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CERONE'S GENERALIZATIONS OF STEFFENSEN'S INEQUALITY

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ABSTRACT. In this paper, generalizations of Steffensen's inequality with bounds involving any two subintervals motivated by Cerone's generalizations are given. Furthermore, weaker conditions for Cerone's generalization as well as for new generalizations obtained in this paper are given. Moreover, functionals defined as the difference between the left-hand and the right-hand side of these generalizations are studied and new Stolarsky type means related to them are obtained.

1. Introduction

Since its appearance in 1918, Steffensen's inequality has been applied to a wide range of topics across mathematics and statistics. Well-known Steffensen's inequality reads (see [10]):

Theorem 1.1. *Suppose that f is nonincreasing and g is integrable on $[a,b]$ with $0 \le g \le 1$ and $\lambda = \int_a^b g(t)\,dt$. Then, we have*

$$\int_{b-\lambda}^{b} f(t)\,dt \le \int_a^b f(t)g(t)\,dt \le \int_a^{a+\lambda} f(t)\,dt. \tag{1.1}$$

The inequalities are reversed for f nondecreasing.

Over the years, Steffensen's inequality has been generalized in many ways. Some of these generalizations were given by Cerone, Mercer, Pečarić, Wu and Srivastava (see [1], [3], [6], [11], respectively). Extensive overviews of generalizations of Steffensen's inequality can be found in [5] and [9].

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First, let us recall Cerone's generalization of Steffensen's inequality which allows bounds involving any two subintervals instead of restricting them to include the end points. Cerone's generalization is given in the following theorem (see [1]).

Theorem 1.2. *Let $f, g\colon [a,b] \to \mathbb{R}$ be integrable functions on $[a,b]$, and let f be nonincreasing. Further, let $0 \le g \le 1$ and $\lambda = \int_a^b g(t)\,dt = d_i - c_i$, where $[c_i, d_i] \subseteq [a,b]$ for $i = 1,2$ and $d_1 \le d_2$. Then*

$$\int_{c_2}^{d_2} f(t)\,dt - r(c_2,d_2) \le \int_a^b f(t)g(t)\,dt \le \int_{c_1}^{d_1} f(t)\,dt + R(c_1,d_1) \tag{1.2}$$

holds, where

$$r(c_2,d_2) = \int_{d_2}^{b} \big(f(c_2) - f(t)\big)g(t)\,dt \ge 0$$

and

$$R(c_1,d_1) = \int_{a}^{c_1} \big(f(t) - f(d_1)\big)g(t)\,dt \ge 0.$$

As noted by Cerone in [1], if in Theorem 1.2 we take

$$c_1 = a \quad \text{and so} \quad d_1 = a + \lambda, \quad \text{then} \quad R(a, a+\lambda) = 0.$$

Further, taking

$$d_2 = b \quad \text{and so} \quad c_2 = b - \lambda, \quad \text{then} \quad r(b-\lambda, b) = 0.$$

Thus, we obtain Steffensen's inequality.

Since

$$\lambda = \int_a^b g(t)\,dt \quad \text{and} \quad 0 \le g \le 1,$$

then

$$c_2 = b - \lambda \ge a \quad \text{and} \quad d_1 = a + \lambda \le b \quad \text{giving} \quad [c_i, d_i] \subseteq [a,b].$$

Hence, Theorem 1.2 is a generalization of Steffensen's inequality for two equal length subintervals that are not necessarily at the ends of $[a,b]$.

The aim of this paper is to give generalizations of Steffensen's inequality with bounds involving any two subintervals motivated by generalizations given in [8]. Moreover, the aim is also to give weaker conditions for Cerone's generalization as well as for new generalizations obtained in this paper.

First, let us recall some notions; *log* denotes the natural logarithm function, *id* is the identity function on the actual set, and by dx we denote the Lebesgue measure on $\mathbb{R}$.

2. Main results

To generalize Cerone's result for the function f/k, we need the following lemma.

Lemma 2.1. *Let k be a positive integrable function on $[a,b]$, and let $f,g,h\colon[a,b]\to\mathbb{R}$ be integrable functions on $[a,b]$. Further, let $[c,d]\subseteq[a,b]$ with $\int_c^d h(t)k(t)\,dt = \int_a^b g(t)k(t)\,dt$, and let $z\in[a,b]$. Then, the following identity holds*

$$\begin{aligned}\int_c^d f(t)h(t)dt - \int_a^b f(t)g(t)\,dt =& \int_a^c \left(\frac{f(z)}{k(z)} - \frac{f(t)}{k(t)}\right) g(t)k(t)\,dt\\ &+ \int_c^d \left(\frac{f(t)}{k(t)} - \frac{f(z)}{k(z)}\right) k(t)\big[h(t)-g(t)\big]dt\\ &+ \int_d^b \left(\frac{f(z)}{k(z)} - \frac{f(t)}{k(t)}\right) g(t)k(t)\,dt. \qquad (2.1)\end{aligned}$$

P r o o f. We have

$$\begin{aligned}&\int_c^d f(t)h(t)\,dt - \int_a^b f(t)g(t)\,dt\\ &= \int_c^d k(t)\big[h(t)-g(t)\big]\frac{f(t)}{k(t)}\,dt - \left[\int_a^c \frac{f(t)}{k(t)}g(t)k(t)\,dt + \int_d^b \frac{f(t)}{k(t)}g(t)k(t)\,dt\right]\\ &= \int_a^c \left(\frac{f(z)}{k(z)} - \frac{f(t)}{k(t)}\right)g(t)k(t)\,dt + \int_c^d \left(\frac{f(t)}{k(t)} - \frac{f(z)}{k(z)}\right)k(t)\big[h(t)-g(t)\big]dt\\ &\quad + \int_d^b \left(\frac{f(z)}{k(z)} - \frac{f(t)}{k(t)}\right)g(t)k(t)\,dt\\ &\quad + \frac{f(z)}{k(z)}\left[\int_c^d k(t)h(t)\,dt - \int_a^c g(t)k(t)\,dt - \int_c^d k(t)g(t)\,dt - \int_d^b g(t)k(t)\,dt\right]. \qquad (2.2)\end{aligned}$$

Since

$$\int_c^d k(t)h(t)\,dt = \int_a^b k(t)g(t)\,dt,$$

we have

$$\frac{f(z)}{k(z)}\left[\int_c^d k(t)h(t)\,dt - \int_a^c g(t)k(t)\,dt - \int_c^d k(t)g(t)\,dt - \int_d^b g(t)k(t)\,dt\right] = 0.$$

Hence, (2.1) follows from (2.2). □

In the following theorems we will give a generalization of Cerone's result for the function f/k.

Theorem 2.1. *Let k be a positive integrable function on $[a,b]$, and let $f,g,h:[a,b]\to\mathbb{R}$ be integrable functions on $[a,b]$ such that f/k is nonincreasing. Further, let $0\le g\le h$ and $\int_c^d h(t)k(t)\,dt=\int_a^b g(t)k(t)\,dt$, where $[c,d]\subseteq[a,b]$. Then*

$$\int_a^b f(t)g(t)\,dt \le \int_c^d f(t)h(t)\,dt + R_g(c,d) \tag{2.3}$$

holds, where

$$R_g(c,d)=\int_a^c \left(\frac{f(t)}{k(t)}-\frac{f(d)}{k(d)}\right) g(t)k(t)\,dt \ge 0. \tag{2.4}$$

If f/k is a nondecreasing function, then the inequalities in (2.3) and (2.4) are reversed.

P r o o f. Since f/k is nonincreasing, k is positive and $0\le g\le h$, we have

$$\int_c^d \left(\frac{f(t)}{k(t)}-\frac{f(d)}{k(d)}\right) k(t)\big[h(t)-g(t)\big]\,dt \ge 0, \tag{2.5}$$

$$\int_d^b \left(\frac{f(d)}{k(d)}-\frac{f(t)}{k(t)}\right) g(t)k(t)\,dt \ge 0 \tag{2.6}$$

and $R_g(c,d)\ge 0$. Now, from (2.1) for $z=d$, (2.5) and (2.6), we have

$$\begin{aligned}&\int_c^d f(t)h(t)dt-\int_a^b f(t)g(t)dt+\int_a^c \left(\frac{f(t)}{k(t)}-\frac{f(d)}{k(d)}\right) g(t)k(t)\,dt=\\ &\int_c^d \left(\frac{f(t)}{k(t)}-\frac{f(d)}{k(d)}\right) k(t)\big[h(t)-g(t)\big]\,dt+\int_d^b \left(\frac{f(d)}{k(d)}-\frac{f(t)}{k(t)}\right) g(t)k(t)\,dt\ge 0.\end{aligned} \tag{2.7}$$

Hence, (2.3) holds. □

Theorem 2.2. *Let k be a positive integrable function on $[a,b]$, and let $f,g,h:[a,b]\to\mathbb{R}$ be integrable functions on $[a,b]$ such that f/k is nonincreasing. Further, let $0\le g\le h$ and $\int_c^d h(t)k(t)dt=\int_a^b g(t)k(t)\,dt$, where $[c,d]\subseteq[a,b]$.*

Then

$$\int_c^d f(t)h(t)\,dt - r_g(c,d) \le \int_a^b f(t)g(t)\,dt \tag{2.8}$$

holds, where

$$r_g(c,d) = \int_d^b \left(\frac{f(c)}{k(c)} - \frac{f(t)}{k(t)}\right) g(t)k(t)\,dt \ge 0. \tag{2.9}$$

If f/k is a nondecreasing function, then the inequalities in (2.8) *and* (2.9) *are reversed.*

Proof. Since f/k is nonincreasing, k is positive and $0 \le g \le h$, we have

$$\int_a^c \left(\frac{f(t)}{k(t)} - \frac{f(c)}{k(c)}\right) k(t)g(t)\,dt \ge 0, \tag{2.10}$$

$$\int_c^d \left(\frac{f(c)}{k(c)} - \frac{f(t)}{k(t)}\right) k(t)\big[h(t) - g(t)\big]\,dt \ge 0 \tag{2.11}$$

and $r_g(c,d) \ge 0$. Now, from (2.1) for $z = c$, (2.10) and (2.11), we have

$$\int_a^b f(t)g(t)\,dt - \int_c^d f(t)h(t)\,dt + \int_d^b \left(\frac{f(c)}{k(c)} - \frac{f(t)}{k(t)}\right) g(t)k(t)\,dt =$$

$$\int_a^c \left(\frac{f(t)}{k(t)} - \frac{f(c)}{k(c)}\right) g(t)k(t)\,dt + \int_c^d \left(\frac{f(c)}{k(c)} - \frac{f(t)}{k(t)}\right) k(t)\big[h(t) - g(t)\big]\,dt \ge 0. \tag{2.12}$$

Hence, (2.8) holds. □

Remark 2.1. If we take $c = a$ and $d = a + \lambda$ in Theorem 2.1, we obtain a Mercer's generalization of the right-hand Steffensen's inequality (see [3, Theorem 3]). If we take $c = b - \lambda$ and $d = b$ in Theorem 2.2, we obtain a similar generalization of the left-hand Steffensen's inequality which is obtained in [8] from a generalization given by Pečarić in [6] (see [8, Theorem 2.7]).

In [8], the authors proved a generalization of Wu and Srivastava refinement of Steffensen's inequality for the nonincreasing function f/k. In the following theorems we will generalize these results to obtain bounds which involve any two subintervals.

Theorem 2.3. *Let k be a positive integrable function on $[a,b]$, and let $f,g,h:[a,b]\to\mathbb{R}$ be integrable functions on $[a,b]$ such that f/k is nonincreasing. Further, let $0\le g\le h$ and $\int_c^d h(t)k(t)\,dt=\int_a^b g(t)k(t)\,dt$, where $[c,d]\subseteq[a,b]$. Then*

$$\begin{aligned}\int_a^b f(t)g(t)\,dt &\le \int_c^d f(t)h(t)\,dt - \int_c^d \left(\frac{f(t)}{k(t)}-\frac{f(d)}{k(d)}\right)k(t)\big[h(t)-g(t)\big]\,dt + R_g(c,d)\\ &\le \int_c^d f(t)h(t)\,dt + R_g(c,d)\end{aligned} \tag{2.13}$$

holds, where $R_g(c,d)$ is defined by (2.4)*.*

If f/k is a nondecreasing function, then the inequality in (2.13) *is reversed.*

Proof. Similar to the proof of Theorem 2.1. □

Theorem 2.4. *Let k be a positive integrable function on $[a,b]$, and let $f,g,h:[a,b]\to\mathbb{R}$ be integrable functions on $[a,b]$ such that f/k is nonincreasing. Further, let $0\le g\le h$ and $\int_c^d h(t)k(t)\,dt=\int_a^b g(t)k(t)\,dt$, where $[c,d]\subseteq[a,b]$. Then*

$$\begin{aligned}&\int_c^d f(t)h(t)\,dt - r_g(c,d)\\ &\le \int_c^d f(t)h(t)\,dt + \int_c^d \left(\frac{f(c)}{k(c)}-\frac{f(t)}{k(t)}\right)k(t)\big[h(t)-g(t)\big]\,dt - r_g(c,d)\\ &\le \int_a^b f(t)g(t)\,dt\end{aligned} \tag{2.14}$$

holds, where $r_g(c,d)$ is defined by (2.9)*.*

If f/k is a nondecreasing function, then the inequality in (2.14) *is reversed.*

Proof. Similar to the proof of Theorem 2.2. □

Remark 2.2. If we take $c=a$ and $d=a+\lambda$ in Theorem 2.3, or $c=b-\lambda$ and $d=b$ in Theorem 2.4, we obtain generalizations of Wu and Srivastava refinement of Steffensen's inequality given in [8].

In [4], Milovanović and Pečarić gave weaker conditions for the function g in Steffensen's inequality. Motivated by their result, we will give weaker conditions for the function g in our previous theorems.

Theorem 2.5. *Let k be a positive integrable function on $[a,b]$, and let $f,g,h:[a,b]\to\mathbb{R}$ be integrable functions on $[a,b]$ such that f/k is nonincreasing. Let $\int_c^d h(t)k(t)\,dt=\int_a^b g(t)k(t)\,dt$, where $[c,d]\subseteq[a,b]$. If*

$$\int_c^x k(t)g(t)\,dt\le\int_c^x k(t)h(t)\,dt,\qquad c\le x\le d \tag{2.15}$$

and

$$\int_x^b k(t)g(t)\,dt\ge 0,\qquad d\le x\le b, \tag{2.16}$$

then

$$\int_a^b f(t)g(t)\,dt\le\int_c^d f(t)h(t)\,dt+\int_a^c\left(\frac{f(t)}{k(t)}-\frac{f(d)}{k(d)}\right)g(t)k(t)\,dt. \tag{2.17}$$

P r o o f. Using the identity (2.7) and applying integration by parts, we obtain

$$\int_c^d f(t)h(t)\,dt-\int_a^b f(t)g(t)\,dt+\int_a^c\left(\frac{f(t)}{k(t)}-\frac{f(d)}{k(d)}\right)g(t)k(t)\,dt=$$

$$-\int_c^d\left(\int_c^x k(t)\left[h(t)-g(t)\right]dt\right)d\left(\frac{f(x)}{k(x)}\right)-\int_d^b\left(\int_x^b g(t)k(t)\,dt\right)d\left(\frac{f(x)}{k(x)}\right)\ge 0, \tag{2.18}$$

when (2.16) holds. □

Taking $c=a$ and $d=a+\lambda$ in Theorem 2.5, we obtain the following theorem.

Theorem 2.6. *Let k be a positive integrable function on $[a,b]$, let $f,g,h:[a,b]\to\mathbb{R}$ be integrable functions on $[a,b]$ such that f/k is nonincreasing. Let λ be defined by $\int_a^{a+\lambda} h(t)k(t)\,dt=\int_a^b g(t)k(t)\,dt$. If*

$$\int_a^x k(t)g(t)\,dt\le\int_a^x k(t)h(t)\,dt,\qquad a\le x\le a+\lambda \tag{2.19}$$

and

$$\int_x^b k(t)g(t)\,dt\ge 0,\qquad a+\lambda\le x\le b, \tag{2.20}$$

then

$$\int_a^b f(t)g(t)\,dt\le\int_a^{a+\lambda} f(t)h(t)\,dt. \tag{2.21}$$

Remark 2.3. In [8], the authors proved that, for a non-negative function h, the conditions (2.19) and (2.20) are equivalent to

$$\int_a^x k(t)g(t)\,dt \le \int_a^x k(t)h(t)\,dt \quad \text{and} \quad \int_x^b k(t)g(t)\,dt \ge 0, \quad \text{for all } x \in [a,b].$$

Hence, in Theorem 2.6, we obtain the sufficient conditions given in [8, Theorem 2.17].

Theorem 2.7. *Let k be a positive integrable function on $[a,b]$, and $f,g,h : [a,b] \to \mathbb{R}$ be integrable functions on $[a,b]$ such that f/k is nonincreasing. Let $\int_c^d h(t)k(t)\,dt = \int_a^b g(t)k(t)\,dt$, where $[c,d] \subseteq [a,b]$. If*

$$\int_x^d k(t)g(t)\,dt \le \int_x^d k(t)h(t)\,dt, \quad c \le x \le d \tag{2.22}$$

and

$$\int_a^x k(t)g(t)\,dt \ge 0, \qquad a \le x \le c, \tag{2.23}$$

then

$$\int_c^d f(t)h(t)\,dt - \int_d^b \left(\frac{f(c)}{k(c)} - \frac{f(t)}{k(t)}\right) g(t)k(t)\,dt \le \int_a^b f(t)g(t)\,dt. \tag{2.24}$$

Proof. Using the identity (2.12) and applying integration by parts, we obtain

$$\int_a^b f(t)g(t)\,dt - \int_c^d f(t)h(t)\,dt + \int_d^b \left(\frac{f(c)}{k(c)} - \frac{f(t)}{k(t)}\right) g(t)k(t)\,dt =$$

$$-\int_a^c \left(\int_a^x g(t)k(t)\,dt\right) d\left(\frac{f(x)}{k(x)}\right) - \int_c^d \left(\int_x^d k(t)\left[h(t) - g(t)\right]dt\right) d\left(\frac{f(x)}{k(x)}\right) \ge 0, \tag{2.25}$$

when (2.23) holds. □

Taking $c = b - \lambda$ and $d = b$ in Theorem 2.7, we obtain the following theorem.

Theorem 2.8. *Let k be a positive integrable function on $[a,b]$, and let $f,g,h : [a,b] \to \mathbb{R}$ be integrable functions on $[a,b]$ such that f/k is nonincreasing. Let λ be defined by*

$$\int_{b-\lambda}^b h(t)k(t)\,dt = \int_a^b g(t)k(t)\,dt.$$

If

$$\int_x^b k(t)g(t)\,dt \le \int_x^b k(t)h(t)\,dt, \quad b-\lambda \le x \le b \tag{2.26}$$

and

$$\int_a^x k(t)g(t)\,dt \ge 0, \qquad a \le x \le b-\lambda, \tag{2.27}$$

then

$$\int_{b-\lambda}^b f(t)h(t)\,dt \le \int_a^b f(t)g(t)\,dt. \tag{2.28}$$

Remark 2.4. In [8], the authors proved that for a non-negative function h, the conditions (2.26) and (2.27) are equivalent to

$$\int_x^b k(t)g(t)\,dt \le \int_x^b k(t)h(t)\,dt \quad \text{and} \quad \int_a^x k(t)g(t)\,dt \ge 0, \quad \text{for all } x \in [a,b].$$

Hence, in Theorem 2.8, we obtain the sufficient conditions given in [8, Theorem 2.18].

Taking $k \equiv 1$ and $h \equiv 1$ in Theorems 2.5 and 2.7, we obtain weaker conditions for the function g in Cerone's generalization of Steffensen's inequality.

Theorem 2.9. *Let $f, g\colon [a,b] \to \mathbb{R}$ be integrable functions on $[a,b]$ such that f is nonincreasing. Let $\lambda = d - c = \int_a^b g(t)\,dt$, where $[c,d] \subseteq [a,b]$. If*

$$\int_c^x g(t)\,dt \le x - c, \quad c \le x \le d \quad \textit{and} \quad \int_x^b g(t)\,dt \ge 0, \quad d \le x \le b,$$

then

$$\int_a^b f(t)g(t)\,dt \le \int_c^d f(t)\,dt + \int_a^c \big(f(t) - f(d)\big)g(t)\,dt. \tag{2.29}$$

Theorem 2.10. *Let $f, g\colon [a,b] \to \mathbb{R}$ be integrable functions on $[a,b]$ such that f is nonincreasing. Let $\lambda = d - c = \int_a^b g(t)\,dt$, where $[c,d] \subseteq [a,b]$. If*

$$\int_x^d g(t)\,dt \le d - x, \quad c \le x \le d \quad \textit{and} \quad \int_a^x g(t)\,dt \ge 0, \quad a \le x \le c, \tag{2.30}$$

then

$$\int_c^d f(t)\,dt - \int_d^b \big(f(c) - f(t)\big)g(t)\,dt \le \int_a^b f(t)g(t)\,dt. \tag{2.31}$$

In the following theorems, we will give weaker conditions for Theorem 2.3 and Theorem 2.4.

Theorem 2.11. *Let k be a positive integrable function on $[a,b]$, and let $f,g,h:[a,b]\to\mathbb{R}$ be integrable functions on $[a,b]$ such that f/k is nonincreasing. Let $\int_c^d h(t)k(t)\,dt = \int_a^b g(t)k(t)\,dt$, where $[c,d]\subseteq[a,b]$. If* (2.15) *and* (2.16) *hold, then*

$$\begin{aligned}\int_a^b f(t)g(t)\,dt \le& \int_c^d f(t)h(t)\,dt + \int_a^c \left(\frac{f(t)}{k(t)} - \frac{f(d)}{k(d)}\right) g(t)k(t)\,dt \\ &- \int_c^d \left(\frac{f(t)}{k(t)} - \frac{f(d)}{k(d)}\right) k(t)\big[h(t)-g(t)\big]\,dt \\ \le& \int_c^d f(t)h(t)\,dt + \int_a^c \left(\frac{f(t)}{k(t)} - \frac{f(d)}{k(d)}\right) g(t)k(t)\,dt. \end{aligned} \tag{2.32}$$

P r o o f. Using the identity (2.1) for $z=d$ and applying integration by parts, we obtain

$$\begin{aligned}&\int_c^d f(t)h(t)\,dt - \int_a^b f(t)g(t)\,dt + \int_a^c \left(\frac{f(t)}{k(t)} - \frac{f(d)}{k(d)}\right) g(t)k(t)\,dt \\ &\quad - \int_c^d \left(\frac{f(t)}{k(t)} - \frac{f(d)}{k(d)}\right) k(t)\big[h(t)-g(t)\big]\,dt \\ &= \int_d^b \left(\frac{f(d)}{k(d)} - \frac{f(t)}{k(t)}\right) g(t)k(t)\,dt \\ &= -\int_d^b \left(\int_x^b g(t)k(t)\,dt\right) d\left(\frac{f(x)}{k(x)}\right) \ge 0 \end{aligned} \tag{2.33}$$

when

$$\int_x^b k(t)g(t)\,dt \ge 0, \qquad d\le x\le b. \tag{2.34}$$

Furthermore,

$$\int_c^d \left(\frac{f(t)}{k(t)} - \frac{f(d)}{k(d)}\right) k(t)\big[h(t)-g(t)\big]\,dt = \\ -\int_c^d \left(\int_c^x k(t)\big[h(t)-g(t)\big]\,dt\right) d\left(\frac{f(x)}{k(x)}\right) \geq 0 \quad (2.35)$$

when

$$\int_c^x k(t)g(t)\,dt \leq \int_c^x k(t)h(t)\,dt, \qquad c \leq x \leq d. \quad (2.36)$$

Hence, (2.32) holds when (2.16) holds. □

Theorem 2.12. *Let k be a positive integrable function on $[a,b]$, and let $f,g,h:[a,b]\to\mathbb{R}$ be integrable functions on $[a,b]$ such that f/k is nonincreasing. Let $\int_c^d h(t)k(t)\,dt = \int_a^b g(t)k(t)\,dt$, where $[c,d]\subseteq[a,b]$. If*

$$\int_x^b k(t)g(t)\,dt \geq 0 \qquad \textit{for} \quad d \leq x \leq b, \quad (2.37)$$

then

$$\int_a^b f(t)g(t)\,dt \leq \int_c^d f(t)h(t)\,dt + \int_a^c \left(\frac{f(t)}{k(t)} - \frac{f(d)}{k(d)}\right) g(t)k(t)\,dt \\ - \int_c^d \left(\frac{f(t)}{k(t)} - \frac{f(d)}{k(d)}\right) k(t)\big[h(t)-g(t)\big]\,dt. \quad (2.38)$$

If we additionally have

$$\int_c^x k(t)g(t)\,dt \leq \int_c^x k(t)h(t)\,dt \qquad \textit{for} \quad c \leq x \leq d, \quad (2.39)$$

then (2.32) *holds.*

P r o o f. Similar to the proof of Theorem 2.11. □

Theorem 2.13. *Let k be a positive integrable function on $[a,b]$, and let $f,g,h:[a,b]\to\mathbb{R}$ be integrable functions on $[a,b]$ such that f/k is nonincreasing. Let $\int_c^d h(t)k(t)\,dt=\int_a^b g(t)k(t)\,dt$, where $[c,d]\subseteq[a,b]$. If* (2.22) *and* (2.23) *hold, then*

$$\begin{aligned}
&\int_c^d f(t)h(t)\,dt-\int_d^b\left(\frac{f(c)}{k(c)}-\frac{f(t)}{k(t)}\right)g(t)k(t)\,dt\\
&\le\int_c^d f(t)h(t)\,dt-\int_d^b\left(\frac{f(c)}{k(c)}-\frac{f(t)}{k(t)}\right)g(t)k(t)\,dt\\
&\quad+\int_c^d\left(\frac{f(c)}{k(c)}-\frac{f(t)}{k(t)}\right)k(t)\big[h(t)-g(t)\big]\,dt\\
&\le\int_a^b f(t)g(t)\,dt. \qquad (2.40)
\end{aligned}$$

P r o o f. Similar to the proof of Theorem 2.11 using the identity (2.1) for $z=c$. □

Theorem 2.14. *Let k be a positive integrable function on $[a,b]$, and let $f,g,h:[a,b]\to\mathbb{R}$ be integrable functions on $[a,b]$ such that f/k is nonincreasing. Let $\int_c^d h(t)k(t)\,dt=\int_a^b g(t)k(t)\,dt$, where $[c,d]\subseteq[a,b]$. If*

$$\int_a^x k(t)g(t)\,dt\ge 0 \qquad \text{for} \quad a\le x\le c, \qquad (2.41)$$

then

$$\begin{aligned}
&\int_c^d f(t)h(t)\,dt-\int_d^b\left(\frac{f(c)}{k(c)}-\frac{f(t)}{k(t)}\right)g(t)k(t)\,dt\\
&\quad+\int_c^d\left(\frac{f(c)}{k(c)}-\frac{f(t)}{k(t)}\right)k(t)\big[h(t)-g(t)\big]\,dt\le\int_a^b f(t)g(t)\,dt. \qquad (2.42)
\end{aligned}$$

If we additionally have

$$\int_x^d k(t)g(t)\,dt\le\int_x^d k(t)h(t)\,dt \qquad \text{for} \quad c\le x\le d, \qquad (2.43)$$

then (2.40) *holds.*

Proof. Similar to the proof of Theorem 2.13. □

3. n-exponential convexity and exponential convexity

First, let us recall some definitions and properties of exponentially convex functions. For more details, see [2] and [7].

Definition 3.1. A function $\psi\colon I \to \mathbb{R}$ is *n-exponentially convex in the Jensen sense* on I if

$$\sum_{i,j=1}^{n} \xi_i \xi_j \psi\left(\frac{x_i + x_j}{2}\right) \geq 0 \tag{3.1}$$

holds for all choices $\xi_i \in \mathbb{R}$ and $x_i \in I$, $i = 1, \dots, n$.

A function $\psi\colon I \to \mathbb{R}$ is *n-exponentially convex* if it is n-exponentially convex in the Jensen sense and continuous on I.

Remark 3.1. n-exponentially convex function in the Jensen sense is k-exponentially convex in the Jensen sense for every $k \in \mathbb{N}$, $k \leq n$.

Definition 3.2. A function $\psi\colon I \to \mathbb{R}$ is *exponentially convex in the Jensen sense* on I if it is n-exponentially convex in the Jensen sense for all $n \in \mathbb{N}$.

A function $\psi\colon I \to \mathbb{R}$ is *exponentially convex* if it is exponentially convex in the Jensen sense and continuous.

Remark 3.2. A positive function is log-convex in the Jensen sense if and only if it is 2-exponentially convex in the Jensen sense.

A positive function is log-convex if and only if it is 2-exponentially convex.

Proposition 3.1. *If f is a convex function on I and if $x_1 \leq y_1$, $x_2 \leq y_2$, $x_1 \neq x_2$, $y_1 \neq y_2$, then the following inequality*

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq \frac{f(y_2) - f(y_1)}{y_2 - y_1}$$

is valid.

If the function f is concave, the inequality is reversed.

Definition 3.3. Let f be a real-valued function defined on $[a, b]$. The nth *order divided difference* of f at distinct points $x_0, x_1, \dots, x_n$ in $[a, b]$ is defined recursively by

$$[x_j; f] = f(x_j), \qquad j = 0, \dots, n,$$

and

$$[x_0, x_1, \dots, x_n; f] = \frac{[x_1, \dots, x_n; f] - [x_0, \dots, x_{n-1}; f]}{x_n - x_0}.$$

Remark 3.3. The value $[x_0, x_1, \dots, x_n; f]$ is independent from the order of the points $x_0, \dots, x_n$. The previous definition can be extended to include the case in which some or all of the points coincide by assuming that $x_0 \le \dots \le x_n$ and letting

$$[\underbrace{x, \dots, x}_{j+1 \text{ times}}; f] = \frac{f^{(j)}(x)}{j!},$$

provided that $f^{(j)}$ exists.

Motivated by inequalities (2.3), (2.8), (2.13) and (2.14), under the assumptions of Theorems 2.1, 2.2, 2.3 and 2.4, we define the following functionals:

$$L_1(f) = \int_a^b f(t)g(t)\,dt - \int_c^d f(t)h(t)\,dt - R_g(c,d), \tag{3.2}$$

$$L_2(f) = \int_c^d f(t)h(t)\,dt - r_g(c,d) - \int_a^b f(t)g(t)\,dt, \tag{3.3}$$

$$\begin{aligned} L_3(f) = &\int_a^b f(t)g(t)\,dt - \int_c^d f(t)h(t)\,dt \\ &+ \int_c^d \left(\frac{f(t)}{k(t)} - \frac{f(d)}{k(d)}\right) k(t)\big[h(t) - g(t)\big]\,dt - R_g(c,d), \end{aligned} \tag{3.4}$$

$$\begin{aligned} L_4(f) = &\int_c^d f(t)h(t)\,dt \\ &+ \int_c^d \left(\frac{f(c)}{k(c)} - \frac{f(t)}{k(t)}\right) k(t)\big[h(t) - g(t)\big]\,dt - r_g(c,d) - \int_a^b f(t)g(t)\,dt. \end{aligned} \tag{3.5}$$

Remark 3.4. The functionals L_1 and L_2 can also be considered under the assumptions of Theorems 2.5 and 2.7, respectively. The functional L_3 can also be considered under the conditions of Theorems 2.11 and 2.12, and the functional L_4 can also be considered under the assumptions of Theorems 2.13 and 2.14.

Remark 3.5. $L_i(f) \ge 0$, $i = 1, \dots, 4$ for all nondecreasing functions f/k.

Now, we give mean value theorems for defined functionals.

Theorem 3.1. *Let f, g, h and k be integrable functions on $[a,b]$ with $k > 0$, and let $f/k \in C^1[a,b]$. Further, let $0 \le g \le h$ and*

$$\int_c^d h(t)k(t)\,dt = \int_a^b g(t)k(t)\,dt, \quad \text{where} \quad [c,d] \subseteq [a,b].$$

Then, there exists $\xi \in [a, b]$ *such that*

$$\begin{aligned}
&\int_a^b f(t)g(t)\,dt - \int_c^d f(t)h(t)\,dt - R_g(c,d) \\
&= \frac{f'(\xi)k(\xi) - f(\xi)k'(\xi)}{k^2(\xi)} \\
&\quad \times \left[\int_a^b tk(t)g(t)\,dt - \int_c^d tk(t)h(t)\,dt - \int_a^c (t-d)k(t)g(t)\,dt \right],
\end{aligned} \tag{3.6}$$

that is,

$$L_1(f) = \frac{f'(\xi)k(\xi) - f(\xi)k'(\xi)}{k^2(\xi)} \cdot L_1(id \cdot k).$$

P r o o f. Since $\left(\frac{f}{k}\right)'$ is continuous on $[a, b]$, there exist

$$m = \min_{x\in[a,b]} \frac{f'(x)k(x) - f(x)k'(x)}{k^2(x)} \quad \text{and} \quad M = \max_{x\in[a,b]} \frac{f'(x)k(x) - f(x)k'(x)}{k^2(x)}.$$

Now, we consider the functions $F_1, F_2 \colon [a, b] \to \mathbb{R}$ defined by

$$F_1(x) = Mxk(x) - f(x) \quad \text{and} \quad F_2(x) = f(x) - mxk(x).$$

Note that F_1/k, F_2/k are nondecreasing functions, so, by Remark 3.5, we have

$$L_1(F_1) \ge 0, \quad L_1(F_2) \ge 0.$$

Further, from Theorem 2.1, we get

$$L_1(f) \le M L_1(id \cdot k), \tag{3.7}$$

$$L_1(f) \ge m\, L_1(id \cdot k), \tag{3.8}$$

that is,

$$m\, L_1(id \cdot k) \le L_1(f) \le M L_1(id \cdot k).$$

If $L_1(id \cdot k) = 0$, then $L_1(f) = 0$, and (3.6) holds for all $\xi \in [a, b]$. Otherwise,

$$m \le \frac{L_1(f)}{L_1(id \cdot k)} \le M.$$

Since $\big(f(x)/k(x)\big)'$ is continuous, there exists $\xi \in [a, b]$ such that (3.6) holds and the proof is complete. $\square$

Theorem 3.2. *Let f, g, h and k be integrable functions on $[a, b]$ with $k > 0$, and let $f/k \in C^1[a, b]$. Further, let $0 \le g \le h$ and $\int_c^d h(t)k(t)\,dt = \int_a^b g(t)k(t)\,dt$, where $[c, d] \subseteq [a, b]$. Then, there exists $\xi \in [a, b]$ such that*

$$L_i(f) = \frac{f'(\xi)k(\xi) - f(\xi)k'(\xi)}{k^2(\xi)} \cdot L_i(id \cdot k), \qquad i = 2, 3, 4.$$

Proof. Similar to the proof of Theorem 3.1. □

Theorem 3.3. *Let $f, \hat{f}, g, h$ and k be integrable functions on $[a, b]$ with $k > 0$, and let $f(x)/k(x), \hat{f}(x)/k(x) \in C^1[a, b]$ such that $\hat{f}'(x)k(x) - \hat{f}(x)k'(x) \neq 0$ for every $x \in [a, b]$. Further, let $0 \le g \le h$ and $\int_c^d h(t)k(t)\,dt = \int_a^b g(t)k(t)\,dt$, where $[c, d] \subseteq [a, b]$. Then, there exists $\xi \in [a, b]$ such that*

$$\frac{L_i(f)}{L_i(\hat{f})} = \frac{f'(\xi)k(\xi) - f(\xi)k'(\xi)}{\hat{f}'(\xi)k(\xi) - \hat{f}(\xi)k'(\xi)}, \qquad i = 1, \dots, 4. \tag{3.9}$$

Proof. For the functionals L_i, $i = 1, \dots, 4$, we define

$$\Phi_i(t) = f(t)L_i(\hat{f}) - \hat{f}(t)L_i(f).$$

Note that

$$\frac{\Phi_i(t)}{k(t)} = \frac{f(t)}{k(t)}L_i(\hat{f}) - \frac{\hat{f}(t)}{k(t)}L_i(f) \in C^1[a, b].$$

By Theorems 3.1 and 3.2, there exists $\xi \in [a, b]$ such that

$$L_i(\Phi_i) = \frac{\Phi_i'(\xi)k(\xi) - \Phi_i(\xi)k'(\xi)}{k^2(\xi)}\, L_i(id \cdot k).$$

From $L_i(\Phi_i) = 0$ it follows that

$$\Phi_i'(\xi)k(\xi) - \Phi_i(\xi)k'(\xi) = 0,$$

That is,

$$\big[f'(\xi)k(\xi) - f(\xi)k'(\xi)\big]L_i(\hat{f}) - \big[\hat{f}'(\xi)k(\xi) - \hat{f}(\xi)k'(\xi)\big]L_i(f) = 0,$$

and (3.9) is proved. □

Now, we will use an idea from [2] to give an elegant method of producing n-exponentially convex functions and exponentially convex functions applying defined functionals on a given family with the same property. The following theorem and corollary are the same as in [8]; only for other functionals and for the reader's convenience, we will recall them without proof. In the following, I and J will denote intervals in $\mathbb{R}$.

Theorem 3.4. *Let k be a positive function, and $\Omega = \{f_p/k : p \in J\}$ be a family of functions defined on I such that the function $p \mapsto [x_0, x_1; f_p/k]$ is n-exponentially convex in the Jensen sense on J for mutually different points $x_0, x_1 \in I$. Let L_i, $i = 1, \ldots, 4$ be linear functionals defined by (3.2)–(3.5). Then, $p \mapsto L_i(f_p)$ is an n-exponentially convex function in the Jensen sense on J.*

If the function $p \mapsto L_i(f_p)$ is continuous on J, then it is n-exponentially convex on J.

Remark 3.6. If in Theorem 3.4 we have that $p \mapsto [x_0, x_1; f_p/k]$ is exponentially convex in the Jensen sense on J, then $p \mapsto L_i(f_p)$ is an exponentially convex function in the Jensen sense on J. If $p \mapsto L_i(f_p)$ is continuous on J, then it is exponentially convex on J.

Corollary 3.1. *Let k be a positive function, and $\Omega = \{f_p/k : p \in J\}$ be a family of functions defined on I such that the function $p \mapsto [x_0, x_1; f_p/k]$ is 2-exponentially convex in the Jensen sense on J for mutually different points $x_0, x_1 \in I$. Let L_i, $i = 1, \ldots, 4$ be linear functionals defined by (3.2)–(3.5). Then, the following statements hold:*

(i) *If the function $p \mapsto L_i(f_p)$ is continuous on J, then it is a 2-exponentially convex function on J. If $p \mapsto L_i(f_p)$ is additionally strictly positive, then it is also log-convex on J. Furthermore, the following inequality holds true:*

$$\big[L_i(f_s)\big]^{t-r} \le \big[L_i(f_r)\big]^{t-s}\big[L_i(f_t)\big]^{s-r} \tag{3.10}$$

for every choice $r, s, t \in J$, such that $r < s < t$.

(ii) *If the function $p \mapsto L_i(f_p)$ is strictly positive and differentiable on J, then, for every $p, q, u, v \in J$ such that $p \le u$ and $q \le v$, we have*

$$M_{p,q}(L_i, \Omega) \le M_{u,v}(L_i, \Omega), \tag{3.11}$$

where

$$M_{p,q}(L_i, \Omega) = \begin{cases} \left(\frac{L_i(f_p)}{L_i(f_q)}\right)^{\frac{1}{p-q}}, & p \ne q; \\ \exp\left(\frac{\frac{d}{dp}L_i(f_p)}{L_i(f_p)}\right), & p = q \end{cases} \tag{3.12}$$

for $f_p/k, f_q/k \in \Omega$.

Remark 3.7. The results from Theorem 3.4 and Corollary 3.1 still hold when $x_0 = x_1 \in I$. This follows from Remark 3.3.

4. Applications to Stolarsky type means

In this section, we will apply general results from the previous section to several families of functions which fulfill conditions of the obtained general results to get other exponentially convex functions and Stolarsky means.

Example 4.1. Let k be a positive integrable function, and let

$$\Omega_1 = \{f_p/k \colon (0,\infty) \to \mathbb{R} \colon p \in \mathbb{R}\}$$

be a family of functions where f_p is defined by

$$f_p(x) = \begin{cases} \frac{x^p}{p} k(x), & p \neq 0; \\ \log x\, k(x), & p = 0. \end{cases}$$

Since

$$\frac{d}{dx} \frac{f_p(x)}{k(x)} = x^{p-1} = e^{(p-1)\log x} > 0 \quad \text{for } \ x > 0,$$

then f_p/k is a nondecreasing function for $x > 0$ and $p \mapsto \frac{d}{dx} \frac{f_p(x)}{k(x)}$ is exponentially convex by definition. Similarly as in the proof of Theorem 3.4, we have that $p \mapsto [x_0, x_1; f_p/k]$ is exponentially convex (and so, exponentially convex in the Jensen sense). Using Remark 3.6, we conclude that $p \mapsto L_i(f_p)$, $i = 1, \dots, 4$ are exponentially convex in the Jensen sense. It is easy to verify that these mappings are continuous (although the mapping $p \mapsto f_p$ is not continuous for $p = 0$), so, they are exponentially convex.

For this family of functions, $M_{p,q}(L_1, \Omega_1)$ from (3.12) becomes for $p \neq q$:

$$M_{p,q}(L_1,\Omega_1) = \left(\frac{q}{p} \frac{\int_a^b t^p k(t)g(t)dt - \int_c^d t^p k(t)h(t)dt - \int_a^c (t^p - d^p)k(t)g(t)dt}{\int_a^b t^q k(t)g(t)dt - \int_c^d t^q k(t)h(t)dt - \int_a^c (t^q - d^q)k(t)g(t)dt} \right)^{\frac{1}{p-q}},$$

for $p \neq 0$:

$$M_{p,p}(L_1,\Omega_1) = \exp\left(\frac{\int_a^b t^p \log t k(t)g(t)dt - \int_c^d t^p \log t k(t)h(t)dt - \int_a^c (t^p \log t - d^p \log d)k(t)g(t)dt}{\int_a^b t^p k(t)g(t)dt - \int_c^d t^p k(t)h(t)dt - \int_a^c (t^p - d^p)k(t)g(t)dt} - \frac{1}{p} \right),$$

$$M_{0,0}(L_1,\Omega_1) = \exp\left(\frac{1}{2} \frac{\int_a^b \log^2 t k(t)g(t)dt - \int_c^d \log^2 t k(t)h(t)dt - \int_a^c (\log^2 t - \log^2 d)k(t)g(t)dt}{\int_a^b \log t k(t)g(t)dt - \int_c^d \log t\, k(t)h(t)dt - \int_a^c (\log t - \log d)k(t)g(t)dt} \right).$$

For other functionals, an explicit shape of $M_{p,q}(L_i,\Omega_1)$, $i=2,3,4$, can be obtained in a similar way from the general functional notation given by

$$M_{p,q}(L_i,\Omega_1)=\begin{cases}\left(\frac{L_i(f_p)}{L_i(f_q)}\right)^{\frac{1}{p-q}}, & p\neq q;\\ \exp\left(\frac{L_i(f_p\frac{f_0}{k})}{L_i(f_p)}-\frac{1}{p}\right), & p=q\neq 0;\\ \exp\left(\frac{L_i(f_0\frac{f_0}{k})}{2L_i(f_0)}\right), & p=q=0.\end{cases}$$

Applying Theorem 3.3 for functions $f_p/k, f_q/k\in\Omega_1$, we obtain that there exists $\xi\in[a,b]$ such that

$$\xi^{p-q}=\frac{L_i(f_p)}{L_i(f_q)},\qquad i=1,\dots,4.$$

Since the function $\xi\mapsto\xi^{p-q}$ is invertible for $p\neq q$, we have

$$a\le\left(\frac{L_i(f_p)}{L_i(f_q)}\right)^{\frac{1}{p-q}}\le b,\quad i=1,\dots,4$$

which together with the fact that $M_{p,q}(L_i,\Omega_1)$ is continuous, symmetric and monotonic shows that $M_{p,q}(L_i,\Omega_1)$, $i=1,\dots,4$ are means.

Example 4.2. Let k be a positive integrable function and let

$$\Omega_2=\left\{g_p/k\colon\mathbb{R}\to(0,\infty)\colon p\in\mathbb{R}\right\}$$

be a family of functions where g_p is defined by

$$g_p(x)=\begin{cases}\frac{e^{px}}{p}k(x), & p\neq 0;\\ x\,k(x), & p=0.\end{cases}$$

Since $\frac{d}{dx}\frac{g_p(x)}{k(x)}=e^{px}>0$, g_p/k is a nondecreasing function on $\mathbb{R}$ for every $p\in\mathbb{R}$ and $p\mapsto\frac{d}{dx}\frac{g_p(x)}{k(x)}$ is exponentially convex by definition. As in Example 4.1, we conclude that $p\mapsto L_i(g_p)$, $i=1,\dots,4$, are exponentially convex.

For this family, from (3.12), we have
for $p\neq q$:

$$M_{p,q}(L_1,\Omega_2)=$$
$$\left(\frac{q}{p}\frac{\int_a^b e^{pt}k(t)g(t)dt-\int_c^d e^{pt}k(t)h(t)dt-\int_a^c(e^{pt}-e^{dt})k(t)g(t)dt}{\int_a^b e^{qt}k(t)g(t)dt-\int_c^d e^{qt}k(t)h(t)dt-\int_a^c(e^{qt}-e^{dt})k(t)g(t)dt}\right)^{\frac{1}{p-q}},$$

for $p \neq 0$:

$$M_{p,q}(L_1,\Omega_2) = \exp\left(\frac{\int_a^b e^{pt}tk(t)g(t)dt - \int_c^d e^{pt}tk(t)h(t)dt - \int_a^c (e^{pt}t - e^{dt}t)k(t)g(t)dt}{\int_a^b e^{pt}k(t)g(t)dt - \int_c^d e^{pt}k(t)h(t)dt - \int_a^c (e^{pt} - e^{dt})k(t)g(t)dt} - \frac{1}{p}\right),$$

$$M_{0,0}(L_1,\Omega_2) = \exp\left(\frac{1}{2}\frac{\int_a^b t^2k(t)g(t)dt - \int_c^d t^2k(t)h(t)dt - \int_a^c (t^2 - d^2)k(t)g(t)dt}{\int_a^b tk(t)g(t)dt - \int_c^d tk(t)h(t)dt - \int_a^c (t-d)k(t)g(t)dt}\right).$$

From (3.11), it follows that $M_{p,q}(L_1,\Omega_2)$ is monotonic in parameters p and q.

For other functionals, an explicit shape of $M_{p,q}(L_i,\Omega_2)$, $i = 2,3,4$, can be obtained in a similar way from the general functional notation given by

$$M_{p,q}(L_i,\Omega_2) = \begin{cases} \left(\frac{L_i(g_p)}{L_i(g_q)}\right)^{\frac{1}{p-q}}, & p \neq q; \\ \exp\left(\frac{L_i(id \cdot g_p)}{L_i(g_p)} - \frac{1}{p}\right), & p = q \neq 0; \\ \exp\left(\frac{L_i(id \cdot g_0)}{2L_i(g_0)}\right), & p = q = 0. \end{cases}$$

Theorem 3.3 applied to the functions $g_p/k, g_q/k \in \Omega_2$ and functionals L_i, $i = 1,\dots,4$, implies that

$$S_{p,q}(L_i,\Omega_2) = \log M_{p,q}(L_i,\Omega_2)$$

satisfies $a \leq S_{p,q}(L_i,\Omega_2) \leq b$, so $S_{p,q}(L_i,\Omega_2)$ is a mean, and by (3.11), it is monotonic.

EXAMPLE 4.3. Let k be a positive integrable function and let

$$\Omega_3 = \left\{\phi_p/k \colon (0,\infty) \to (0,\infty) : p \in (0,\infty)\right\}$$

be a family of functions, where ϕ_p is defined by

$$\phi_p(x) = \begin{cases} \frac{-p^{-x}}{\log p}k(x), & p \neq 1; \\ xk(x), & p = 1. \end{cases}$$

Since $\frac{d}{dx}\frac{\phi_p(x)}{k(x)} = p^{-x} > 0$ for $p, x \in (0,\infty)$, ϕ_p/k is a nondecreasing function for $x > 0$. $\frac{d}{dx}\frac{\phi_p(x)}{k(x)} = p^{-x}$ is the Laplace transformation of a non-negative function, that is, $p^{-x} = \frac{1}{\Gamma(x)}\int_0^\infty e^{-pt}t^{x-1}\,dt$, so $p \mapsto \frac{d}{dx}\frac{\phi_p(x)}{k(x)}$ is exponentially convex on $(0,\infty)$. As in Example 4.1, we conclude that $p \mapsto L_i(\phi_p)$, $i = 1,\dots,4$, are exponentially convex.

For this family of functions, from (3.12), we have
for $p \neq q$:

$$M_{p,q}(L_1, \Omega_3) = \left(\frac{\log q}{\log p} \frac{\int_a^b p^{-t} k(t)g(t)dt - \int_c^d p^{-t} k(t)h(t)dt - \int_a^c (p^{-t} - p^{-d})k(t)g(t)dt}{\int_a^b q^{-t} k(t)g(t)dt - \int_c^d q^{-t} k(t)h(t)dt - \int_a^c (q^{-t} - q^{-d})k(t)g(t)dt} \right)^{\frac{1}{p-q}},$$

for $p \neq 1$:

$$M_{p,p}(L_1, \Omega_3) = \exp\left(\frac{-1}{p} \frac{\int_a^b tp^{-t} k(t)g(t)dt - \int_c^d tp^{-t} k(t)h(t)dt - \int_a^c (tp^{-t} - tp^{-d})k(t)g(t)dt}{\int_a^b p^{-t} k(t)g(t)dt - \int_c^d p^{-t} k(t)h(t)dt - \int_a^c (p^{-t} - p^{-d})k(t)g(t)dt} - \frac{1}{p \log p} \right),$$

$$M_{1,1}(L_1, \Omega_3) = \exp\left(\frac{-1}{2} \frac{\int_a^b t^2 k(t)g(t)dt - \int_c^d t^2 k(t)h(t)dt - \int_a^c (t^2 - d^2)k(t)g(t)dt}{\int_a^b tk(t)g(t)dt - \int_c^d tk(t)h(t)dt - \int_a^c (t-d)k(t)g(t)dt} \right).$$

For other functionals, an explicit shape can be obtained from a general functional notation given by

$$M_{p,q}(L_i, \Omega_3) = \begin{cases} \left(\frac{L_i(\phi_p)}{L_i(\phi_q)} \right)^{\frac{1}{p-q}}, & p \neq q; \\ \exp\left(\frac{-L_i(id \cdot \phi_p)}{p\, L_i(\phi_p)} - \frac{1}{p \log p} \right), & p = q \neq 1; \\ \exp\left(\frac{-L_i(id \cdot \phi_1)}{2 L_i(\phi_1)} \right), & p = q = 1. \end{cases}$$

Again, using Theorem 3.3, it follows that

$$S_{p,q}(L_i, \Omega_3) = -L(p,q) \log M_{p,q}(L_i, \Omega_3)$$

satisfies $a \leq S_{p,q}(L_i, \Omega_3) \leq b$, so $S_{p,q}(L_i, \Omega_3)$ is a mean, and by (3.11), monotonic. $L(p,q)$ is the logarithmic mean defined by

$$L(p,q) = \begin{cases} \frac{p-q}{\log p - \log q}, & p \neq q; \\ p, & p = q. \end{cases}$$

Example 4.4. Let k be a positive integrable function and let

$$\Omega_4 = \left\{ \psi_p / k \colon (0,\infty) \to (0,\infty) : p \in (0,\infty) \right\}$$

be a family of functions, where ψ_p is defined by

$$\psi_p(x) = \frac{-e^{-x\sqrt{p}}}{\sqrt{p}} k(x).$$

For every $p > 0$, ψ_p are nondecreasing functions for $x > 0$. Again, we conclude, $p \mapsto \frac{d}{dx}\frac{\psi_p(x)}{k(x)} = e^{-x\sqrt{p}}$ is the Laplace transform of a non-negative function, so it is exponentially convex on $(0,\infty)$. As in Example 4.1, we conclude that $p \mapsto L_i(\psi_p)$, $i = 1,\dots,4$, are exponentially convex. For this family of functions, from (3.12), we have

for $p \neq q$:

$M_{p,q}(L_1,\Omega_4) =$

$$\left(\frac{\sqrt{q}}{\sqrt{p}}\,\frac{\int_a^b e^{-t\sqrt{p}}k(t)g(t)dt - \int_c^d e^{-t\sqrt{p}}k(t)h(t)dt - \int_a^c (e^{-t\sqrt{p}} - e^{-d\sqrt{p}})k(t)g(t)dt}{\int_a^b e^{-t\sqrt{q}}k(t)g(t)dt - \int_c^d e^{-t\sqrt{q}}k(t)h(t)dt - \int_a^c (e^{-t\sqrt{q}} - e^{-d\sqrt{q}})k(t)g(t)dt}\right)^{\frac{1}{p-q}},$$

$M_{p,p}(L_1,\Omega_4) =$

$$\exp\left(\frac{-1}{2\sqrt{p}}\,\frac{\int_a^b te^{-t\sqrt{p}}k(t)g(t)dt - \int_c^d te^{-t\sqrt{p}}k(t)h(t)dt - \int_a^c t(e^{-t\sqrt{p}} - e^{-d\sqrt{p}})k(t)g(t)dt}{\int_a^b e^{-t\sqrt{p}}k(t)g(t)dt - \int_c^d e^{-t\sqrt{p}}k(t)h(t)dt - \int_a^c (e^{-t\sqrt{p}} - e^{-d\sqrt{p}})k(t)g(t)dt} - \frac{1}{2p}\right).$$

For other functionals, an explicit shape of

$$M_{p,q}(L_i,\Omega_4), \quad i = 2,3,4,$$

can be obtained in a similar way from the general functional notation given by

$$M_{p,q}(L_i,\Omega_4) = \begin{cases} \left(\frac{L_i(\psi_p)}{L_i(\psi_q)}\right)^{\frac{1}{p-q}}, & p \neq q; \\ \exp\left(\frac{-L_i(id\cdot\psi_p)}{2\sqrt{p}\,L_i(\psi_p)} - \frac{1}{2p}\right), & p = q. \end{cases}$$

Theorem 3.3 applied to the functions $\psi_p/k, \psi_q/k \in \Omega_4$ and functionals L_i, $i = 1,\dots,4$, implies that

$$S_{p,q}(L_i,\Omega_4) = -\big(\sqrt{p} + \sqrt{q}\big)\log M_{p,q}(L_i,\Omega_4),$$

satisfies

$$a \leq S_{p,q}(L_i,\Omega_4) \leq b,$$

so $S_{p,q}(L_i,\Omega_4)$ is a mean, and by (3.11), it is monotonic.

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