weak limit $h_{n_k}(x) \to f(x)$.

2^0. For every $h(x)$ possessing at points $\beta_1 = 0 < \beta_2 < \cdots < \beta_{s-1} < \beta_s = 1$ the values $\alpha_1 = 0 < \alpha_2 < \cdots < \alpha_{s-1} < \alpha_s = 1$, respectively, and being linear in each interval $[\beta_i, \beta_{i+1}]$, we can define a sequence of integer intervals $[m_k^{(1)}, n_k]$, $k = 1, 2, \dots$, and their divisions

$$m_k^{(1)} < m_k^{(2)} < \cdots < m_k^{(s-1)} < m_k^{(s)} < n_k$$

in which we can define integers

$$x_{m_k^{(1)}} < x_{m_k^{(2)}} < \cdots < x_{m_k^{(s-1)}} < x_{m_k^{(s)}} < x_{n_k}$$

such that for $i = 1, 2, \dots, s$ we have

(i) $\frac{x_{m_k^{(i)}}}{x_{n_k}} \to \beta_i$,

(ii) $\frac{m_k^{(i)}}{n_k} \to \alpha_i$,

(iii) $x_{m_k^{(i)}} - x_{m_k^{(i-1)}} \geq m_k^{(i)} - m_k^{(i-1)}$,

(iv) $x_{n_k} - x_{m_k^{(s)}} \geq n_k - m_k^{(s)}$.

For other $n \in \left[m_k^{(1)}, n_k\right]$ we define x_n linearly, i.e., for $n \in [m_k^{(i-1)}, m_k^{(i)}]$ we put

(v)

$$x_n = x_{m_k^{(i-1)}} + \left[\left(n - m_k^{(i-1)}\right) \frac{x_{m_k^{(i)}} - x_{m_k^{(i-1)}}}{m_k^{(i)} - m_k^{(i-1)}}\right].$$

Directly from (i), (ii) and (v) it follows that

$$\frac{\#\left\{n \in [m_k^{(1)}, n_k]; \frac{x_n}{x_{n_k}} < x\right\}}{n_k} \to h(x) \quad \text{for } x \in (0,1) \quad \text{as } k \to \infty. \tag{52}$$

See the following Fig. 1 and Fig. 2.

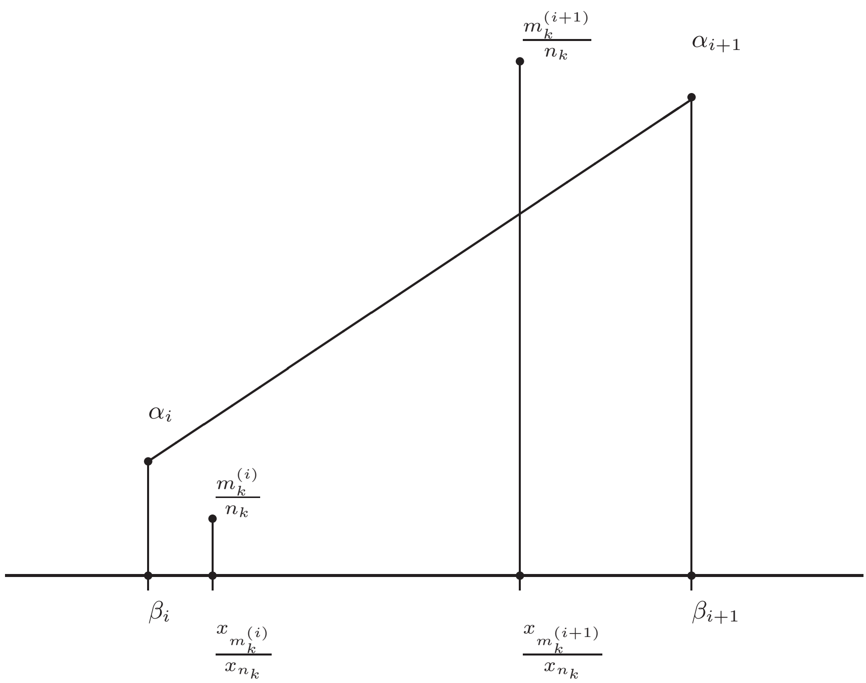


FIGURE 1. A part of graph of $h(x)$ and (i)–(ii) properties.

Note that, in this step, the intervals $[m_k^{(1)}, n_k]$, $k = 1, 2, \dots$, can intersect. For necessity of pairwise disjointness we use the next step.

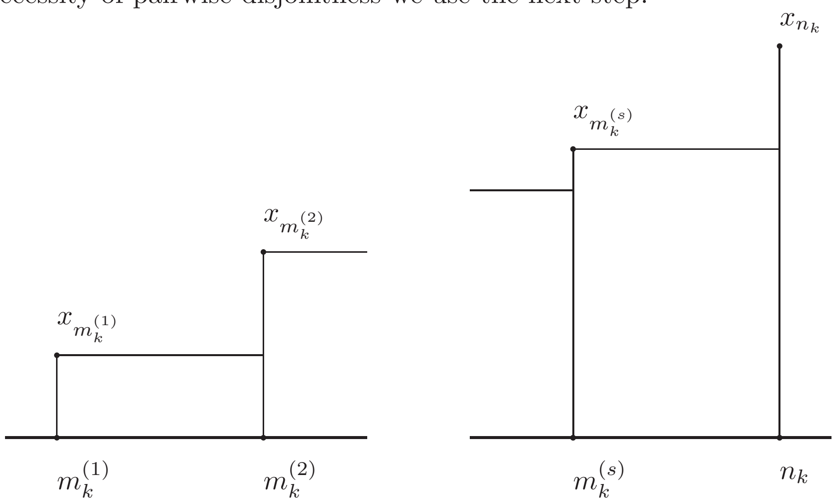


Figure 2. (iii)–(iv) properties.

3^0. One solution $[m_k^{(1)}, n_k]$, $k = 1, 2, \dots$ in 2^0 gives infinitely many solutions by the following: Let $A_k < B_k$ be two positive integer sequences. Replace $[m_k^{(1)}, n_k]$ by $[A_k m_k^{(1)}, A_k n_k]$ with division

$$A_k m_k^{(1)} < A_k m_k^{(2)} < \cdots < A_k m_k^{(s-1)} < A_k m_k^{(s)} < A_k n_k$$

and define the values of x_n as

$$x_{A_k m_k^{(i)}} = B_k x_{m_k^{(i)}},$$

$i = 1, 2, \dots, s$ and $x_{A_k n_k} = B_k x_{n_k}$. Then the limits (i) and (ii) again hold

$$\frac{x_{A_k m_k^{(i)}}}{x_{A_k n_k}} = \frac{B_k x_{m_k^{(i)}}}{B_k x_{n_k}} \to \beta_i, \quad \frac{A_k m_k^{(i)}}{A_k n_k} \to \alpha_i.$$

Also (iii) and (iv) hold, since

$$\begin{aligned} x_{A_k m_k^{(i)}} - x_{A_k m_k^{(i-1)}} &= B_k x_{m_k^{(i)}} - B_k x_{m_k^{(i-1)}} \\ &\geq B_k \left(m_k^{(i)} - m_k^{(i-1)} \right) \geq A_k m_k^{(i)} - A_k m_k^{(i-1)}. \end{aligned}$$

4^0. Let $h_i(x)$, $i = 1, 2, \dots$ be a dense set of d.f.s in H and for $h_i(x) = h(x)$ rewrite the interval $[m_k^{(1)}, n_k]$ in 2^0 as $[m_k^{(1,i)}, n_k^{(i)}]$. Order these intervals to infinite matrix $\mathbb{A}$

$$\begin{array}{l}
[m_1^{(1,1)}, n_1^{(1)}], [m_2^{(1,1)}, n_2^{(1)}], \dots, [m_k^{(1,1)}, n_k^{(1)}], \dots \\
[m_1^{(1,2)}, n_1^{(2)}], [m_2^{(1,2)}, n_2^{(2)}], \dots, [m_k^{(1,2)}, n_k^{(2)}], \dots \\
\dots \\
[m_1^{(1,i)}, n_1^{(i)}], [m_2^{(1,i)}, n_2^{(i)}], \dots, [m_k^{(1,i)}, n_k^{(i)}], \dots \\
\dots
\end{array}$$

and reorder it to a linear sequence by diagonals, i.e., to

$$[m_1^{(1,1)}, n_1^{(1)}], [m_1^{(1,2)}, n_1^{(2)}], [m_2^{(1,1)}, n_2^{(1)}], \dots$$

and denote it as a new sequence $[m_k^{(1)}, n_k]$, $k = 1, 2, \dots$ Since these intervals can intersect we use in 3^0 suitable $A_k < B_k$, $k = 1, 2, \dots$ such that the resulting sequence is disjoint and

(vi) $x_{m_{k+1}^{(1)}} - x_{n_k} \geq m_{k+1}^{(1)} - n_k$,

(vii) $x_{m_1^{(1)}} \geq m_1^{(1)}$.

For n which are not in the intervals $\left[m_k^{(1)}, n_k\right]$, $k = 1, 2, \dots$ we can define x_n linearly. Now, if from n_k, $k = 1, 2, \dots$ we select n'_k corresponding to ith line of $\mathbb{A}$, then $F(X_{n'_k}, x) \to h_i(x)$ for $x \in [0, 1]$.

5^0. Finally, we give a solution of (i)–(iv) in 2^0. We start with increasing sequence of indices n_k, $k = 1, 2, \dots$, and let $\lambda > 1$ and put (integer parts are omitted)

$$\begin{aligned}
x_{n_k} &= \lambda n_k, \\
x_{m_k^{(i)}} &= \beta_i \lambda n_k, \\
m_k^{(i)} &= \alpha_i n_k.
\end{aligned}$$

For (iv) we need

$$\begin{aligned}
x_{m_k^{(i)}} - x_{m_k^{(i-1)}} = \beta_i \lambda n_k - \beta_{i-1} \lambda n_k &= \lambda(\beta_i - \beta_{i-1}) n_k \\
&\geq m_k^{(i)} - m_k^{(i-1)} = (\alpha_i - \alpha_{i-1}) n_k
\end{aligned}$$

which gives assumption $\lambda > \max \frac{\alpha_i - \alpha_{i-1}}{\beta_i - \beta_{i-1}}$. □

Note that by Theorem 23 there exists an integer sequence $1 \leq x_1 < x_2 < \cdots$ such that $G(X_n)$ contains all d.f.s. Especially, for every sequence $y_n \in [0, 1)$, $n = 1, 2, \dots$, there exists an X_n such that $G(y_n) \subset G(X_n)$.

4.10. $g(x) \in G(X_n)$ with constant intervals

Theorem 24 ([23]). *Assume that $\underline{d} > 0$. If there exists an interval $(u, v) \subset [0, 1]$ such that every $g \in G(X_n)$ has a constant value on (u, v) (may be different), then every $g \in G(X_n)$ has infinitely many intervals with constant values such that g increases at their endpoints.*

P r o o f. Since

$$x_i < xx_m \Longleftrightarrow x_i < \left(x\frac{x_m}{x_n}\right)x_n,$$

then we have (1)

$$F(X_m, x) = \frac{n}{m}F\left(X_n, x\frac{x_m}{x_n}\right),$$

for every $m \leq n$ and $x \in [0, 1)$. Using the Helly selection principle, we can select a subsequence (m_k, n_k) of the sequence (m, n) such that $F(X_{n_k}) \to g(x)$, $F(X_{m_k}) \to \tilde{g}(x)$ as $k \to \infty$; furthermore $x_{m_k}/x_{n_k} \to \beta$ and $n_k/m_k \to \alpha$, but α may be infinity. Assuming $\beta > 0$ and $g(\beta - 0) > 0$, we have $\alpha < \infty$ and (3)

$$\tilde{g}(x) = \alpha g(x\beta) \text{ a.e. on } [0, 1].$$

Thus, if $\tilde{g}(x)$ has a constant value on (u, v), then $g(x)$ must be constant on the interval $(u\beta, v\beta)$. Furthermore, if $\underline{d} > 0$, then for every $g \in G(X_n)$ we have (7)

$$(\underline{d}/\overline{d})x \leq g(x) \leq (\overline{d}/\underline{d})x$$

for every $x \in [0, 1]$. Thus, there exists a sequence $\beta_k \in (0, 1)$ such that $\beta_k \searrow 0$ and $g(x)$ increases in β_k, $g(\beta_k) > 0$, $k = 1, 2, \dots$ For such β_k, $g(x)$, applying the Helly principle, we can find sequences α_k and $\tilde{g}_k(x) \in G(X_n)$ such that

$$\tilde{g}_k(x) = \alpha_k g(x\beta_k)$$

a.e. on $[0, 1]$. Every $\tilde{g}_k(x)$ has a constant value on the interval (u, v), hence, $g(x)$ must be constant on the intervals $(u\beta_k, v\beta_k)$ for $k = 1, 2, \dots$ □

4.11. Transformation of X_n by $1/x \bmod 1$

The mapping $1/x \bmod 1$ transforms the block X_n to the block

$$Z_n = \left(\frac{x_n}{x_1}, \frac{x_n}{x_2}, \dots, \frac{x_n}{x_n}\right) \bmod 1.$$

For example, the block sequence $X_n = \left(\frac{1}{n}, \frac{2}{n}, \dots, \frac{n}{n}\right)$, $n = 1, 2, \dots$ which is u.d. is transformed to the block sequence

$$Z_n = \left(\frac{n}{1}, \frac{n}{2}, \dots, \frac{n}{n}\right) \bmod 1, \qquad n = 1, 2, \dots$$

which has a.d.f.

$$g(x) = \int_0^1 \frac{1-t^x}{1-t}\,\mathrm{d}t = \sum_{n=1}^{\infty} \frac{x}{n(n+x)} = \gamma_0 + \frac{\Gamma'(1+x)}{\Gamma(1+x)},$$

where γ_0 is Euler's constant. This was proved by G. Pólya, (see I. J. Schoenberg [17]). The following theorem, which generalizes [12, p. 56, Th. 7.6] describes a relation between $G(X_n)$ and $G(Z_n)$.

Theorem 25 ([9, Th. 7]). *If every $g(x) \in G(X_n)$ is continuous on $[0,1]$, then*

$$G(Z_n) = \left\{ \tilde{g}(x) = \sum_{n=1}^{\infty} g(1/n) - g\big(1/(n+x)\big); g(x) \in G(X_n) \right\}.$$

Proof. For $f(x) = 1/x \bmod 1$ we have $f^{-1}([0,t)) = \cup_{i=1}^{\infty}\big(1/(t+i), 1/i\big]$. Thus $F(Z_n,t) = \sum_{i=1}^{\infty}\big(F(X_n,1/i) - F(X_n, 1/(t+i))\big)$.

1^0. Assume that $F(X_{n_k},x) \to g(x)$, where $g(x)$ is everywhere continuous on $[0,1]$. Thus

$$\sum_{i=1}^{K}\Big(F(X_{n_k},1/i) - F\big(X_{n_k},1/(t+i)\big)\Big) \to \sum_{i=1}^{K}\Big(g(1/i) - g\big(1/(t+i)\big)\Big),$$

$$\sum_{i=K+1}^{\infty}\Big(F(X_{n_k},1/i) - F\big(X_{n_k},1/(t+i)\big)\Big) \le F\big(X_{n_k},1/(K+1)\big)$$

$$\to g\big(1/(K+1)\big) \to 0.$$

Thus $F(Z_{n_k},t) \to \tilde{g}(t) = \sum_{i=1}^{\infty}\big(g(1/i) - g(1/(t+i))\big)$ for $t \in [0,1]$.

2^0. Assume that $F(Z_{n_k},t) \to \tilde{g}(t)$ weakly. From n_k there can be selected n'_k such that $F(X_{n'_k},x) \to g(x)$. Assuming continuity of $g(x)$, we apply 1^0. □

5. Examples

Example 1 ([24]). Put $x_n = p_n$, the nth prime and denote

$$X_n = \left(\frac{2}{p_n}, \frac{3}{p_n}, \dots, \frac{p_{n-1}}{p_n}, \frac{p_n}{p_n}\right).$$

The sequence of blocks X_n is u.d. and therefore the ratio sequence p_m/p_n, $m=1,2,\dots,n$, $n=1,2,\dots$ is u.d. in $[0,1]$. This generalizes a result of A. Schinzel (cf. W. Sierpiński (1964, p. 155)). Note that from u.d. of X_n applying for the L^2 discrepancy of X_n we get the following interesting limit

$$\lim_{n\to\infty} \frac{1}{n^2 p_n} \sum_{i,j=1}^{n} |p_i - p_j| = \frac{1}{3}.$$

Example 2 ([24, Ex. 11.1]). Let γ, δ, and a be given real numbers satisfying $1 \le \gamma < \delta \le a$. Let x_n be an increasing sequence of all integer points lying in the intervals

$$(\gamma, \delta), (\gamma a, \delta a), \dots, (\gamma a^k, \delta a^k), \dots$$

Then $G(X_n) = \{g_t(x); t \in [0,1]\}$, where $g_t(x)$ has constant values

$$g_t(x) = \frac{1}{a^i(1 + t(a-1))} \quad \text{for } x \in \frac{(\delta, a\gamma)}{a^{i+1}(t\delta + (1-t)\gamma)}, \qquad i = 0,1,2,\dots$$

and on the component intervals it has a constant derivative

$$g_t'(x) = \frac{t\delta + (1-t)\gamma}{(\delta - \gamma)(\frac{1}{a-1} + t)} \quad \text{for } x \in \frac{(\gamma, \delta)}{a^{i+1}(t\delta + (1-t)\gamma)}, \qquad i = 0,1,2,\dots$$
$$\text{and } x \in \left(\frac{\gamma}{t\delta + (1-t)\gamma}, 1\right),$$

where

$$F(X_{n_k}, x) \to g_t(x) \text{ for } n_k \quad \text{for which} \quad x_{n_k} = \left[a^k\gamma + ta^k(\delta - \gamma)\right]. \tag{53}$$

Here we write $(xz, yz) = (x,y)z$ and $(x/z, y/z) = (x,y)/z$. Then the set $G(X_n)$ has the following properties:

1^0. Every $g \in G(X_n)$ is continuous.

2^0. Every $g \in G(X_n)$ has infinitely many intervals with constant values, i.e., with $g'(x) = 0$, and in the infinitely many complement intervals it has a constant derivative $g'(x) = c$, where $\frac{1}{\overline{d}} \le c \le \frac{1}{\underline{d}}$ and for lower $\underline{d}$ and upper $\overline{d}$ asymptotic density of x_n we have

$$\underline{d} = \frac{(\delta - \gamma)}{\gamma(a-1)}, \quad \overline{d} = \frac{(\delta - \gamma)a}{\delta(a-1)}.$$

3^0. The graph of every $g \in G(X_n)$ lies in the intervals

$$\left[\frac{1}{a}, 1\right] \times \left[\frac{1}{a}, 1\right] \cup \left[\frac{1}{a^2}, \frac{1}{a}\right] \times \left[\frac{1}{a^2}, \frac{1}{a}\right] \cup \dots$$

Moreover, the graph g in $\left[\frac{1}{a^k}, \frac{1}{a^{k-1}}\right] \times \left[\frac{1}{a^k}, \frac{1}{a^{k-1}}\right]$ is similar to the graph of g in $\left[\frac{1}{a^{k+1}}, \frac{1}{a^k}\right] \times \left[\frac{1}{a^{k+1}}, \frac{1}{a^k}\right]$ with coefficient $\frac{1}{a}$. Using the parametric expression, it can be written for all $x \in \left(\frac{1}{a^{i+1}}, \frac{1}{a^i}\right)$ that $g_t(x) = \frac{g_t(a^i x)}{a^i}$, $i = 0,1,2,\dots$

4^0. $G(X_n)$ is connected and the upper distribution function $\overline{g}(x) = g_0(x) \in G(X_n)$ and the lower distribution function $\underline{g}(x) \notin G(X_n)$. The graph of $\underline{g}(x)$ on $\left[\frac{1}{a}, 1\right] \times \left[\frac{1}{a}, 1\right]$ coincides with the graph of

$$y(x) = \left(1 + \frac{1}{\underline{d}}\left(\frac{1}{x} - 1\right)\right)^{-1}$$

on $\left[\frac{\gamma}{\delta},1\right]$, further, on $\left[\frac{1}{a},\frac{\gamma}{\delta}\right]$ we have $\underline{g}(x)=\frac{1}{a}$.

5^0. $G(X_n)=\left\{\frac{g_0(x\beta)}{g_0(\beta)};\beta\in\left[\frac{1}{a},\frac{\delta}{a\gamma}\right]\right\}$.

For the proofs of 1^0. – 5^0. we only note:
Assume that $x_n\in a^k(\gamma,\delta)$, $i,i+1,i+2,\dots\in a^j(\gamma,\delta)$ for some $j<k$, and let $F(X_n,x)\to g(x)$ for some sequence of n. Then $g(x)$ has a constant derivative in the intervals containing $\frac{i}{x_n},\frac{i+1}{x_n},\frac{i+2}{x_n},\dots$, since

$$\frac{\frac{1}{n}}{\frac{i+1}{x_n}-\frac{i}{x_n}}=\frac{x_n}{n},$$

and thus $\frac{x_n}{n}$ must be convergent to $g'(x)$, so $\frac{1}{\overline{d}}\le g'(x)\le\frac{1}{\underline{d}}$. For

$$x_n=\left[ta^k\delta+(1-t)a^k\gamma\right]$$

we can find

$$\begin{aligned}g'(x)=\lim_{n\to\infty}\frac{x_n}{n}&=\lim_{k\to\infty}\frac{a^k(t\delta+(1-t)\gamma)}{\sum_{j=0}^{k-1}a^j(\delta-\gamma)+a^k(t\delta+(1-t)\gamma)-a^k\gamma}\\&=\frac{t\delta+(1-t)\gamma}{(\delta-\gamma)\left(\frac{1}{a-1}+t\right)}.\end{aligned}$$

Using Theorem 18 and [3, Ex. 3] we shall add the following properties moreover:

6^0. By definition (5) of the local asymptotic density d_g and by (53) for $g(x)=g_t(x)$ we have

$$\begin{aligned}d_{g_t}=\lim_{k\to\infty}\frac{n_k}{x_{n_k}}&=\lim_{k\to\infty}\frac{\sum_{i=0}^{k-1}a^i(\delta-\gamma)+ta^k(\delta-\gamma)}{a^k\gamma+ta^k(\delta-\gamma)}\\&=\frac{(\delta-\gamma)(1+t(a-1))}{(a-1)(\gamma+t(\delta-\gamma))}\end{aligned}\tag{54}$$

and for $t=0$ we have $d_{g_0}=\underline{d}$ and for $t=1$ we have $d_{g_1}=\overline{d}$ and we see

$$g_t'(x)=\frac{1}{d_{g_t}}\tag{55}$$

for x with the constant derivative of $g_t(x)$.

7^0. For the function $h_{1,g}(x)$ defined in (26), putting $g(x)=g_t(x)$, we have:

$$\frac{\underline{d}}{d_{g_t}}=\frac{\gamma+t(\delta-\gamma)}{\gamma(1+t(a-1))},\ \frac{1-d_{g_t}}{1-\underline{d}}=\frac{\gamma}{\gamma+t(\delta-\gamma)},$$

$$\frac{\underline{d}}{d_{g_t}}\frac{1-d_{g_t}}{1-\underline{d}}=\frac{1}{1+t(a-1)}.$$

Then

$$h_{1,g_t}(x) = \begin{cases} x\frac{\gamma+t(\delta-\gamma)}{\gamma(1+t(a-1))} & \text{for } x \in \left(0, \frac{\gamma}{\gamma+t(\delta-\gamma)}\right), \\ x\frac{1}{d_{g_t}} + 1 - \frac{1}{d_{g_t}}, & \text{for } x \in \left(\frac{\gamma}{\gamma+t(\delta-\gamma)}, 1\right), \end{cases} \tag{56}$$

see the following figure.

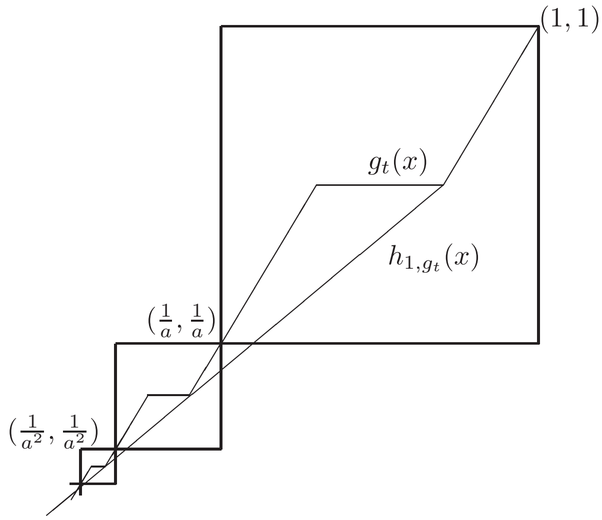


Figure: $g_t(x)$ and $h_{1,g_t}(x)$.

8^0. In the proof of the upper bound (29) we have proved that $1-\int_0^1 h_{1,g}(x)\,\mathrm{d}x$ is maximal for $d_g = \min(\sqrt{\underline{d}}, \overline{d})$. Let $t_0 \in [0,1]$ be such that $d_{g_{t_0}} = \min(\sqrt{\underline{d}}, \overline{d})$ and t_0 can be computed by inverse formula to (54)

$$t = \frac{d_{g_t}(a-1)\gamma - (\delta-\gamma)}{(\delta-\gamma)(a-1)(1-d_{g_t})}. \tag{57}$$

9^0. Let $P(t)$ be the area in $\left[\frac{1}{a}, 1\right] \times \left[\frac{1}{a}, 1\right]$ bounded by the graph of $g_t(x)$. Then

$$\begin{aligned} \int_0^1 g_t(x)\,\mathrm{d}x &= P(t)\frac{1}{1-\frac{1}{a^2}} + \frac{1}{a+1} \\ &= \frac{1}{2} + \frac{1}{2}.\frac{1}{(a+1)}.\frac{(\gamma a - \delta)}{(1+t(a-1))(\gamma+t(\delta-\gamma))} \\ &\quad + \frac{1}{2}.\frac{t(\delta-\gamma a)}{(1+t(a-1))(\gamma+t(\delta-\gamma))} \end{aligned} \tag{58}$$

and since $g_0(x) = \overline{g}(x)$ we have that the $\max_{t\in[0,1]} \int_0^1 g_t(x)\,\mathrm{d}x$ is attained at $t = 0$. Using derivative of $P(t)$ it can be see that the $\min_{t\in[0,1]} \int_0^1 g_t(x)\,\mathrm{d}x$

is attained at $t = 1$. It also follows from the fact that for $x_{n+1} = x_n + 1$ we have

$$\frac{1}{n+1}\sum_{i=1}^{n+1}\frac{x_i}{x_{n+1}} - \frac{1}{n}\sum_{i=1}^{n}\frac{x_i}{x_n}$$
$$= \frac{1}{n+1} - \left(\frac{1}{x_n+1} + \frac{1}{n+1}.\frac{1}{1+\frac{1}{x_n}}\right)\left(\frac{1}{n}\sum_{i=1}^{n}\frac{x_i}{x_n}\right) > 0$$

because $c_1(x) \notin G(X_n)$ and thus $\limsup_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\frac{x_i}{x_n} < 1$. Now, denoting the index n_k for $x_{n_k} = [a^k\delta]$, the lim sup of $\frac{1}{n}\sum_{i=1}^{n}\frac{x_i}{x_n}$ is attained over $n = n_k$, $k = 0, 1, 2, \dots$ and for such n_k we have $F(X_{n_k}, x) \to g_1(x)$ for $x \in [0,1]$.

10^0. Thus we have

$$\liminf_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\frac{x_i}{x_n} = 1 - \int_0^1 g_0(x)\,\mathrm{d}x = \frac{1}{2} - \frac{1}{2}.\frac{1}{(a+1)}\left(\frac{\gamma a - \delta}{\gamma}\right), \tag{59}$$

$$\limsup_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\frac{x_i}{x_n} = 1 - \int_0^1 g_1(x)\,\mathrm{d}x = \frac{1}{2} + \frac{1}{2}.\frac{1}{(a+1)}\left(\frac{\gamma a - \delta}{\delta}\right). \tag{60}$$

The upper bound (29) coincides with the maximal value of $1 - \int_0^1 h_{1,g}(x)\,\mathrm{d}x$ attained for $d_g = \min(\sqrt{\underline{d}}, \overline{d})$. Since $1 - \int_0^1 g_1(x)\,\mathrm{d}x$ is maximal for all $1 - \int_0^1 g_t(x)\,\mathrm{d}x$, $t \in [0,1]$ and $1 - \int_0^1 g_1(x)\,\mathrm{d}x \le 1 - \int_0^1 h_{1,g_1}(x)\,\mathrm{d}x$ then the upper bound (60) satisfies (29).

11^0. Using explicit formulas

$$\underline{d} = \frac{(\delta-\gamma)}{\gamma(a-1)}, \quad \overline{d} = \frac{(\delta-\gamma)a}{\delta(a-1)} \tag{61}$$

for asymptotic densities we see again that (59) and (60) satisfy (28) and (29), respectively, in Theorem 19.

Example 3 ([9, Ex. 2]). Let x_n and y_n, $n = 1, 2, \dots$, be two strictly increasing sequences of positive integers such that for the related block sequences $X_n = \left(\frac{x_1}{x_n}, \dots, \frac{x_n}{x_n}\right)$ and $Y_n = \left(\frac{y_1}{y_n}, \dots, \frac{y_n}{y_n}\right)$, we have singleton for both $G(X_n) = \{g_1(x)\}$ and $G(Y_n) = \{g_2(x)\}$. Furthermore, let n_k, $k = 1, 2, \dots$, be an increasing sequence of positive integers such that $N_k = \sum_{i=1}^{k} n_i$ satisfies $\frac{n_k}{N_k} \to 1$. Denote by z_n the following increasing sequence of positive integers composed by blocks (here we use the notation $a(b, c, d, \dots) = (ab, ac, ad, \dots)$)

$$(x_1, \dots, x_{n_1}), x_{n_1}(y_1, \dots, y_{n_2}), x_{n_1}y_{n_2}(x_1, \dots, x_{n_3}), x_{n_1}y_{n_2}x_{n_3}(y_1, \dots, y_{n_4}), \dots$$

Then the sequence of blocks $Z_n = \left(\frac{z_1}{z_n}, \dots, \frac{z_n}{z_n}\right)$ has the set of d.f.s

$$
\begin{aligned}
G(Z_n) = \{g_1(x), g_2(x), c_0(x)\} &\cup \{g_1(xy_n); n = 1, 2, \dots\}\\
&\cup \{g_2(xx_n); n = 1, 2, \dots\}\\
&\cup \left\{\frac{1}{1+\alpha}c_0(x) + \frac{\alpha}{1+\alpha}g_1(x); \alpha \in [0, \infty)\right\}\\
&\cup \left\{\frac{1}{1+\alpha}c_0(x) + \frac{\alpha}{1+\alpha}g_2(x); \alpha \in [0, \infty)\right\},
\end{aligned}
$$

where $g_1(xy_n) = 1$ if $xy_n \ge 1$, similarly for $g_2(xx_n)$.

P r o o f. For every $n = 1, 2, \dots$ there exists an integer k such that

$$N_{k-1} < n \le N_k$$

(here $N_0 = 0$). Put $n' = n - N_{k-1}$. For every n we have

$$
z_n = \begin{cases} x_{n_1} y_{n_2} \dots x_{n_{k-1}} y_{n'} & \text{if } k \text{ is even,}\\ x_{n_1} y_{n_2} \dots y_{n_{k-1}} x_{n'} & \text{if } k \text{ is odd.} \end{cases}
$$

Firstly we assume that k is even. Then Z_n has the form

$$
\begin{aligned}
&Z_n =\\
&\left(\dots, \frac{x_{n_1} y_{n_2} \dots y_{n_{k-2}}(x_1, \dots, x_{n_{k-1}})}{x_{n_1} y_{n_2} \dots x_{n_{k-1}} y_{n'}}, \frac{x_{n_1} y_{n_2} \dots x_{n_{k-1}}(y_1, \dots, y_{n'})}{x_{n_1} y_{n_2} \dots x_{n_{k-1}} y_{n'}}\right) =\\
&\left(\dots, \frac{1}{x_{n_{k-1}} y_{n'}}\left(\frac{y_1}{y_{n_{k-2}}}, \dots, \frac{y_{n_{k-2}}}{y_{n_{k-2}}}\right), \frac{1}{y_{n'}}\left(\frac{x_1}{x_{n_{k-1}}}, \dots, \frac{x_{n_{k-1}}}{x_{n_{k-1}}}\right), \left(\frac{y_1}{y_{n'}}, \dots, \frac{y_{n'}}{y_{n'}}\right)\right)
\end{aligned}
$$

and thus for $x > \frac{1}{x_{n_{k-1}}}$ we have

$$
\begin{aligned}
F(Z_n, x) &= \frac{N_{k-2} + n_{k-1} F(X_{n_{k-1}}, x y_{n'}) + n' F(Y_{n'}, x)}{N_{k-1} + n'}\\
&= \frac{N_{k-2}}{N_{k-1} + n'} + \frac{\frac{n_{k-1}}{N_{k-1}}}{1 + \frac{n'}{N_{k-1}}} F(X_{n_{k-1}}, x y_{n'}) + \frac{1}{1 + \frac{N_{k-1}}{n'}} F(Y_{n'}, x).
\end{aligned}
$$

If $n \to \infty$, then the first term tends to zero. If $F(Z_n, x) \to g(x)$ for some sequence of n, we can select a subsequence of n's such that $\frac{n'}{N_{k-1}} \to \alpha$ for some $\alpha \in [0, \infty)$, or $\frac{n'}{N_{k-1}} \to \infty$. For such n' we distinguish the following cases:

(a) If $n' =$ constant, then

$$\frac{\frac{n_{k-1}}{N_{k-1}}}{1+\frac{n'}{N_{k-1}}} F(X_{n_{k-1}}, xy_{n'}) \to g_1(xy_{n'})(\text{ here } g_1(xy_{n'}) = 1 \text{ for } xy_{n'} > 1)$$

$$\frac{1}{1+\frac{N_{k-1}}{n'}} F(Y_{n'}, x) \to 0$$

and thus $F(Z_n, x) \to g_1(xy_{n'})$.

(b) If $n' \to \infty$, then $F(X_{n_{k-1}}, xy_{n'}) \to 1$; precisely $F(X_{n_{k-1}}, xy_{n'}) \to c_0(x)$.

(b1) If $\frac{n'}{N_{k-1}} \to 0$, then $F(Z_n, x) \to c_0(x)$.

(b2) If $\frac{n'}{N_{k-1}} \to \alpha \in (0, \infty)$, then $F(Z_n, x) \to \frac{1}{1+\alpha} c_0(x) + \frac{\alpha}{1+\alpha} g_2(x)$.

(b3) If $\frac{n'}{N_{k-1}} \to \infty$, then $F(Z_n, x) \to 0 + g_2(x)$.

For k-odd we use a similar computation. □

Now, identify $x_n = y_n$ and select x_n such that $g_1(x) = x$ (e.g., $x_n = n$ or $x_n = p_n$, the nth prime) and put $n_k = 2^{k^2}$ for $k = 1, 2, \dots$. Then the set of all d.f.s

$$\begin{aligned} G(Z_n) = &\big\{g_1(x), c_0(x)\big\} \cup \big\{g_1(xx_n); n = 1, 2, \dots\big\} \\ &\cup \left\{\frac{1}{1+\alpha} c_0(x) + \frac{\alpha}{1+\alpha} g_1(x); \alpha \in [0, \infty)\right\} \end{aligned}$$

is disconnected, as it can be seen in the figure on the page 174.

Example 4. Let x_n, $n = 1, 2, \dots$, be an increasing sequence of positive integers for which there exists a sequence n_k, $k = 1, 2, \dots$, of positive integers such that (as $k \to \infty$)

(i) $\frac{n_{k-1}}{n_k} \to 0$,

(ii) $\frac{n_k}{x_{n_k}} \to 0$,

(iii) $\frac{x_{n_{k-1}}}{x_{n_k}} \to 0$, and

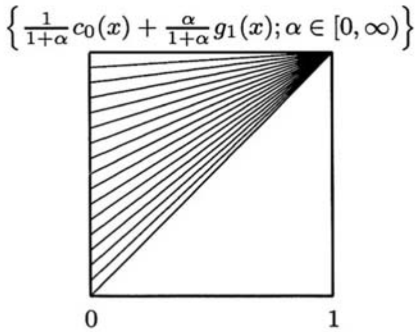
(iv) $x_{n_k - i} = x_{n_k} - i$ for $i = 0, 1, \dots, n_k - n_{k-1} - 1$.

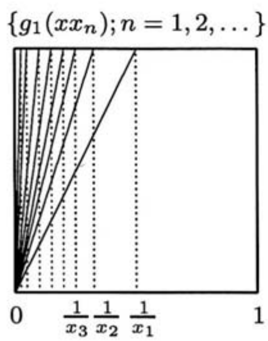
Then the sequence of blocks

$$X_n = \left(\frac{x_1}{x_n}, \frac{x_2}{x_n}, \dots, \frac{x_n}{x_n}\right)$$

has

$$G(X_n) = \big\{h_\alpha(x); \alpha \in [0, 1]\big\}.$$





P r o o f. For given $\theta \in [0,1]$ and $n = n_k - \big[\theta(n_k - n_{k-1})\big]$ and by (iv) we have

$$x_n = x_{n_k} - \big[\theta(n_k - n_{k-1})\big].$$

For $i \le n$ we distinguish two cases: $x_i \in (x_{n_{k-1}}, x_n]$ and $x_i \le x_{n_{k-1}}$.

(I) For $x_i \in (x_{n_{k-1}}, x_n]$ we have

$$\frac{x_i}{x_n} \in \left[\frac{x_{n_k} - (n_k - n_{k-1}) + 1}{x_{n_k} - [\theta(n_k - n_{k-1})]}, 1\right] \to [1,1]$$

as $n \to \infty$ and for any $\theta \in [0,1]$. The number of such x_i's is

$$(n_k - n_{k-1}) - \big[\theta(n_k - n_{k-1})\big] = (1-\theta)(n_k - n_{k-1}) + O(1).$$

(II) For $x_i \le x_{n_{k-1}}$ we have

$$\frac{x_i}{x_n} \in \left[0, \frac{x_{n_{k-1}}}{x_{n_k} - [\theta(n_k - n_{k-1})]}\right] \to [0,0].$$

We thus get, for any $x \in (0,1)$ and any sufficiently large n,

$$F(X_n, x) = \frac{n_{k-1}}{n} = \frac{n_{k-1}}{n_{k-1} + (1-\theta)(n_k - n_{k-1}) + O(1)}.$$

This gives:

(a) If $\theta \le \varepsilon_0 < 1$, for some fixed ε_0, then

$$F(X_n, x) \to c_1(x).$$

(b) If $\theta = 1$, then

$$F(X_n, x) \to c_0(x).$$

(c) For any $\alpha \in (0,1)$ there exists a sequence $\theta_k \to 1$, as $k \to \infty$, such that

$$\frac{n_{k-1}}{n_{k-1} + (1-\theta_k)(n_k - n_{k-1})} \to \alpha,$$

and in this case

$$F(X_n, x) \to h_\alpha(x).$$

□

Note that the sequences $n_k = 2^{k^2}$ and $x_{n_k} = 2^{(k+1)^2}$ satisfy the assumptions (i), (ii), (iii) and (iv). We also see that $G(X_n)$ is connected but

$F(X_{n_k+1}, x) \to c_0(x)$, and
$F(X_{n_k}, x) \to c_1(x)$,

a.e. on $[0,1]$ and thus $\rho(t_{n_k+1}, t_{n_k}) \to 1$. Using the permutation $\pi : \mathbb{N} \to \mathbb{N}$

$$1, 2, \dots, n_1, n_2, n_2 - 1, n_2 - 2, \dots, n_1 + 1, n_2 + 1, n_2 + 2, \dots n_3, n_4, n_4 - 1,$$
$$n_4 - 2, \dots, n_3 + 1, n_4 + 1, n_4 + 2, \dots, n_5, n_6, n_6 - 1, n_6 - 2, \dots, n_5 + 1, \dots$$

we have $\rho(t_{\pi(n+1)}, t_{\pi(n)}) \to 0$ as $n \to \infty$, because the "neighbouring" d.f. of $t_{\pi(n)}$ satisfies the scheme

$$c_1(x), c_1(x), \dots, c_0(x), c_0(x), \dots, c_1(x), c_1(x), \dots, c_0(x), c_0(x), \dots,$$
$$c_1(x), c_1(x), \dots, c_0(x), c_0(x), \dots$$

Example 5. In [8] is proved that $\frac{x_n}{x_{n+1}} \to 1$ does not imply that $G(X_n)$ is a singleton. This is a negative answer to the Problem 1.9.2 in [20].

Let a_k, n_k, $k = 1, 2, \dots$, and x_n, $n = 1, 2, \dots$ be three increasing integer sequences and $h_1 < h_2$ be two positive integers. Assume that

(i) $\frac{n_k}{n_{k+1}} \to 0$ for $k \to \infty$;
(ii) $\frac{a_k}{n_{k+1}} \to 0$ for $k \to \infty$;
(iii) for odd k we have
$$a_k^{h_2} \le x_{n_k} = (a_{k-1} + n_k - n_{k-1})^{h_1} \le (a_k + 1)^{h_2} \text{ and}$$
$$x_i = (a_k + i - n_k)^{h_2} \quad \text{for } n_k < i \le n_{k+1};$$
(iv) for even k we have
$$a_k^{h_1} \le x_{n_k} = (a_{k-1} + n_k - n_{k-1})^{h_2} \le (a_k + 1)^{h_1} \text{ and}$$
$$x_i = (a_k + i - n_k)^{h_1} \quad \text{for } n_k < i \le n_{k+1}.$$

Then $\frac{x_n}{x_{n+1}} \to 1$ and the set $G(X_n)$ of all distribution functions of the sequence of blocks X_n is $G(X_n) = G_1 \cup G_2 \cup G_3 \cup G_4$, where

$G_1 = \left\{x^{\frac{1}{h_2}}.t; t \in [0,1]\right\}$,
$G_2 = \left\{x^{\frac{1}{h_2}}(1-t) + t; t \in [0,1]\right\}$,
$G_3 = \left\{\max(0, x^{\frac{1}{h_1}} - (1 - x^{\frac{1}{h_1}})u); u \in [0,\infty)\right\}$ and
$G_4 = \left\{\min(1, x^{\frac{1}{h_1}}.v); v \in [1,\infty)\right\}$.

In [24, Th. 5.2, p. 762] = Theorem 15, it is proved that the condition $\frac{x_n}{x_{n+1}} \to 1$ implies the connectivity of $G(X_n)$

P r o o f. **1.** Firstly we prove that for any $h_1 < h_2$ the sequences a_k, n_k, x_n satisfying (i)–(iv) exist:

For $i = 1, \dots, n_1$ we put $x_i = i^{h_1}$ and then we find a_1 such that $a_1^{h_2} \le x_{n_1} \le (a_1+1)^{h_2}$. If we have selected, for an odd step k, all a_i, $i = 1, 2, \dots, k-1$, x_i, $i = 1, 2, \dots, n_k$, then we find a_k such that $a_k^{h_2} \le x_{n_k} < (a_k+1)^{h_2}$, and then we put $x_i = (a_k + i - n_k)^{h_2}$ for $n_k < i \le n_{k+1}$, where we choose n_{k+1} sufficiently large to satisfy the limits (i) and (ii). For an even step k we proceed similarly replacing h_2 by h_1.

2. In contrary to the independence of a_k and n_{k+1} we have

$$\frac{a_k}{n_k^{\frac{h_1}{h_2}}} \to 1 \ \text{ for odd } k \to \infty, \qquad \frac{a_k}{n_k^{\frac{h_2}{h_1}}} \to 1 \ \text{ for even } k \to \infty. \tag{62}$$

This follows from (iii) and (iv), directly, e.g., from (iii) we have

$$\frac{a_k^{h_2}}{n_k^{h_1}} < \left(\frac{a_{k-1}}{n_k} + 1 - \frac{n_{k-1}}{n_k}\right)^{h_1} < \frac{(a_k+1)^{h_2}}{n_k^{h_1}}.$$

As an application of (62) we have

$$\frac{a_k}{n_k} \to 0 \ \text{ for odd } k \to \infty, \qquad \frac{a_k}{n_k} \to \infty \ \text{ for even } k \to \infty. \tag{63}$$

3. Now we prove $\frac{x_i}{x_{i+1}} \to 1$ as $i \to \infty$. Let $i \in (n_k, n_{k+1})$ and let, e.g., k be odd. Then by (iii)

$$\frac{x_i}{x_{i+1}} = \left(1 - \frac{1}{a_k + i + 1 - n_k}\right)^{h_2} > \left(1 - \frac{1}{a_k}\right)^{h_2}$$

and for $i = n_k$ again

$$\frac{x_{n_k}}{x_{n_k+1}} > \frac{a_k^{h_2}}{(a_k+1)^{h_2}} > \left(1 - \frac{1}{a_k}\right)^{h_2}$$

which implies the limit 1 as odd $k \to \infty$. Similarly for even k.

4. Let $N \in [n_k, n_{k+1}]$ be an integer sequence (we shall omit the index in N_k) for $k \to \infty$. For $x \in (0,1)$ we have

$$\begin{aligned} F(X_N, x) &= \frac{\#\{1 \le i \le n_{k-1}; \frac{x_i}{x_N} < x\}}{N} \\ &\quad + \frac{\#\{n_{k-1} < i \le n_k; \frac{x_i}{x_N} < x\}}{N} + \frac{\#\{n_k < i \le N; \frac{x_i}{x_N} < x\}}{N} \\ &= o(1) + \frac{A}{N} + \frac{B}{N}. \end{aligned} \tag{64}$$

To compute $\frac{A}{N}$ for odd k we use

$$\frac{x_i}{x_N} = \frac{(a_{k-1} + i - n_{k-1})^{h_1}}{(a_k + N - n_k)^{h_2}} < x \Longleftrightarrow i - n_{k-1} < x^{\frac{1}{h_1}} (a_k + N - n_k)^{\frac{h_2}{h_1}} - a_{k-1}$$

and we have

$$\frac{A}{N}=\frac{\min(n_k-n_{k-1},\max(0,[x^{\frac{1}{h_1}}(a_k+N-n_k)^{\frac{h_2}{h_1}}-a_{k-1}]))}{N}. \tag{65}$$

Similarly, for even k

$$\frac{A}{N}=\frac{\min(n_k-n_{k-1},\max(0,[x^{\frac{1}{h_2}}(a_k+N-n_k)^{\frac{h_1}{h_2}}-a_{k-1}]))}{N}. \tag{66}$$

For $\frac{B}{N}$ and odd k we use

$$\frac{x_i}{x_N}=\left(\frac{a_k+i-n_k}{a_k+N-n_k}\right)^{h_2}<x \Longleftrightarrow i-n_k<x^{\frac{1}{h_2}}(a_k+N-n_k)-a_k$$

which gives

$$\frac{B}{N}=\frac{\min(N-n_k,\max(0,[x^{\frac{1}{h_2}}(a_k+N-n_k)-a_k]))}{N}. \tag{67}$$

Similarly, for even k we have

$$\frac{B}{N}=\frac{\min(N-n_k,\max(0,[x^{\frac{1}{h_1}}(a_k+N-n_k)-a_k]))}{N}. \tag{68}$$

In the following we will distinguish three cases

$$\frac{n_k}{N}\to t>0,\quad \frac{n_k}{N}\to 0\quad\text{and}\quad\frac{N}{n_{k+1}}\to 0,\quad\text{and}\quad\frac{N}{n_{k+1}}\to t>0.$$

5. Now, let $\frac{n_k}{N}\to t>0$ as $k\to\infty$.

a) Assume that k is odd and compute the limit of $\frac{A}{N}$ by (65). We have $\frac{n_k-n_{k-1}}{N}\to t$ and if $t<1$ we see

$$x^{\frac{1}{h_1}}\left(\frac{a_k}{N^{\frac{h_1}{h_2}}}+\frac{N}{N^{\frac{h_1}{h_2}}}\left(1-\frac{n_k}{N}\right)\right)^{\frac{h_2}{h_1}}-\frac{a_{k-1}}{N}\to\infty$$

since $\frac{N}{N^{\frac{h_1}{h_2}}}$ for $h_1<h_2$ is unbounded and by (62)

$$\frac{a_k}{N^{\frac{h_1}{h_2}}}=\frac{a_k}{n_k^{\frac{h_1}{h_2}}}\left(\frac{n_k}{N}\right)^{\frac{h_1}{h_2}}\to t^{\frac{h_1}{h_2}}$$

is bounded. Thus, for $0<t<1$, we have

$$\frac{A}{N}\to t\qquad\text{for odd}\quad k\to\infty. \tag{69}$$

a1) Let for the moment $t=1$. We have $\frac{a_k}{n_k^{\frac{h_1}{h_2}}}\to 1$ and

$$x^{\frac{1}{h_1}}\left(\frac{a_k}{N^{\frac{h_1}{h_2}}}+\frac{N-n_k}{N^{\frac{h_1}{h_2}}}\right)^{\frac{h_2}{h_1}}-\frac{a_{k-1}}{N}\to x^{\frac{1}{h_1}}(1+u)^{\frac{h_2}{h_1}}$$

assuming the limit $\frac{N-n_k}{N^{\frac{h_1}{h_2}}}\to u$, where $u\in[0,\infty)$ can be arbitrary. Put $v=(1+u)^{\frac{h_2}{h_1}}$. Thus for $t=1$ and corresponding $v\in[1,\infty)$ we have

$$\frac{A}{N}\to\min(1,x^{\frac{1}{h_1}}v)\qquad\text{for odd}\quad k\to\infty. \tag{70}$$

If $\frac{N-n_k}{N^{\frac{h_1}{h_2}}}\to\infty$, then

$$\frac{A}{N}\to 1\qquad\text{for odd}\quad k\to\infty. \tag{71}$$

b) Now, again $0<t\le 1$. For even k in (66) we have

$$x^{\frac{1}{h_2}}\left(\frac{a_k}{N^{\frac{h_2}{h_1}}}+\frac{N}{N^{\frac{h_2}{h_1}}}\left(1-\frac{n_k}{N}\right)\right)^{\frac{h_1}{h_2}}-\frac{a_{k-1}}{N}\to x^{\frac{1}{h_2}}.t$$

since by (62)

$$\frac{a_k}{N^{\frac{h_2}{h_1}}}=\frac{a_k}{n_k^{\frac{h_2}{h_1}}}\left(\frac{n_k}{N}\right)^{\frac{h_2}{h_1}}\to t^{\frac{h_2}{h_1}}.$$

Thus

$$\frac{A}{N}\to x^{\frac{1}{h_2}}.t\qquad\text{for even}\quad k\to\infty. \tag{72}$$

c) For the limit $\frac{B}{N}$ as odd $k\to\infty$ we compute (67) by using $\frac{N-n_k}{N}\to 1-t$ and

$$x^{\frac{1}{h_2}}\left(\frac{a_k}{N}+1-\frac{n_k}{N}\right)-\frac{a_k}{N}\to x^{\frac{1}{h_2}}(1-t)$$

since by (63) we have $\frac{a_k}{N}=\frac{a_k}{n_k}\frac{n_k}{N}\to 0$. Thus

$$\frac{B}{N}\to x^{\frac{1}{h_2}}(1-t)\qquad\text{for odd}\quad k\to\infty. \tag{73}$$

d) Again by (63), for even k we have $\frac{a_k}{N}=\frac{a_k}{n_k}\frac{n_k}{N}\to\infty$, then (assuming $x<1$)

$$x^{\frac{1}{h_1}}\left(\frac{a_k}{N}+1-\frac{n_k}{N}\right)-\frac{a_k}{N}\to-\infty.$$

Thus

$$\frac{B}{N}\to 0\qquad\text{for even}\quad k\to\infty. \tag{74}$$

e) Summing up (69), (72), (73) and (74) we find, for every $x\in(0,1)$,

$$F(X_N,x)\to\begin{cases}x^{\frac{1}{h_2}}(1-t)+t & \text{for odd } k\to\infty,\\ x^{\frac{1}{h_2}}.t & \text{for even } k\to\infty\end{cases} \tag{75}$$

for $\frac{n_k}{N}\to t$, $0<t<1$. For $\frac{n_k}{N}\to t=1$, $\frac{N-n_k}{N^{\frac{h_1}{h_2}}}\to u$ and $v=(1+u)^{\frac{h_2}{h_1}}$ we have applying (70)

$$F(X_N,x)\to\min(1,x^{\frac{1}{h_1}}.v)\qquad\text{for odd}\quad k\to\infty,\tag{76}$$

and for $\frac{N-n_k}{N^{\frac{h_1}{h_2}}}\to\infty$ we have

$$F(X_N,x)\to c_0(x)\qquad\text{for odd}\quad k\to\infty,\tag{77}$$

where $c_0(x)=1$ for $x\in(0,1)$.

6. In the case $\frac{n_k}{N}\to 0$ and $\frac{N}{n_{k+1}}\to 0$ we have $\frac{A}{N}=o(1)$ and then it suffices to compute the limit $\frac{B}{N}$ by (67) or (68).

a) Assume that odd $k\to\infty$. Since $\frac{N-n_k}{N}\to 1$ and by (63) we have $\frac{a_k}{N}=\frac{a_k}{n_k}\frac{n_k}{N}\to 0$ and thus

$$x^{\frac{1}{h_2}}\left(\frac{a_k}{N}+1-\frac{n_k}{N}\right)-\frac{a_k}{N}\to x^{\frac{1}{h_2}}.\tag{78}$$

b) Assume that even $k\to\infty$. In this case (by (62) and (ii)) we have

$$\frac{a_k}{N}=\frac{a_k}{n_k^{\frac{h_2}{h_1}}}\frac{n_k^{\frac{h_2}{h_1}}}{N},\quad\frac{a_k}{n_k^{\frac{h_2}{h_1}}}\to 1,\quad\frac{a_k}{n_{k+1}}\to 0,\qquad\text{then}\quad\frac{n_k^{\frac{h_2}{h_1}}}{n_{k+1}}\to 0.$$

Thus, for any $u\in[0,\infty)$ we can find a subsequence of N such that

$$\frac{n_k^{\frac{h_2}{h_1}}}{N}\to u.\tag{79}$$

Then

$$x^{\frac{1}{h_1}}\left(\frac{a_k}{N}+1-\frac{n_k}{N}\right)-\frac{a_k}{N}\to x^{\frac{1}{h_1}}-(1-x^{\frac{1}{h_1}})u.\tag{80}$$

c) Summing up (78) and (80) we find for every $x\in(0,1)$

$$F(X_N,x)\to\begin{cases}x^{\frac{1}{h_2}} & \text{for odd } k\to\infty,\\ \max(0,x^{\frac{1}{h_1}}-(1-x^{\frac{1}{h_1}})u) & \text{for even } k\to\infty\end{cases}\tag{81}$$

for $\frac{n_k}{N}\to 0$, $\frac{N}{n_{k+1}}\to 0$ and for $u\in(0,\infty)$ satisfying (79) if k is even. If $\frac{n_k^{\frac{h_2}{h_1}}}{N}\to\infty$ then

$$F(X_N,x)\to c_1(x)\qquad\text{for even}\quad k\to\infty,\tag{82}$$

where $c_1(x)=0$ for $x\in(0,1)$.

7. Finally, let $\frac{N}{n_{k+1}} \to t > 0$. Then $\frac{a_k}{N} \to 0$, because (ii) $\frac{a_k}{n_{k+1}} \to 0$. Computing the limit $\frac{B}{N}$ by (67) or (68) we find

$$F(N_N, x) \to \begin{cases} x^{\frac{1}{h_2}} & \text{for odd } k \to \infty, \\ x^{\frac{1}{h_1}} & \text{for even } k \to \infty. \end{cases} \tag{83}$$

8. Now, assume that $F(X_N, x) \to g(x)$ for some sequence of $N \in [n_k, n_{k+1}]$, i.e., $g(x) \in G(X_n)$. Then we can find subsequence of N (denoting again as N) such that $\frac{n_k}{N}$, $\frac{N-n_k}{N^{\frac{h_1}{h_2}}}$, $\frac{N}{n_{k+1}}$, and $\frac{n_k^{\frac{h_2}{h_1}}}{N}$ converge. Consequently $g(x)$ is contained in the collection of (75), (76), (77), (81), (82) and (83).

Thus the proof is finished. □

L. Mišík (2004, personal communication) found the following sequence x_n for which $c_1(x) \in G(X_n)$ and $c_0(x) \notin G(X_n)$ and consequently the implication Q.7 in [9] does not hold.

Example 6. Let x_n, $n = 1, 2, \dots$, be an increasing sequence of positive integers which satisfies the following conditions

(i) if $n_k = (k+1)(k-1)!2^{\frac{k(k-1)}{2}}$ for $k = 1, 2, \dots$, then $x_{n_k} = (k+1)n_k$,

(ii) if $n'_k = k(k-2)!2^{\frac{k(k-1)}{2}}$ then $x_{n'_k} = k^2 n'_k$,

(iii) if $n = 2^i n_{k-1} + j$, $0 \le j < 2^i n_{k-1}$ and $0 \le i < k-1$ for $k = 1, 2, \dots$, then $x_n = x_{n_{k-1}}(i+1)2^i + (i+3)kj$ (i.e., $n \in [n_{k-1}, n'_k]$),

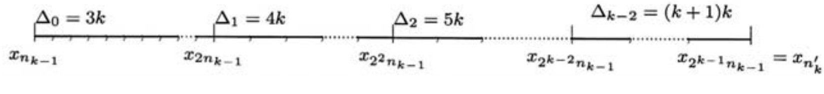
(iv) if $n \in [n'_k, n_k]$ for $k = 1, 2, \dots$, then $x_n = x_{n'_k} + n - n'_k$.

Then for the sequence of blocks

$$X_n = \left(\frac{x_1}{x_n}, \frac{x_2}{x_n}, \dots, \frac{x_n}{x_n} \right)$$

we have $c_1(x) \in G(X_n)$ but $c_0(x) \notin G(X_n)$.[5]

Proof. We start with the following figure:



Here for n running through $[2^i n_{k-1}, 2^{i+1} n_{k-1}]$, the x_n is equi-distributed in $[x_{2^i n_{k-1}}, x_{2^{i+1} n_{k-1}}]$ with difference Δ_i, where $i = 0, 1, \dots, k-2$.

[5] This and the Theorem 13 imply that $G(X_n) \not\subset \{c_\alpha(x); \alpha \in [0,1]\}$.

1^0. Using the definition of x_n we can see that $\frac{x_{n'_k}}{x_{n_k}} \to 1$ and $\frac{n'_k}{n_k} \to 0$ and thus we have $c_1(x) \in G(X_n)$.

2^0. On the contrary, assume that there exists increasing sequence $m'_l < m_l$, $l = 1, 2, \dots$, such that $m'_l \in [n_{k-1}, n_k]$, $k = k(l)$, (i) $\frac{x_{m'_l}}{x_{m_l}} \to 0$ and (ii) $\frac{m'_l}{m_l} \to 1$ as $l \to \infty$.

a) If $[2^j n_{k-1}, 2^{j+1} n_{k-1}] \subset [m'_l, m_l]$ for some $0 \le j \le k-2$, then

$$\frac{m'_l}{m_l} \le \frac{2^j n_{k-1}}{2^{j+1} n_{k-1}} = \frac{1}{2}$$

which contradicts (ii).

b) If $[m'_l, m_l] \subset [2^j n_{k-1}, 2^{j+2} n_{k-1}]$, then

$$\frac{x_{m'_l}}{x_{m_l}} \ge \frac{x_{2^j n_{k-1}}}{x_{2^{j+2} n_{k-1}}} = \frac{(j+1)2^j}{(j+3)2^{j+2}} = \left(1 - \frac{2}{j+3}\right)\frac{1}{4}$$

which contradicts (i).

c) If $[n'_k, n_k] \subset [m'_l, m_l]$, then

$$\frac{m'_l}{m_l} \le \frac{n'_k}{n_k} \to 0$$

which contradicts (ii).

d) If $m'_l \in [2^{k-2} n_{k-1}, n'_k]$ and $m_l \in [n'_k, n_k]$, i.e., $m_l = n'_k + i$, then (because $n'_k = 2^{k-1} m_{k-1}$ and $x_{m_l} = x_{n'_k} + i$)

$$\frac{x_{m'_l}}{x_{m_l}} \ge \frac{x_{2^{k-2} n_{k-1}}}{x_{m_l}} = \frac{x_{2^{k-2} n_{k-1}}}{x_{2^{k-1} n_{k-1}}} \cdot \frac{x_{n'_k}}{x_{m_l}} = \left(\frac{k-1}{k}\right) \cdot \frac{1}{2} \cdot \frac{1}{1 + \frac{i}{x_{n'_k}}},$$

$$\frac{m'_l}{m_l} \le \frac{n'_k}{m_l} = \frac{1}{1 + \frac{i}{n'_k}}.$$

Furthermore, (i) implies $\frac{i}{n'_k} \to 0$ and (ii) implies $\frac{i}{x_{n'_k}} = \frac{i}{k^2 n'_k} \to \infty$ which is impossible.

e) If $[2n_k, 2^2 n_k] \subset [m'_l, m_l]$ then

$$\frac{m'_l}{m_l} \le \frac{2n_k}{2^2 n_k} = \frac{1}{2}$$

which contradicts (ii).

f) Finally, assume that $m'_l \in [n'_k, n_k]$ and $m_l \in [n_k, 2n_k]$. Since $x_{2n_k} = 4x_{n_k}$, we have

$$\frac{x_{m'_l}}{x_{m_l}} \ge \frac{x_{n'_k}}{x_{2n_k}} = \frac{x_{n'_k}}{4x_{n_k}} \to \frac{1}{4}$$

which contradicts (i). $\square$

6. Historical remarks [21, 1.8.23]

For every $n = 1, 2, \dots$, let

$$X_n = (x_{n,1}, \dots, x_{n,N_n})$$

be a finite sequence in $[0, 1]$. The infinite sequence

$$\omega = (x_{1,1}, \dots, x_{1,N_1}, x_{2,1}, \dots, x_{2,N_2} \dots),$$

abbreviated as $\omega = (X_n)_{n=1}^{\infty}$, will be called *a block sequence* associated with the sequence of single blocks X_n, $n = 1, 2, \dots$ We will distinguish between block sequences and sequences of individual blocks. For the block sequence $\omega = (y_n)_{n=1}^{\infty}$ we can use the step d.f. $F_N(x)$ defined as

$$F_N(x) = \frac{\#\{n \le N; y_n < x\}}{N}$$

for $x \in [0, 1)$, and $F_N(1) = 1$. For individual blocks X_n, we define

$$F(X_n, x) = \frac{\#\{i \le N_n; x_{n,i} < x\}}{N_n}$$

for $x \in [0, 1)$ and $F(X_n, 1) = 1$.

A d.f. g is a d.f. of the sequence y_n if there exists an increasing sequence of positive integers $N_1, N_2, \dots$ such that

$$\lim_{k\to\infty} F_{N_k}(x) = g(x)$$

a.e. on $[0, 1]$.

A d.f. g is a d.f. of the sequence of single blocks X_n, if there exists an increasing sequence of positive integers $n_1, n_2, \dots$ such that

$$\lim_{k\to\infty} F(X_{n_k}, x) = g(x)$$

a.e. on $[0, 1]$.

Denote by $G(y_n)$ the set of all d.f. of the sequence y_n and denote by $G(X_n)$ the set of all d.f. of the sequence of single blocks X_n.

In the literature various types of blocks were published:

I. J. Schoenberg [17] introduced and studied the asymptotic distribution function (abbreviating a.d.f.) of X_n with $N_n = n$. For the definition see Section 2. He gave some criteria and mentioned a result of G. Pólya that

$$X_n = \left(\frac{n}{1}, \frac{n}{2}, \dots, \frac{n}{n}\right) \bmod 1$$

has a.d.f. $g(x) = \int_0^1 \frac{1-t^x}{1-t}\, dt$. E. Hlawka in the monograph [10, p. 57–60], called sequences of single blocks X_n, for $N_n = n$, *double sequences* and, for general N_n,

N_n*-double sequences*. As examples he included a proof of uniform distribution (abbreviating u.d.) for

$$X_n = \left(\frac{1}{n}, \frac{2}{n}, \dots, \frac{n}{n}\right), \quad \text{and} \quad X_n = \left(\frac{1}{n}, \frac{a_2}{n}, \dots, \frac{a_{\phi(n)}}{n}\right),$$

where $a_1 = 1 < a_2 < \cdots < a_{\phi(n)}$, g.c.d. $(a_i, n) = 1$ and $\phi(n)$ denotes Euler's function. U.d. for related block sequences $\omega = (X_n)_{n=1}^{\infty}$ is given in the monograph of L. Kuipers and H. Niederreiter [12, Lemma 4.1, Example 4.1, p. 136]. G. Myerson [13, p. 172] called a sequence of blocks X_n (without any ordering in X_n) a sequence of sets. The same terminology is used by H. Niederreiter in his book [14]. Myerson called the associated block sequence ω (X_n with some order) an *underlying sequence* and established criteria for u.d. of X_n. The sequence of single blocks X_n with $N_n = n$ is also called a *triangular array*. R. F. Tichy [25] gave some examples of u.d. of such X_n.

Let x_n be an increasing sequence of positive integers. Extending a result of S. Knapowski [11], Š. Porubsky, T. Šalát and O. Strauch [15] have investigated a sequence of blocks X_n of the type

$$X_n = \left(\frac{1}{x_n}, \frac{2}{x_n}, \dots, \frac{x_n}{x_n}\right).$$

They obtained a complete theory for the uniform distribution of the related block sequence $\omega = (X_n)_{n=1}^{\infty}$.

As we see in this paper we have concentrated only on the sequence of blocks X_n, $n = 1, 2, \dots$, with blocks

$$X_n = \left(\frac{x_1}{x_n}, \frac{x_2}{x_n}, \dots, \frac{x_n}{x_n}\right).$$

Finally, denote by $\mathbb{N}$ the set of all positive integers and if a subset $A \subset \mathbb{N}$ is given, define the ratio set $R(A)$ as $R(A) = \{a/b; a, b \in A\}$. Main result [22]: For every $A \subset \mathbb{N}$, if the lower asymptotic density $\underline{d}(A) \geq 1/2$ then the ratio set $R(A)$ is everywhere dense in $[0, \infty)$. Conversely, if $0 \leq \gamma < 1/2$ then there exists an $A \subset \mathbb{N}$ such that $\underline{d}(A) = \gamma$ and $R(A)$ is not everywhere dense in $[0, \infty)$.

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