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OSCILLATION CRITERIA FOR THIRD ORDER NEUTRAL NONLINEAR DYNAMIC EQUATIONS WITH DISTRIBUTED DEVIATING ARGUMENTS ON TIME SCALES

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ABSTRACT. Some new oscillation criteria for third order neutral nonlinear dynamic equations with distributed deviating arguments on time scales are established. The obtained results extend, improve and correlate many known oscillation results for third order dynamic equations.

1. Introduction

This paper is concerned with the oscillatory behavior of third order neutral nonlinear dynamic equations with distributed deviating arguments on time scales

$$\left(a(t)\left(\left(x(t)+\int_c^d p(t,\theta)x\big(h(t,\theta)\big)\Delta\theta\right)^{\Delta\Delta}\right)^{\alpha}\right)^{\Delta}+\int_a^b q(t,\tau)x^{\lambda}\big(g(t,\tau)\big)\Delta\tau=0 \tag{1.1}$$

and

$$\left(a(t)\left(\left(x^{\beta}\big(h_1(t)\big)-r(t)x^{\gamma}\big(h_2(t)\big)\right)^{\Delta\Delta}\right)^{\alpha}\right)^{\Delta}+\int_a^b q(t,\tau)x^{\lambda}\big(g(t,\tau)\big)\Delta\tau=0, \tag{1.2}$$

on an arbitrary time scale $\mathbb{T}\subseteq\mathbb{R}$ with $\sup\mathbb{T}=\infty$, $0<a<b$ and $0<c<d$. We assume that:

(i) α, β, γ and λ are the ratios of positive odd integers;

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(ii) $a, r\colon \mathbb{T} \to [0,\infty)$, $a(t) > 0$ are real valued, rd-continuous functions and

$$\int_{t_0}^{\infty} a^{-1/\alpha}(s)\Delta s = \infty, \qquad t_0 \in \mathbb{T}; \tag{1.3}$$

(iii) $q\colon \mathbb{T} \times [a,b] \to [0,\infty)$ and $p\colon \mathbb{T} \times [c,d] \to [0,\infty)$ are real valued, rd-continuous functions;

(iv) $g : \mathbb{T} \times [a,b] \to \mathbb{T}$ and $h : \mathbb{T} \times [c,d] \to \mathbb{T}$ are nonincreasing with respect to second variable,

$$g(t,\tau) \le t \quad \text{and} \quad \lim_{t\to\infty} g(t,\tau) = \infty, \qquad \tau \in [a,b],$$

and

$$h(t,\theta) \le t \quad \text{and} \quad \lim_{t\to\infty} h(t,\theta) = \infty, \qquad \theta \in [c,d];$$

(v) $h_i : \mathbb{T} \to \mathbb{T}$ for $i = 1, 2$, are real valued, rd-continuous nondecreasing functions such that $h_i(t) \le t$ for $t \ge t_0 \in \mathbb{T}$ and $\lim_{t\to\infty} h_i(t) = \infty$ for $i = 1, 2$.

We recall that a solution x of the equation (1.1) (respectively the equation (1.2)) is said to be nonoscillatory if there exists $t_0 \in \mathbb{T}$ such that $x(t)x\big(\sigma(t)\big) > 0$ for all $t \in [t_0,\infty)_{\mathbb{T}}$; otherwise, it is said to be oscillatory. The equation (1.1) (respectively the equation (1.2)) is said to be oscillatory if all its extendible solutions are oscillatory.

Neutral differential equations appear in modelling of the networks containing lossless transmission lines, in the study of vibrating masses attached to an elastic bar, as the Euler equation in some variational problems. In the theory of automatic control and in neuro-mechamial systems in which inertia plays an important role; see [11].

In recent years, there has been much research activity concerning the oscillation theory and applications of dynamic equations, see [1]–[10], [13]–[18] and the references contained therein. Particularly, the study content of oscillatory criteria of first and second dynamic equations on time scales is rich. In contrast, the study of oscillation criteria of third order dynamic equations is relatively less. Some interesting results have been obtained concerning the oscillatory and asymptotic behavior of some special cases of the equations (1.1) and (1.2); see [9], [12]. To the best of our knowledge, the oscillatory behavior of (1.1) and (1.2) have not been studied up to now.

The purpose of this paper is to establish some new criteria for the equations (1.1) and (1.2) by using the approach to reduce the problem is such a way that specific oscillation results for first and second order dynamic equations can be adapted for the third order case. In Section 2, we investigate the oscillatory behaviour of the equation (1.1) while Section 3 is devoted to study of oscillatory properties of the equation (1.2). The obtained results extend, improve and correlate many of the known oscillation results appeared in the literature that deal with special cases of the equations (1.1) and (1.2).

2. Oscillation of the equation (1.1)

In this section we begin with the following lemmas that are essential in the proofs of our results. For simplicity in what follows, whenever we write "$t \geq t_1$" we mean "$t \in [t_1, \infty) \cap \mathbb{T}$". It will be convenient to set

$$y(t) := x(t) + \int_c^d p(t,\theta) x\big(h(t,\theta)\big)\Delta\theta. \tag{2.1}$$

Equation (1.1) can be written as

$$\Big(a(t)\left(y^{\Delta\Delta}(t)\right)^{\alpha}\Big)^{\Delta} + \int_a^b q(t,\tau) x^{\lambda}\big(g(t,\tau)\big)\Delta\tau = 0. \tag{2.2}$$

Lemma 2.1. *Let the condition* (1.3) *hold and assume that x is an eventually positive solution of the equation* (1.1)*. Then there are only two possible cases for the behaviour of y for large $t \geq t_0$:*

(I) $y(t) > 0$, $y^{\Delta}(t) > 0$, $y^{\Delta\Delta}(t) > 0$ and $\left(a(t)\left(y^{\Delta\Delta}(t)\right)^{\alpha}\right)^{\Delta} \leq 0$;

(II) $y(t) > 0$, $y^{\Delta}(t) < 0$, $y^{\Delta\Delta}(t) > 0$ and $\left(a(t)\left(y^{\Delta\Delta}(t)\right)^{\alpha}\right)^{\Delta} \leq 0$.

Lemma 2.2. *Let condition* (1.3) *hold and assume that x is an eventually positive solution of equation* (1.1) *and the corresponding y satisfies Case (I) of Lemma 2.1. If*

$$0 \leq P(t) := \int_c^d p(t,\theta)\Delta\theta \leq p^* < 1, \tag{2.3}$$

then

$$x(t) \geq (1-p^*)y(t), \qquad \textit{eventually.} \tag{2.4}$$

Proof. Since x is an eventually positive solution of the equation (1.1) and the corresponding y is satisfying Case (I) of Lemma 2.1. Then there exists a $t_1 \geq t_0 \in \mathbb{T}$ such that

$$x(t) > 0,\ x\big(h(t,\theta)\big) > 0 \quad \text{and} \quad y^{\Delta}(t) > 0 \qquad \text{for} \quad t \geq t_1 \text{ and } \theta \in [c,d].$$

Now,

$$\begin{aligned} x(t) &= y(t) - \int_c^d p(t,\theta)x\big(h(t,\theta)\big)\Delta\theta \\ &\geq y(t) - \int_c^d p(t,\theta)y\big(h(t,\theta)\big)\Delta\theta \\ &\geq y(t) - \left(\int_c^d p(t,\theta)\Delta\theta\right) y\,(t) \\ &= \left(1 - \int_c^d p(t,\theta)\Delta\theta\right) y\,(t) \qquad \text{for} \quad t \geq t_1. \end{aligned}$$

This completes the proof. □

Lemma 2.3. *Let the condition* (1.3) *hold and assume that x is an eventually positive solution of the equation* (1.1) *and the corresponding y satisfies Case (I) of Lemma 2.1. Then for $t \geq t_1 \in \mathbb{T}$*

$$y(t) \geq \Big(a(t)\left(y^{\Delta\Delta}(t)\right)^{\alpha}\Big)^{1/\alpha} \int_{t_1}^{t}\int_{t_1}^{u} a^{-1/\alpha}(s)\Delta s\Delta u. \tag{2.5}$$

P r o o f. Since x is an eventually positive solution of the equation (1.1) and the corresponding y is satisfying Case (I) of Lemma 2.1. Then there exists $t_1 \geq t_0 \in \mathbb{T}$ such that

$$x(t) > 0 \quad \text{and} \quad x\big(h(t,\theta)\big) > 0 \qquad \text{for} \quad t \geq t_1 \text{ and } \theta \in [c,d].$$

By using the fact that $a(t)\big(y^{\Delta\Delta}(t)\big)^{\alpha}$ is decreasing for $t \geq t_1$, we have

$$\begin{aligned} y^{\Delta}(t) &\geq y^{\Delta}(t) - y^{\Delta}(t_1) \\ &= \int_{t_1}^{t} \Big(a(s)\big(y^{\Delta\Delta}(s)\big)^{\alpha}\Big)^{1/\alpha} a^{-1/\alpha}(s)\Delta s \\ &\geq \Big(a(t)\big(y^{\Delta\Delta}(t)\big)^{\alpha}\Big)^{1/\alpha} \int_{t_1}^{t} a^{-1/\alpha}(s)\Delta s. \end{aligned}$$

Integrating this inequality from t_1 to t, we obtain the desired result. □

Let

$$Q(t) := \int_a^b q(t,\tau)\Delta\tau, \quad g_1(t) := g(t,a) \quad \text{and} \quad g_2(t) := g(t,b). \tag{2.6}$$

In the following result, we employ the following auxiliary equation

$$z^{\Delta}(t) + (1-p^*)^{\lambda} Q(t) \left(\int_{t_1}^{g_2(t)} \int_{t_1}^{u} a^{-1/\alpha}(s)\Delta s \Delta u \right)^{\lambda} z^{\lambda/\alpha}\big(g_2(t)\big) = 0 \tag{2.7}$$

for $t \ge t_1$, $t_1 \in [t_0,\infty)_{\mathbb{T}}$.

Lemma 2.4. *Let the conditions* (1.3) *and* (2.3) *hold and the equation* (2.7) *be oscillatory, then the equation* (1.1) *has no eventually positive solution x such that y defined by* (2.1) *satisfies Case (I) of Lemma 2.1.*

Proof. Let $t_0 \in \mathbb{T}$ be sufficiently large such that $x(t) > 0$, $x\big(h(t,\theta)\big) > 0$ and $x\big(g(t,\tau)\big) > 0$ for $t \ge t_0$, $\theta \in [c,d]$ and $\tau \in [a,b]$ and assume that y satisfies Case (I) of Lemma 2.1. Using (2.4) and (2.5) in the equation (2.2), we get

$$z^{\Delta}(t) + (1-p^*)^{\lambda}\, Q(t) \left(\int_{t_1}^{g_2(t)} \int_{t_1}^{u} a^{-1/\alpha}(s)\Delta s \Delta u \right)^{\lambda} z^{\lambda/\alpha}\big(g_2(t)\big) \le 0, \tag{2.8}$$

for $t_1 \in [t_0,\infty)_{\mathbb{T}}$, where $z(t) := a(t)\left(y^{\Delta\Delta}(t)\right)^{\alpha} > 0$. Integrating (2.8) from t to $u \ge t$ and letting $u \to \infty$, we have

$$z(t) \ge G\big(t, z(t)\big),$$

where

$$G\big(t,z(t)\big) := (1-p^*)^{\lambda} \int_t^{\infty} Q(v) \left(\int_{t_1}^{g_2(v)} \int_{t_1}^{u} a^{-1/\alpha}(s)\Delta s \Delta u \right)^{\lambda} z^{\lambda/\alpha}\big(g_2(v)\big)\Delta v.$$

Now, we define a sequence of successive approximations $\{w_j(t)\}$ as follows:

$$\begin{aligned} w_0(t) &:= z(t), \\ w_{j+1}(t) &:= G\big(t, w_j(t)\big), \qquad j = 0,1,2,\dots \end{aligned}$$

It is easy to show that

$$0 < w_j(t) \le z(t) \quad \text{and} \quad w_{j+1}(t) \le w_j(t), \qquad j = 0,1,2,\dots$$

Then, the sequence $\{w_j(t)\}$ is nonincreasing and bounded for each $t \ge t_1$. This means that we may define $w(t) := \lim_{j\to\infty} w_j(t) \ge 0$. Since

$$0 \le w(t) \le w_j(t) \le z(t) \qquad \text{for all} \quad j \ge 0.$$

By the Lebesgue's dominated convergence theorem on time scale, one can easily find

$$w(t) = G\big(t, w(t)\big).$$

Therefore,

$$w^{\Delta}(t) = -\left(1-p^*\right)^{\lambda} Q(t) \left(\int_{t_1}^{g_2(t)} \int_{t_1}^{u} a^{-1/\alpha}(s)\Delta s \Delta u\right)^{\lambda} w^{\lambda/\alpha}\big(g_2(t)\big).$$

Hence the equation (2.7) has a positive solution $w(t)$. This completes the proof. $\square$

In the case when $\lambda = \alpha$ in the inequality (2.8), we state the following lemma.

Lemma 2.5 ([1]). *If* $\lambda = \alpha$,

$$\limsup_{t\to\infty} \sup_{\xi\in E} \Big\{\xi e_{-\xi\eta}\big(t, g_2(t)\big)\Big\} < 1,$$

where

$$E := \big\{\xi : \xi > 0,\ 1 - \xi\eta(t)\mu(t) > 0\big\},$$

and

$$\eta(t) := \left(1-p^*\right)^{\alpha} Q(t) \left(\int_{t_1}^{g_2(t)} \int_{t_1}^{u} a^{-1/\alpha}(s)\Delta s \Delta u\right)^{\alpha},$$

then the inequality (2.8) *has no eventually positive solution.*

Lemma 2.6. *Let the condition* (1.3) *hold and assume that $x(t)$ is an eventually positive solution the equation* (1.1) *and the corresponding y satisfies Case (I) of Lemma 2.1. Then there exists $t_1 \in [t_0,\infty)_{\mathbb{T}}$ such that*

$$y(t) \geq y^{\Delta}(t) A(t,t_1) \qquad \text{for} \quad t \in (t_1,\infty)_{\mathbb{T}}, \tag{2.9}$$

where

$$A(t,t_1) := \frac{\int_{t_1}^{t}\int_{t_1}^{u} a^{-1/\alpha}(s)\Delta s\ \Delta u}{\int_{t_1}^{t} a^{-1/\alpha}(s)\Delta s}.$$

P r o o f. Let $t_0 \in \mathbb{T}$ be sufficiently large such that $x(t) > 0$, $x\big(h(t,\theta)\big) > 0$ and $x\big(g(t,\tau)\big) > 0$ for $t \geq t_0$, $\theta \in [c,d]$ and $\tau \in [a,b]$ and assume that y satisfies Case (I) of Lemma 2.1 for $t \geq t_0$. From Lemma 2.3, we have

$$y^{\Delta}(t) \geq \Big(a(t)\left(y^{\Delta\Delta}(t)\right)^{\alpha}\Big)^{1/\alpha} \int_{t_1}^{t} a^{-1/\alpha}(s)\Delta s \qquad \text{for} \quad t \geq t_1 \geq t_0.$$

Note that

$$\left[\frac{y^{\Delta}(t)}{\int_{t_1}^{t} a^{-1/\alpha}(s)\Delta s}\right]^{\Delta} = \frac{a^{-1/\alpha}(t)\Big[\big(a(t)\left(y^{\Delta\Delta}(t)\right)^{\alpha}\big)^{1/\alpha}\int_{t_1}^{t} a^{-1/\alpha}(s)\Delta s - y^{\Delta}(t)\Big]}{\int_{t_1}^{t} a^{-1/\alpha}(s)\Delta s \int_{t_1}^{\sigma(t)} a^{-1/\alpha}(s)\Delta s},$$

we have

$$\left[\frac{y^{\Delta}(t)}{\int_{t_1}^{t} a^{-1/\alpha}(s)\Delta s}\right]^{\Delta} < 0 \qquad \text{for} \quad t \in (t_1, \infty)_{\mathbb{T}}.$$

Then

$$\begin{aligned} y(t) &\geq y(t) - y(t_1) \\ &= \int_{t_1}^{t} \frac{y^{\Delta}(u)}{\int_{t_1}^{u} a^{-1/\alpha}(s)\Delta s} \left(\int_{t_1}^{u} a^{-1/\alpha}(s)\Delta s\right) \Delta u \\ &\geq \frac{y^{\Delta}(t)}{\int\limits_{t_1}^{t} a^{-1/\alpha}(s)\Delta s} \int_{t_1}^{t}\int_{t_1}^{u} a^{-1/\alpha}(s)\Delta s \Delta u \\ &= y^{\Delta}(t) A(t, t_1). \end{aligned}$$

This completes the proof. $\square$

For $g_2(t) > t_0$, we set

$$\bar{Q}(t) := (1 - p^*)^{\lambda} A^{\lambda}\big(g_2(t), t_0\big) Q(t).$$

The hypotheses of next two lemmas include knowledge of the behaviour of the solution of the second order dynamic equation

$$\Big(a(t)\left(z^{\Delta}(t)\right)^{\alpha}\Big)^{\Delta} + \bar{Q}(t) z^{\lambda}\big(g_2(t)\big) = 0, \tag{2.10}$$

Lemma 2.7. *If the conditions* (1.3) *and* (2.3) *hold and the equation* (2.10) *is oscillatory, then the equation* (1.1) *has no eventually positive solution x such that y is defined by* (2.1) *which satisfies Case (I) of Lemma 2.1.*

Proof. Let $x(t)$ be an eventually positive solution of the equation (1.1), say $x(t) > 0$ and $x\big(h(t,\theta)\big) > 0$ and $x\big(g(t,\tau)\big) > 0$ for $t \geq t_1$ for some $t_1 \in [t_0, \infty)_{\mathbb{T}}$, $\theta \in [c, d]$ and $\tau \in [a, b]$ and assume that y satisfies Case (I) of Lemma 2.1. From (2.9), there exist a constant k_1, $0 < k_1 < 1$ and a $t_2 > t_1$ such that

$$y(t) \geq A(t, t_1) y^{\Delta}(t) \qquad \text{for} \quad t \geq t_2. \tag{2.11}$$

From (2.4), we see that

$$x(t) \geq (1 - p^*)\, y(t) \qquad \text{for} \quad t \geq t_3, \tag{2.12}$$

for some $t_3 \geq t_2$. Hence, there exists $t_4 \geq t_3$ such that

$$y\big(g_2(t)\big) \geq A\big(g_2(t), t_1\big) y^{\Delta}\big(g_2(t)\big) \qquad \text{for} \quad t \geq t_4. \tag{2.13}$$

Using (2.12) and (2.13) in the equation (2.2) we have

$$\Big(a(t)\left(z^{\Delta}(t)\right)^{\alpha}\Big)^{\Delta} + \bar{Q}(t) z^{\lambda}\big(g_2(t)\big) \leq 0, \tag{2.14}$$

for $t \geq t_4$, where $z(t) := y^{\Delta}(t) > 0$. Integrating (2.14) from t to $u \geq t \geq t_4$ and letting $u \to \infty$, we obtain

$$z^{\Delta}(t) \geq \left(\frac{1}{a(t)} \int_t^{\infty} \bar{Q}(s) z^{\lambda}\big(g_2(s)\big) \Delta s \right)^{1/\alpha}. \tag{2.15}$$

Integrating (2.15) from t_4 to $t \geq t_4$, we obtain

$$z(t) \geq z(t_4) + \int_{t_4}^{t} \left(\frac{1}{a(u)} \int_u^{\infty} \bar{Q}(s) z^{\lambda}\big(g_2(s)\big) \Delta s \right)^{1/\alpha} \Delta u.$$

Next, we define a sequence $\{w_m(t)\}_{m \in \mathbb{N}_0}$ by

$$\begin{aligned} w_0(t) &= z(t), \\ w_{m+1}(t) &= z(t_4) + \int_{t_4}^{t} \left(\frac{1}{a(u)} \int_u^{\infty} \bar{Q}(s) z^{\lambda}\big(g_2(s)\big) \Delta s \right)^{1/\alpha} \Delta u, \qquad m \in \mathbb{N}_0. \end{aligned}$$

It is easy to check by induction that $\{w_m(t)\}$ is a well-defined decreasing sequence satisfying

$$z(t_4) \leq w_m(t) \leq z(t) \qquad \text{for} \quad t \geq t_4 \ \text{ and } \ m \in \mathbb{N}_0.$$

Thus, there exists a function w on $[t_4, \infty)_{\mathbb{T}}$ such that

$$\lim_{m \to \infty} w_m(t) = w(t) \quad \text{and} \quad z(t_4) \leq w(t) \leq z(t).$$

By the Lebesgue's dominated convergence theorem on time scale, it follows that

$$w(t) = z(t_4) + \int_{t_4}^{t} \left(\frac{1}{a(u)} \int_u^{\infty} \bar{Q}(s) w^{\lambda}\big(g_2(s)\big) \Delta s \right)^{1/\alpha} \Delta u. \tag{2.16}$$

Differentiating (2.16) twice, we conclude that w is a nonoscillatory solution of the equation (2.10) with the desired property. This completes the proof of the lemma. □

The following lemma makes use of the auxiliary equation

$$\left(\frac{1}{\bar{A}^{\alpha}(t)} \big(v^{\Delta}(t)\big)^{\alpha} \right)^{\Delta} + (1 - p^*)^{\lambda}\, Q(t) v^{\lambda}\big(g_2(t)\big) = 0 \qquad \text{for} \quad t \geq t_2, \tag{2.17}$$

where $\bar{A}(t) := \int_{t_1}^{t} a^{-1/\alpha}(s) \Delta s$ for sufficiently large $t_2 \in (t_1, \infty)_{\mathbb{T}}$ for some $t_1 \in [t_0, \infty)_{\mathbb{T}}$.

Lemma 2.8. *If the conditions* (1.3) *and* (2.3) *hold and the equation* (2.17) *is oscillatory, then the conclusion of Lemma 2.7 holds.*

Proof. Let $x(t)$ be an eventually positive solution of the equation (1.1), say $x(t) > 0$ and $x\big(h(t,\theta)\big) > 0$ and $x\big(g(t,\tau)\big) > 0$ for $t \ge t_1$ for some $t_1 \in [t_0,\infty)_{\mathbb{T}}$, $\theta \in [c,d]$ and $\tau \in [a,b]$ and let y satisfy Case (I) of Lemma 2.1. Hence

$$\begin{aligned} y^{\Delta}(t) &= y^{\Delta}(t_1) + \int_{t_1}^{t} \frac{a^{1/\alpha}(s) y^{\Delta\Delta}(s)}{a^{1/\alpha}(s)} \Delta s \ge a^{1/\alpha}(t) y^{\Delta\Delta}(t) \int_{t_1}^{t} \frac{\Delta s}{a^{1/\alpha}(s)} \\ &= a^{1/\alpha}(t) y^{\Delta\Delta}(t) \bar{A}(t) \qquad \text{for} \quad t \ge t_1. \end{aligned}$$

An integration yields

$$y(t) \ge y(t_1) + \int_{t_1}^{t} \bar{A}(u) a^{1/\alpha}(u) y^{\Delta\Delta}(u) \Delta u. \tag{2.18}$$

Using (2.4) in the equation (2.2), integrating from u to $v \ge u \ge t_1$ and letting $v \to \infty$, we have

$$y^{\Delta\Delta}(u) \ge \frac{1}{a^{1/\alpha}(u)} \left(\int_{u}^{\infty} (1-p^*)^{\lambda} Q(s) y^{\lambda}\big(g_2(s)\big) \Delta s \right)^{1/\alpha}. \tag{2.19}$$

Substituting (2.19) into (2.18) gives

$$y(t) \ge y(t_2) + \int_{t_1}^{t} \bar{A}(u) \left(\int_{u}^{\infty} (1-p^*)^{\lambda} Q(s) y^{\lambda}\big(g_2(s)\big) \Delta s \right)^{1/\alpha} \Delta u$$

Next we define the sequences $\{v_m(t)\}_{m \in \mathbb{N}_0}$ by

$$\begin{aligned} v_0(t) &= y(t), \\ v_{m+1}(t) &= y(t_1) + \int_{t_1}^{t} \bar{A}(u) \left(\int_{u}^{\infty} (1-p^*)^{\lambda} Q(s) y^{\lambda}\big(g_2(s)\big) \Delta s \right)^{1/\alpha} \Delta u, \quad m \in \mathbb{N}_0. \end{aligned}$$

The remainder of the proof is similar to the proof of Lemma 2.7 and is omitted. □

Next, we present the following result.

Lemma 2.9. *Let the conditions* (1.3) *and* (2.3) *hold and assume that x is an eventually positive solution of the equation* (1.1) *and the corresponding y satisfies Case (II) of Lemma 2.1. Then either*

$$x(t) \ge \left(\frac{1 - p^* \delta}{\delta} \right) y(t), \tag{2.20}$$

eventually, where $\delta > 1$ is any constant with $p^*\delta < 1$ and p^* is as in (2.3), *or* $\lim_{t\to\infty} x(t) = 0$.

P r o o f. Pick $t_0 \in \mathbb{T}$ such that

$$x(t) > 0 \quad \text{and} \quad x\big(h(t,\theta)\big) > 0 \quad \text{for} \quad t \geq t_0 \ \text{ and } \ \theta \in [c,d].$$

Also, since $y(t)$ satisfies Case (II) of Lemma 2.1, then there exists a constant k such that

$$\lim_{t\to\infty} y(t) = k < \infty.$$

(i) Assume that $k > 0$, then we have

$$k < y(t) < k\delta, \qquad \text{for all} \quad \delta > 1 \ \text{ and } \ t \geq t_0. \tag{2.21}$$

Now,

$$x(t) = y(t) - \int_c^d p(t,\theta) x\big(h(t,\theta)\big)\Delta\theta,$$

and so

$$x(t) \geq k - kp^*\delta = \left(\frac{1-p^*\delta}{\delta}\right) k\delta \geq \left(\frac{1-p^*\delta}{\delta}\right) y(t) \qquad \text{for} \quad t \geq t_0.$$

(ii) Assume that $k = 0$, then $\lim_{t\to\infty} y(t) = 0$. Since $0 < x(t) \leq y(t)$ on $[t_0,\infty)_{\mathbb{T}}$, then $\lim_{t\to\infty} x(t) = 0$. This completes the proof of the lemma. □

Lemma 2.10. *Let the conditions* (1.3) *and* (2.3) *hold and assume that* $x(t)$ *is an eventually positive solution of the equation* (1.1) *and the corresponding* y *satisfies Case (II) of Lemma 2.1. If*

$$\int_{t_0}^{\infty}\int_{v}^{\infty} \left[\frac{1}{a(u)}\int_u^{\infty} Q(s)\Delta s\right]^{1/\alpha} \Delta u \Delta v = \infty, \tag{2.22}$$

then $\lim_{t\to\infty} x(t) = 0$.

P r o o f. Proceeding as in the proof of Lemma 2.9, we obtain either (2.20) holds or $\lim_{t\to\infty} x(t) = 0$. We suppose that

$$x(t) \geq \left(\frac{1-p^*\delta}{\delta}\right) y(t), \qquad \text{for} \quad t \geq t_1 \geq t_0. \tag{2.23}$$

Using (2.23) in the equation (2.2), we have

$$\begin{aligned} \left(a(t)\left(y^{\Delta\Delta}(t)\right)^{\alpha}\right)^{\Delta} &= -\int_a^b q(t,\tau)\ x^{\lambda}\big(g(t,\tau)\big)\Delta\tau \\ &\leq -\left(\frac{1-p^*\delta}{\delta}\right)^{\lambda} \int_a^b q(t,\tau)\ y^{\lambda}\big(g(t,\tau)\big)\Delta\tau \\ &\leq -\left(\frac{1-p^*\delta}{\delta}\right)^{\lambda} Q(t) y^{\lambda}\big(g_1(t)\big) \qquad \text{for} \quad t \geq t_1 \geq t_0. \end{aligned} \tag{2.24}$$

Integrating this inequality from t to $u \geq t \geq t_1$ and letting $u \to \infty$, we have

$$y^{\Delta\Delta}(t) \geq \left(\frac{1-p^*\delta}{\delta}\right)^{\lambda/\alpha} \left(\frac{1}{a(t)}\int_t^\infty Q(s)y^\lambda\big(g_1(s)\big)\Delta s\right)^{1/\alpha}. \tag{2.25}$$

Using (2.21) in (2.25), we get

$$y^{\Delta\Delta}(t) \geq c\left(\frac{1}{a(t)}\int_t^\infty Q(s)\Delta s\right)^{1/\alpha}, \tag{2.26}$$

where $c := \big(k\big[\frac{1-p^*\delta}{\delta}\big]\big)^{\lambda/\alpha}$. Integrating (2.26) twice, we obtain

$$\infty > y(t_1) \geq c\int_{t_1}^{t}\int_{v}^{\infty}\left(\frac{1}{a(u)}\int_u^\infty Q(s)\Delta s\right)^{1/\alpha}\Delta u\Delta v \to \infty \qquad \text{as } t \to \infty,$$

which is a contradiction. This completes the proof of the lemma. $\square$

Lemma 2.11. *Let g_1 be a nondecreasing on $[t_0,\infty)_{\mathbb{T}}$, conditions* (1.3) *and* (2.3) *hold and assume that $x(t)$ is an eventually positive solution of the equation* (1.1) *and the corresponding y satisfies Case (II) of Lemma 2.1. If*

$$\limsup_{t\to\infty}\int_{g_1(t)}^{t}\left(\frac{1}{a(u)}\int_u^t Q(s)\big[g_1(t)-g_1(s)\big]^\lambda\Delta s\right)^{1/\alpha}\Delta u > \begin{cases} c & if\ \lambda=\alpha,\\ 0 & if\ \lambda<\alpha,\end{cases} \tag{2.27}$$

where Q and g_1 are as in (2.6), *$c := \frac{\delta}{1-p^*\delta}$, p^* and δ are as in Lemma 2.9, then* $\lim_{t\to\infty} x(t) = 0$.

P r o o f. Proceeding as in the proof of Lemma 2.10, we obtain (2.24). We also have

$$\begin{aligned} -y\big(g_1(s)\big) &\leq y\big(g_1(t)\big)-y\big(g_1(s)\big) = \int_{g_1(s)}^{g_1(t)} y^\Delta(\tau)\Delta\tau \\ &\leq y^\Delta\big(g_1(t)\big)\int_{g_1(s)}^{g_1(t)}\Delta\tau = y^\Delta\big(g_1(t)\big)\big[g_1(t)-g_1(s)\big], \end{aligned} \tag{2.28}$$

for $t \geq s \geq t_0$. Integrating (2.24) from u to $t \geq u \geq t_0$, we obtain

$$y^{\Delta\Delta}(u) \geq \left(\frac{1-p^*\delta}{\delta}\right)^{\lambda/\alpha}\left(\frac{1}{a(u)}\int_u^t Q(s)y^\lambda\big(g_1(s)\big)\Delta s\right)^{1/\alpha}. \tag{2.29}$$

Substituting (2.28) into (2.29), we obtain

$$y^{\Delta\Delta}(u) \geq \left(\frac{1-p^*\delta}{\delta}\right)^{\lambda/\alpha}\left(\frac{1}{a(u)}\int_u^t Q(s)\left[g_1(t)-g_1(s)\right]^{\lambda}\Delta s\right)^{1/\alpha}\left(-y^{\Delta}\left(g_1(t)\right)\right)^{\lambda/\alpha}.$$

Integrating from $g_1(t) \geq t_0$ to t gives

$$\begin{aligned}&-y^{\Delta}\left(g_1(t)\right) \geq y^{\Delta}(t) - y^{\Delta}\left(g_1(t)\right) \geq \\ &\left(\frac{1-p^*\delta}{\delta}\right)^{\lambda/\alpha}\left(-y^{\Delta}\left(g_1(t)\right)\right)^{\lambda/\alpha}\int_{g_1(t)}^t\left(\frac{1}{a(u)}\int_u^t Q(s)\left[g_1(t)-g_1(s)\right]^{\lambda}\Delta s\right)^{1/\alpha}\Delta u.\end{aligned}$$

So

$$\begin{aligned}&\left(-y^{\Delta}\left(g_1(t)\right)\right)^{1-\lambda/\alpha} \geq \\ &\qquad\left(\frac{1-p^*\delta}{\delta}\right)^{\lambda/\alpha}\int_{g_1(t)}^t\left(\frac{1}{a(u)}\int_u^t Q(s)\left[g_1(t)-g_1(s)\right]^{\lambda}\Delta s\right)^{1/\alpha}\Delta u. \qquad (2.30)\end{aligned}$$

Taking lim sup as $t \to \infty$ of both sides of the above inequality. If $\lambda = \alpha$, the contradiction is obvious. If $\lambda < \alpha$, then the left hand side of (2.30) is positive and must decrease to zero (to prevent a contradiction to the positivity of $y(t)$). This contradicts (2.27) and completes the proof of the lemma. □

Lemma 2.12. *Let the hypotheses of Lemma 2.11 hold with the condition* (2.27) *be replaced by*

$$\limsup_{t\to\infty}\int_{g_1(t)}^t\int_v^t\left(\frac{1}{a(u)}\int_u^t Q(s)\Delta s\right)^{1/\alpha}\Delta u\Delta v > \begin{cases} c & if\ \ \lambda = \alpha, \\ 0 & if\ \ \lambda < \alpha. \end{cases} \qquad (2.31)$$

Then the conclusion of Lemma 2.11 holds.

Proof. As in the proof of Lemma 2.11, we obtain (2.29) and integrating we have

$$\begin{aligned}-y^{\Delta}(v) &\geq y^{\Delta}(t) - y^{\Delta}(v) \\ &\geq \left(\frac{1-p^*\delta}{\delta}\right)^{\lambda/\alpha} y^{\lambda/\alpha}\left(g_1(t)\right)\int_v^t\left(\frac{1}{a(u)}\int_u^t Q(s)\Delta s\right)^{1/\alpha}\Delta u.\end{aligned}$$

Integrating from $g_1(t) \geq t_0$ to t yields

$$y^{1-\lambda/\alpha}\left(g_1(t)\right) \geq \left(\frac{1-p^*\delta}{\delta}\right)^{\lambda/\alpha}\int_{g_1(t)}^t\int_v^t\left(\frac{1}{a(u)}\int_u^t Q(s)\Delta s\right)^{1/\alpha}\Delta u\Delta v.$$

Taking $\limsup$ as $t \to \infty$ gives a contradiction to the condition (2.31). This completes the proof of the lemma. $\square$

We are now ready to present the main results in this section.

Theorem 2.1. *Let* (1.3), (2.3), (2.22) *and either one of the equations* (2.7), (2.10) *or* (2.17) *be oscillatory, then every solution* $x(t)$ *of the equation* (1.1) *oscillates or satisfies* $\lim_{t\to\infty} x(t) = 0$.

Proof. Let $x(t)$ be an eventually positive solution of the equation (1.1), say

$$x(t) > 0 \quad \text{and} \quad x\big(h(t,\theta)\big) > 0 \quad \text{and} \quad x\big(g(t,\tau)\big) > 0$$

for $t \geq t_1$ for some $t_1 \in [t_0, \infty)_{\mathbb{T}}$, $\theta \in [c, d]$ and $\tau \in [a, b]$. Then $y(t)$ satisfies one of the two cases of Lemma 2.1. By either Lemmas 2.4, 2.7 or 2.8, Case (I) cannot hold. If Case (II) holds, Lemma 2.10 implies $\lim_{t\to\infty} x(t) = 0$. This proves the theorem. $\square$

Next, we establish another new oscillation criteria for the equation (1.1).

Theorem 2.2. *Let* $\lambda \leq \alpha$, g_1 *be a nondecreasing on* $[t_0, \infty)_{\mathbb{T}}$, (1.3) *and* (2.3) *hold. If either one of the equations* (2.7), (2.10) *or* (2.17) *is oscillatory, and condition* (2.27) *or* (2.31) *holds, then every solution* $x(t)$ *of the equation* (1.1) *oscillates or satisfies* $\lim_{t\to\infty} x(t) = 0$.

Proof. Let $x(t)$ be an eventually positive solution of the equation (1.1), say

$$x(t) > 0 \quad \text{and} \quad x\big(h(t,\theta)\big) > 0 \quad \text{and} \quad x\big(g(t,\tau)\big) > 0$$

for $t \geq t_1$ for some $t_1 \in [t_0, \infty)_{\mathbb{T}}$, $\theta \in [c, d]$ and $\tau \in [a, b]$. Then $y(t)$ satisfies one of the two cases of Lemma 2.1. By either Lemmas 2.4, 2.7 or 2.8, Case (I) cannot hold. By Lemma 2.11 (or Lemma 2.12), Case (II) does not hold. This completes the proof of the theorem. $\square$

Remark 1. Our results of this section remain valid of $g(t,\tau)$ is nondecreasing in the second variable. In this case, we replace

$$g_1(t) \quad \text{by} \quad \bar{g}_1(t) = g(t, b)$$

and

$$g_2(t) \quad \text{by} \quad \bar{g}_2(t) = g(t, a).$$

Remark 2. We may apply Lemma 2.5 to equation (2.7) with $\lambda = \alpha$. This details are left to the readers.

3. Oscillation of the equation (1.2)

We begin with the following lemmas that are essential in the proof of our theorems. It will be convenient to set

$$y(t) = x^{\beta}\big(h_1(t)\big) - r(t)x^{\gamma}\big(h_2(t)\big). \tag{3.1}$$

The equation (1.2) can then be written as

$$\Big(a(t)\left(y^{\Delta\Delta}(t)\right)^{\alpha}\Big)^{\Delta} + \int_a^b q\,(t,\tau)\,x^{\lambda}\big(g(t,\tau)\big)\Delta\tau = 0. \tag{3.2}$$

Lemma 3.1. *Let the condition* (1.3) *hold and assume that x is an eventually positive solution of the equation* (1.2). *Then there are only three possible cases for the behaviour of y for large $t \geq t_0$:*

(I) $y(t) > 0,\ y^{\Delta}(t) > 0,\ y^{\Delta\Delta}(t) > 0,$ *and* $\big(a(t)\left(y^{\Delta\Delta}(t)\right)^{\alpha}\big)^{\Delta} \leq 0;$

(II) $y(t) > 0,\ y^{\Delta}(t) < 0,\ y^{\Delta\Delta}(t) > 0,$ *and* $\big(a(t)\left(y^{\Delta\Delta}(t)\right)^{\alpha}\big)^{\Delta} \leq 0;$

(III) $y(t) < 0,\ y^{\Delta}(t) < 0,\ y^{\Delta\Delta}(t) > 0,$ *and* $\big(a(t)\left(y^{\Delta\Delta}(t)\right)^{\alpha}\big)^{\Delta} \leq 0.$

If case (I) or (II) holds, then we find

$$x(t) \geq y^{1/\beta}\left(h_1^{-1}(t)\right). \tag{3.3}$$

Using (3.3) in the equation (3.2), we have

$$\Big(a(t)\left(y^{\Delta\Delta}(t)\right)^{\alpha}\Big)^{\Delta} + \int_a^b q\,(t,\tau)\,y^{\lambda/\beta}\left(h_1^{-1}\circ g(t,\tau)\right)\Delta\tau \leq 0, \tag{3.4}$$

and when Case (III) holds, we see that

$$0 < z(t) = -y(t) = r(t)x^{\gamma}\big(h_2(t)\big) - x^{\beta}\big(h_1(t)\big) \leq r(t)x^{\gamma}\big(h_2(t)\big),$$

and so

$$x(t) \geq \left(\frac{z\left(h_2^{-1}(t)\right)}{r\left(h_2^{-1}(t)\right)}\right)^{1/\gamma}, \tag{3.5}$$

and the equation (3.2) becomes

$$\Big(a(t)\left(z^{\Delta\Delta}(t)\right)^{\alpha}\Big)^{\Delta} \geq \int_a^b q\,(t,\tau)\,r^{-\lambda/\gamma}\big(h_2^{-1}\circ g(t,\tau)\big)\,z^{\lambda/\gamma}\big(h_2^{-1}\circ g(t,\tau)\big)\Delta\tau. \tag{3.6}$$

Next, we assume that

(vi) $\zeta_1(t,\tau) = h_1^{-1}\circ g(t,\tau) \leq t$ for $\tau \in [a,b]$, ζ_1 is nonincreasing with respect to the second variable and $\lim_{t\to\infty}\zeta_1(t,\tau) = \infty$;

(vii) $\zeta_2(t,\tau) = h_2^{-1} \circ g(t,\tau) \le t$ for $\tau \in [a,b]$, ζ_2 is nonincreasing with respect to the second variable and $\lim_{t\to\infty} \zeta_2(t,\tau) = \infty$.

Also, we set

$$Q(t) := \int_a^b q(t,\tau)\Delta\tau,\ \hat{\zeta}_1(t) := \zeta_1(t,a),\ \bar{\zeta}_1(t) := \zeta_1(t,b), \tag{3.7}$$

and

$$\hat{\zeta}_2(t) := \zeta_2(t,a),\ \bar{\zeta}_2(t) := \zeta_2(t,b). \tag{3.8}$$

Now, if y satisfies Case (I) of Lemma 3.1, then (3.4) becomes

$$\left(a(t)\left(y^{\Delta\Delta}(t)\right)^{\alpha}\right)^{\Delta} + Q\,(t)\,y^{\lambda/\beta}\left(\bar{\zeta}_1(t)\right) \le 0, \tag{3.9}$$

where Q is as in (2.6). When y satisfies Case (II) of Lemma 3.1, then (3.4) becomes

$$\left(a(t)\left(y^{\Delta\Delta}(t)\right)^{\alpha}\right)^{\Delta} + Q\,(t)\,y^{\lambda/\beta}\left(\hat{\zeta}_1(t)\right) \le 0. \tag{3.10}$$

If y satisfies Case (III) of Lemma 3.1, then (3.6) takes the form

$$\left(a(t)\left(z^{\Delta\Delta}(t)\right)^{\alpha}\right)^{\Delta} \ge \hat{Q}\,(t)\,z^{\lambda/\gamma}\left(\bar{\zeta}_2(t)\right), \tag{3.11}$$

where

$$\hat{Q}\,(t) := \int_a^b q\,(t,\tau)\,r^{-\lambda/\gamma}\left(\zeta_2(t,\tau)\right)\Delta\tau. \tag{3.12}$$

As direct consequence of Lemmas 2.4, 2.6, 2.7, 2.8, 2.10, 2.11 and 2.12, we get the following results.

Lemma 3.2. *Let the condition* (1.3) *hold and equation*

$$z^{\Delta}(t) + Q(t)\left(\int_{t_1}^{\bar{\zeta}_1(t)}\int_{t_1}^{u} a^{-1/\alpha}(s)\Delta s\Delta u\right)^{\lambda/\beta} z^{\lambda/(\alpha\beta)}\left(g_2(t)\right) = 0 \quad \textit{for } t \ge t_1, \tag{3.13}$$

for $t_1 \in [t_0,\infty)_{\mathbb{T}}$, *is oscillatory, then the equation* (1.2) *has no eventually positive solution* x *such that* y *defined by* (3.1) *satisfies Case (I) of Lemma 3.1.*

Lemma 3.3. *Let the condition* (1.3) *hold and assume that* $x(t)$ *is an eventually positive solution of the equation* (1.2) *and the corresponding* y *satisfies Case (I) of Lemma 3.1. Then there exists* $t_1 \in [t_0,\infty)_{\mathbb{T}}$ *such that* (2.9) *holds.*

Lemma 3.4. *If the condition* (1.3) *holds and the equation*

$$\left(a(t)\left(z^{\Delta}(t)\right)^{\alpha}\right)^{\Delta} + \bar{Q}(t)z^{\lambda}\left(g_2(t)\right) = 0, \tag{3.14}$$

where for $\bar{\zeta}_1(t) > t_0$, *we set*

$$\bar{Q}(t) := A^{\lambda}\left(\bar{\zeta}_1(t), t_0\right)Q(t),$$

is oscillatory, then the equation (1.2) *has no eventually positive solution x such that y is defined by* (3.1) *satisfies Case (I) of Lemma 3.1.*

Lemma 3.5. *If the condition* (1.3) *hold and the equation*

$$\left(\frac{1}{\bar{A}^{\alpha}(t)}\left(v^{\Delta}(t)\right)^{\alpha}\right)^{\Delta}+Q(t)v^{\lambda}\left(g_2(t)\right)=0 \qquad \text{for} \quad t\geq t_2, \tag{3.15}$$

where $\bar{A}(t):=\int_{t_1}^{t}a^{-1/\alpha}(s)\Delta s$ for sufficiently large $t_2\in(t_1,\infty)_{\mathbb{T}}$ for some $t_1\in[t_0,\infty)_{\mathbb{T}}$, is oscillatory, then the conclusion of Lemma 3.4 holds.

Lemma 3.6. *Let the condition* (1.3) *hold and assume that $x(t)$ is an eventually positive solution of the equation* (1.2) *and the corresponding y satisfies Case (II) of Lemma 3.1. If* (2.22) *holds, then $\lim_{t\to\infty}x(t)=0$.*

Lemma 3.7. *Let $\hat{\zeta}_1$ be a nondecreasing on $[t_0,\infty)_{\mathbb{T}}$, the condition* (1.3) *hold and assume that $x(t)$ is an eventually positive solution of the equation* (1.2) *and the corresponding y satisfies Case (II) of Lemma 3.1. If*

$$\limsup_{t\to\infty}\int_{\hat{\zeta}_1(t)}^{t}\left(\frac{1}{a(u)}\int_{u}^{t}Q(s)\left[\hat{\zeta}_1(t)-\hat{\zeta}_1(s)\right]^{\lambda/\beta}\Delta s\right)^{1/\alpha}\Delta u>\begin{cases}1 & if\ \ \lambda=\alpha\beta,\\ 0 & if\ \ \lambda<\alpha\beta,\end{cases} \tag{3.16}$$

then $\lim_{t\to\infty}x(t)=0$.

Lemma 3.8. *Let the hypotheses of Lemma 3.7 hold with the condition* (3.16) *be replaced by*

$$\limsup_{t\to\infty}\int_{\hat{\zeta}_1(t)}^{t}\int_{v}^{t}\left(\frac{1}{a(u)}\int_{u}^{t}Q(s)\Delta s\right)^{1/\alpha}\Delta u\Delta v>\begin{cases}1 & if\ \lambda=\alpha\beta,\\ 0 & if\ \lambda<\alpha\beta.\end{cases} \tag{3.17}$$

Then the conclusion of Lemma 3.7 holds.

In the following two lemmas, we consider the second order delay dynamic equation

$$\left(a(t)\left(w^{\Delta}(t)\right)^{\alpha}\right)^{\Delta}=\bar{d}\left(\bar{\zeta}_2(t)\right)^{\lambda/\gamma}\hat{Q}\left(t\right)w^{\lambda/\gamma}\left(\bar{\zeta}_2(t)\right), \tag{3.18}$$

where $\bar{\zeta}_2$ and $\hat{Q}$ are as in (3.8) and (3.12) respectively, a, α, γ, λ are as in the equation (1.1) and $\bar{d}>0$ is a constant.

Lemma 3.9. *Let $\bar{\zeta}_2$ be a nondecreasing on $[t_0,\infty)_{\mathbb{T}}$ and the condition* (1.3) *hold. If*

$$\limsup_{t\to\infty}\int_{\bar{\zeta}_2(t)}^{t}\left(\bar{\zeta}_2(s)\right)^{\lambda/\gamma}\hat{Q}\left(s\right)\left(\int_{\bar{\zeta}_2(s)}^{\bar{\zeta}_2(t)}a^{-1/\alpha}(\tau)\Delta\tau\right)^{\lambda\gamma}\Delta s>\begin{cases}\frac{1}{\bar{d}} & if\ \ \lambda=\alpha\gamma,\\ 0 & if\ \ \lambda<\alpha\gamma,\end{cases} \tag{3.19}$$

then all bounded solutions of the equation (3.18) *are oscillatory.*

Proof. Let $w(t)$ be a bounded nonoscillatory solution of the equation (3.18), say $w(t) > 0$ for $t \geq t_1$ for some $t_1 \geq t_0 \in \mathbb{T}$. Then there exists $t_2 \in [t_1, \infty)_{\mathbb{T}}$ such that

$$w(t) > 0, \ w^{\Delta}(t) < 0 \quad \text{and} \quad \Big(a(t)\left(w^{\Delta}(t)\right)^{\alpha}\Big)^{\Delta} > 0 \qquad \text{for} \quad t \geq t_2. \tag{3.20}$$

Now for $v \geq u \geq t_2$, we have

$$\begin{aligned} w(u) \quad &\geq \quad w(u) - w(v) = -\int_u^v w^{\Delta}(\tau)\Delta\tau = -\int_u^v a^{-1/\alpha}(\tau)\Big(a(\tau)\left(w^{\Delta}(\tau)\right)^{\alpha}\Big)^{1/\alpha}\Delta\tau \\ &\geq \quad -\Big(a(v)\left(w^{\Delta}(v)\right)^{\alpha}\Big)^{1/\alpha}\int_u^v a^{-1/\alpha}(\tau)\Delta\tau. \end{aligned} \tag{3.21}$$

For $t \geq s \geq t_2$, setting $u = \bar{\zeta}_2(s)$ and $v = \bar{\zeta}_2(t)$ in the inequality (3.21) gives

$$w\big(\bar{\zeta}_2(s)\big) \geq -\Big(a(\bar{\zeta}_2(t))\left(w^{\Delta}(\bar{\zeta}_2(t))\right)^{\alpha}\Big)^{1/\alpha}\int_{\bar{\zeta}_2(s)}^{\bar{\zeta}_2(t)} a^{-1/\alpha}(\tau)\Delta\tau. \tag{3.22}$$

Integrating the equation (3.18) from $\bar{\zeta}_2(t) \geq t_2$ to t, we obtain

$$\begin{aligned} -a\big(\bar{\zeta}_2(t)\big)\Big(w^{\Delta}\left(\bar{\zeta}_2(t)\right)\Big)^{\alpha} &\geq a(t)\big(w^{\Delta}(t)\big)^{\alpha} - a\big(\bar{\zeta}_2(t)\big)\Big(w^{\Delta}\left(\bar{\zeta}_2(t)\right)\Big)^{\alpha} \\ &= \int_{\bar{\zeta}_2(t)}^{t} \bar{d}\left(\bar{\zeta}_2(s)\right)^{\lambda/\gamma}\hat{Q}(s)w^{\lambda/\gamma}\big(\bar{\zeta}_2(s)\big)\Delta s. \end{aligned} \tag{3.23}$$

Using (3.22) in (3.23), one can easily see that

$$\begin{aligned} &-a\big(\bar{\zeta}_2(t)\big)\Big(w^{\Delta}\left(\bar{\zeta}_2(t)\right)\Big)^{\alpha} \\ &\quad \geq \bigg(-a\big(\bar{\zeta}_2(t)\big)\Big(w^{\Delta}\left(\bar{\zeta}_2(t)\right)\Big)^{\alpha}\bigg)^{\frac{\lambda}{\alpha\gamma}}\int_{\bar{\zeta}_2(t)}^{t} \bar{d}\left(\bar{\zeta}_2(s)\right)^{\lambda/\gamma}\hat{Q}(s)\left(\int_{\bar{\zeta}_2(s)}^{\bar{\zeta}_2(t)} a^{-1/\alpha}(\tau)\Delta\tau\right)^{\lambda/\gamma}\Delta s, \end{aligned}$$

or

$$\bigg[-a\big(\bar{\zeta}_2(t)\big)\Big(w^{\Delta}\left(\bar{\zeta}_2(t)\right)\Big)^{\alpha}\bigg]^{1-\frac{\lambda}{\alpha\gamma}} \geq \int_{\bar{\zeta}_2(t)}^{t} \bar{d}\big(\bar{\zeta}_2(s)\big)^{\lambda/\gamma}\hat{Q}(s)\left(\int_{\bar{\zeta}_2(s)}^{\bar{\zeta}_2(t)} a^{-1/\alpha}(\tau)\Delta\tau\right)^{\lambda/\gamma}\Delta s. \tag{3.24}$$

Now take the lim sup as $t \to \infty$ of both sides of the above inequality. If $\lambda = \alpha\gamma$ the contradiction is obvious. If $\lambda < \alpha\gamma$ the left hand side of (3.24) is positive and must decrease to zero (to present a contradiction to the positivity of $w(t)$). This contradicts (3.19) and completes the proof of the lemma. □

Lemma 3.10. *Let $\bar{\zeta}_2$ be a nondecreasing on $[t_0,\infty)_{\mathbb{T}}$ and the condition* (3.19) *in Lemma 3.9 be replaced by*

$$\limsup_{t\to\infty}\int_{\bar{\zeta}_2(t)}^{t}\left(\frac{1}{a(s)}\int_{s}^{t}\bigl(\bar{\zeta}_2(\tau)\bigr)^{\lambda/\gamma}\hat{Q}(\tau)\,\Delta\tau\right)\Delta s>\begin{cases}\bar{d}^{-1/\alpha} & if\ \ \lambda=\alpha\gamma,\\ 0 & if\ \ \lambda<\alpha\gamma.\end{cases}\tag{3.25}$$

Then the conclusion of Lemma 3.9 holds.

Proof. Let $w(t)$ be a bounded nonoscillatory solution of the equation (3.18), say $w(t)>0$ for $t\ge t_1\ge t_0\in\mathbb{T}$. As in the proof of Lemma 3.9, we obtain (3.20) for $t\ge t_2$ for some $t_2\in[t_1,\infty)_{\mathbb{T}}$. Integrating (3.18) from $u\ge t_2$ to $t\ge u$, we have

$$a(t)\bigl(w^{\Delta}(t)\bigr)^{\alpha}-a(u)\bigl(w^{\Delta}(u)\bigr)^{\alpha}=\int_{u}^{t}\bar{d}\bigl(\bar{\zeta}_2(s)\bigr)^{\lambda/\gamma}\hat{Q}(s)w^{\lambda/\gamma}\bigl(\bar{\zeta}_2(s)\bigr)\Delta s,$$

or

$$-w^{\Delta}(u)\ge w^{\frac{\lambda}{\alpha\gamma}}\bigl(\bar{\zeta}_2(t)\bigr)\left(\frac{1}{a(u)}\int_{u}^{t}\bar{d}\bigl(\bar{\zeta}_2(s)\bigr)^{\lambda/\gamma}\hat{Q}(s)\Delta s\right)^{1/\alpha}.$$

Integrating this inequality from $\bar{\zeta}_2(t)$ to t, we obtain

$$w\bigl(\bar{\zeta}_2(t)\bigr)\ge w^{\frac{\lambda}{\alpha\gamma}}\bigl(\bar{\zeta}_2(t)\bigr)\int_{\bar{\zeta}_2(t)}^{t}\left(\frac{1}{a(u)}\int_{u}^{t}\bar{d}\bigl(\bar{\zeta}_2(s)\bigr)^{\lambda/\gamma}\hat{Q}(s)\Delta s\right)^{1/\alpha}\Delta u,$$

or

$$w^{\left(1-\frac{\lambda}{\alpha\gamma}\right)}\bigl(\bar{\zeta}_2(t)\bigr)\ge\int_{\bar{\zeta}_2(t)}^{t}\left(\frac{1}{a(u)}\int_{u}^{t}\bar{d}\bigl(\bar{\zeta}_2(s)\bigr)^{\lambda/\gamma}\hat{Q}(s)\Delta s\right)^{1/\alpha}\Delta u.$$

Taking the lim sup as $t\to\infty$ of both sides of the above inequality, we again obtain a contradiction as in the previous lemma. □

We are now ready to establish the main results of this section.

Theorem 3.1. *Let $\lambda\le\alpha\gamma$, $\lambda\le\alpha\beta$, $\hat{\zeta}_1$ and $\bar{\zeta}_2$, $i=1,2$ be nondecreasing on $[t_0,\infty)_{\mathbb{T}}$ and the condition* (1.3) *hold. If either one of the dynamic equations* (3.13)*,* (3.14) *or* (3.15) *is oscillatory, the condition* (3.16) *or (the condition* (3.17)*) holds and the condition* (3.19) *or (the condition* (3.25)*) is satisfied with $0<\bar{d}<1$, then every solution $x(t)$ of the equation* (1.2) *oscillates or satisfies* $\lim_{t\to\infty}x(t)=0$.

Proof. Let $x(t)$ be an eventually positive solution of the equation (1.2), say $x(t)>0$ and $x\bigl(h_i(t)\bigr)>0$, $i=1,2$ and $x\bigl(g(t,\tau)\bigr)>0$ for $t\ge t_1$

for some $t_1 \in [t_0, \infty)_{\mathbb{T}}$ and $\tau \in [a, b]$. Then $y(t)$ defined by (3.1) satisfies one of the three cases of Lemma 3.1 and satisfies the equation (3.2). If Case (I) holds, then (3.3) holds. Now using (3.3) in the equation (3.2), we obtain the inequality (3.9). By either Lemma 3.2, Lemma 3.4 or Lemma 3.5, Case (I) cannot hold. Similarly, when Case (II) one can easily obtain the inequality (3.10), and by Lemma 3.7 or Lemma 3.8, Case (II) cannot hold. Finally, if Case (III) holds, we let $0 < z(t) = -y(t)$ and proceed as above to obtain the inequality (3.11). It is easy to check that z satisfies

$$z(t) > 0, \quad z^{\Delta}(t) > 0, \quad z^{\Delta\Delta}(t) < 0 \quad \text{and} \quad \Big(a(t)\left(z^{\Delta\Delta}(t)\right)^{\alpha}\Big)^{\Delta} \geq 0,$$

for $t \geq t_2 \geq t_1$. Now, there exists a constant $\bar{d} \in (0,1)$ such that

$$z\left(\bar{\zeta}_2(t)\right) \geq \bar{d}\,\bar{\zeta}_2(t) z^{\Delta}\left(\bar{\zeta}_2(t)\right) \qquad \text{for} \quad t \geq t_3 \geq t_2. \tag{3.26}$$

Using (3.26) in (3.11), we get

$$\Big(a(t)\left(z^{\Delta\Delta}(t)\right)^{\alpha}\Big)^{\Delta} \geq \hat{Q}(t)\Big(\bar{d}\,\bar{\zeta}_2(t) z^{\Delta}\left(\bar{\zeta}_2(t)\right)\Big)^{\lambda/\gamma} \qquad \text{for} \quad t \geq t_3,$$

or

$$\Big(a(t)\left(w^{\Delta}(t)\right)^{\alpha}\Big)^{\Delta} \geq \left(\bar{d}\right)^{\lambda/\gamma}\left(\bar{\zeta}_2(t)\right)^{\lambda/\gamma}\hat{Q}(t) w^{\lambda/\gamma}\left(\bar{\zeta}_2(t)\right) \qquad \text{for} \quad t \geq t_3,$$

where $w(t) := z^{\Delta}(t)$. Proceeding as in the proofs of Lemma 3.9 and Lemma 3.10, we arrive at the desired conclusion completing the proof of the theorem. □

Remark 3. We may note that Theorem 2.1 is also applicable to the equation (1.2). The details are omitted.

When $p(t,\theta) \equiv 0$ in the equation (1.1) or $\beta = 1$, $r(t) = 0$ and $h_1(t) = t$ in the equation (1.2), both equations are reduced to

$$\Big(a(t)\left(x^{\Delta\Delta}(t)\right)^{\alpha}\Big)^{\Delta} + \int_a^b q(t,\tau) x^{\lambda}\big(g(t,\tau)\big)\Delta\tau = 0. \tag{3.27}$$

In this case we have the following new result.

Theorem 3.2. *Let $\lambda \leq \alpha$ and* (1.3) *hold. If either one of the equations* (2.7) *with $p^* \equiv 0$,* (2.10) *or* (2.17) *with $p^* \equiv 0$ is oscillatory and the condition* (2.27) *or (the condition* (2.31)*) holds, then every solution $x(t)$ of the equation* (3.27) *oscillates or satisfies* $\lim_{t\to\infty} x(t) = 0$.

P r o o f. The conclusion follows from Theorem 2.2 or 3.1 and is omitted. □

4. General remarks

(1) The results of this paper are presented in a form that is essentially new and of a high degree of generality.

(2) We note that there are many criteria in the literature of first and second order dynamic equations and so by applying these results to the equations (2.7), (2.10) and (2.17), we can obtain many oscillation results, more that those known in the literature. Here we omit the details.

(3) The results here are valid for various type of time scales, e.g., $\mathbb{T} = \mathbb{R}$, $\mathbb{T} = \mathbb{Z}$, $\mathbb{T} = h\mathbb{Z}$ with $h > 0$, $\mathbb{T} = q^{\mathbb{N}_0}$ with $q > 1$, $\mathbb{T} = \mathbb{N}_0^2$, etc. (see [2]).

(4) We note that our results on the asymptotic behavior of solutions are applicable to the equations (1.1) and (1.2) if $g(t,\tau) \leq t$, $\tau \in [a,b]$ while our oscillation results are applicable to the equations (1.1) and (1.2) if $g(t,\tau) < t$, $\tau \in [a,b]$. Thus as it has been known, there is the delay in the equations (1.1) and (1.2) which can generate oscillations.

(5) Our results of Section 2 are new and our results of Section 3 include, extend and improve the results in [9] and [12].

(6) Finally, it would be of interest to consider the equations (1.1) and (1.2) try to obtain some oscillation criteria if other appropriate conditions on the functions $p(t,\theta)$, $\theta \in [c,d]$, $r(t)$, etc.

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