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SOME REMARKS ON PERMUTATIONS WHICH PRESERVE THE WEIGHTED DENSITY

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ABSTRACT. In this paper we study the conditions (1), (2) and (3) for the permutations which preserve the weighted density. These conditions are motivated by the conditions of Lévy group, originated in [Levy, P.: *Problèmes concrets d'Analyse Fonctionelle*. Gauthier Villars, Paris, 1951], and studied in [Obata, N.: *Density of natural numbers and Lévy group*, J. Number Theory **30** (1988), 288–297]. In the second part we prove that under some conditions for the sequence of weights, for instance for the logarithmic density, the first two conditions can be launched.

Let us denote the set of natural numbers by $\mathbb{N}$, and the set of all permutations of the set $\mathbb{N}$ by $Aut(\mathbb{N})$. Suppose that $\{c_j\}$ is the sequence of positive real numbers such that $\sum_{j=1}^{\infty} c_j = \infty$. This sequence will be called the sequence of weights. Let $A \subset \mathbb{N}$. Put

$$S(A,N) = \sum_{j \le N,\, j \in A} c_j,\ S(N) = \sum_{j \le N} c_j.$$

The value $\overline{d}_c(A) = \limsup_{N\to\infty} \frac{S(A,N)}{S(N)}$ we shall call the upper weight density of A and $\underline{d}_c(A) = \limsup_{N\to\infty} \frac{S(A,N)}{S(N)}$ we shall call the lower weight density of the set A. If there exists $\lim_{N\to\infty} \frac{S(A,N)}{S(N)}$, then we say that the set A has weighted asymptotic density and we will denote it as $d_c(A)$.

The weighted density is object of observations in several works. By D_c we will denote the set of all sets, which has the weighted asymptotic density. In special case, if $c_j = 1$ for all j, then the weighted asymptotic density is just asymptotic density and is denoted by $d(A)$.

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Obata in [OB] characterizes the set of permutations of $\mathbb{N}$, which preserve the asymptotic density

$$G = \left\{ g \in Aut(\mathbb{N}) : \lim_{N\to\infty} \frac{1}{N} |\{1 \le j \le N; g(j) > N\}| = 0 \right\},$$

where $Aut(\mathbb{N})$ is the group of all permutations of $\mathbb{N}$. This set forms a subgroup of the group of all automorphisms of $\mathbb{N}$ with respect to composition and is called the Lévy group.

In the case of weighted asymptotic density, the condition

$$\lim_{N\to\infty} \frac{1}{N} \left| \left\{ 1 \le j \le N; g(j) > N \right\} \right| = 0$$

splits into two conditions:

$$\lim_{N\to\infty} \frac{1}{S(N)} \sum_{\substack{j \le N \\ g(j) > N}} c_j \ = \ 0, \tag{1}$$

$$\lim_{N\to\infty} \frac{1}{S(N)} \sum_{\substack{j > N \\ g(j) \le N}} c_{g(j)} \ = \ 0, \tag{2}$$

but, as the next example shows, it is not sufficient for preserving the weighted asymptotic density.

Example 1. Let the weights be such that $c_{2k} = 1$, $c_{2k+1} = 0.5$ and let $g \in Aut(\mathbb{N})$ be such that $g(2k-1) = 2k$ and $g(2k) = 2k-1$. Then

$$\sum_{\substack{j \le N \\ g(j) > N}} c_j \le 0.5,$$

because just for one element $j \le N$ the $g(j)$ can exceed the number N, so

$$\lim_{N\to\infty} \frac{1}{S(N)} \sum_{\substack{j \le N \\ g(j) > N}} c_j \ = \ 0.$$

Analogously,

$$\lim_{N\to\infty} \frac{1}{S(N)} \sum_{\substack{j > N \\ g(j) \le N}} c_{g(j)} \ = \ 0.$$

But this permutation does not preserve the weighted asymptotic density, because for the set of even numbers A_{2k} we have $d_c(A_{2k}) = \frac{2}{3}$ and the image is the set of odd numbers $g(A_{2k}) = A_{2k+1}$ and $d_c\big(g(A_{2k})\big) = d_c(A_{2k+1}) = \frac{1}{3} \ne d_c(A_{2k})$.

We recall the statement well known as

Stolz Theorem. *Let α_n, β_n be two sequences of positive real numbers, and $\sum \beta_n = \infty$. Then*

$$\lim_{n\to\infty} \frac{\alpha_n}{\beta_n} = L \Rightarrow \lim_{N\to\infty} \frac{\sum_{n\le N} \alpha_n}{\sum_{n\le N} \beta_n} = L.$$

Let G_c be the set of all permutations $g \in Aut(\mathbb{N})$ which satisfy the conditions (1), (2) and

$$\lim_{j\to\infty} \frac{c_{g(j)}}{c_j} = 1. \tag{3}$$

Lemma 1. *G_c is a group with respect to the composition of permutations.*

P r o o f. Clearly, the identic permutation belongs to G_c, and the set of permutations fulfilling the condition (3) satisfies the conditions of the group. Let $f, g \in G_c$. Thus

$$\lim_{N\to\infty} \frac{1}{S(N)} \sum_{\substack{j\le N\\ f(j)>N}} c_j = 0 = \lim_{N\to\infty} \frac{1}{S(N)} \sum_{\substack{j>N\\ f(j)\le N}} c_{f(j)},$$

$$\lim_{N\to\infty} \frac{1}{S(N)} \sum_{\substack{j\le N\\ g(j)>N}} c_j = 0 = \lim_{N\to\infty} \frac{1}{S(N)} \sum_{\substack{j>N\\ g(j)\le N}} c_{g(j)}.$$

We prove that $f \circ g$ satisfies (1) and (2). There holds

$$\lim_{N\to\infty} \frac{1}{S(N)} \sum_{\substack{j\le N\\ (f\circ g)(j)>N}} c_j = \lim_{N\to\infty} \frac{1}{S(N)} \sum_{\substack{j\le N\\ g(f(j))>N}} c_j$$

$$= \lim_{N\to\infty} \frac{1}{S(N)} \left(\sum_{\substack{j\le N\\ g(f(j))>N\\ f(j)\le N}} c_j + \sum_{\substack{j\le N\\ g(f(j))>N\\ f(j)>N}} c_j \right)$$

$$\le \lim_{N\to\infty} \frac{1}{S(N)} \left(\sum_{\substack{f(j)\le N\\ g(f(j))>N}} c_j + \sum_{\substack{j\le N\\ f(j)>N}} c_j \right) = \lim_{N\to\infty} \frac{1}{S(N)} \sum_{\substack{f(j)\le N\\ g(f(j))>N}} c_j,$$

for $f, g \in G_c$.

If the sum

$$\sum_{\substack{N=1, f(j)\le N\\ g(f(j))>N}}^{\infty} c_j < \infty,$$

then

$$\lim_{N\to\infty} \frac{1}{S(N)} \sum_{\substack{f(j)\le N\\ g(f(j))>N}} c_j = 0.$$

If the sum $\sum_{N=1, f(j)\le N, g(f(j))>N}^{\infty} c_j = \infty$, then because $\lim_{j\to\infty} \frac{c_{f(j)}}{c_j} = 1$, using the Stoltz theorem we have

$$\lim_{N\to\infty} \frac{\sum\limits_{\substack{f(j)\le N\\ g(f(j))>N}} c_{f(j)}}{\sum\limits_{\substack{f(j)\le N\\ g(f(j))>N}} c_j} = 1,$$

and so

$$\lim_{N\to\infty} \frac{1}{S(N)} \sum_{\substack{f(j)\le N\\ g(f(j))>N}} c_j = \lim_{N\to\infty} \frac{1}{S(N)} \sum_{\substack{f(j)\le N\\ g(f(j))>N}} c_{f(j)} = 0,$$

for $f, g \in G_c$. Analogously the second limit is equal to zero.

Now we show that for every $f \in G_c$, we have $f^{-1} \in G_c$. Let $f \in G_c$, so

$$\lim_{N\to\infty} \frac{1}{S(N)} \sum_{\substack{j\le N\\ f(j)>N}} c_j = 0,$$

and

$$\lim_{N\to\infty} \frac{1}{S(N)} \sum_{\substack{j>N\\ f(j)\le N}} c_{f(j)} = 0.$$

Using the $f\big(f^{-1}(j)\big) = j$ we have

$$\lim_{N\to\infty} \frac{1}{S(N)} \sum_{\substack{j\le N\\ f^{-1}(j)>N}} c_j$$

$$= \lim_{N\to\infty} \frac{1}{S(N)} \sum_{\substack{f(f^{-1}(j))\le N\\ f^{-1}(j)>N}} c_{f(f^{-1}(j))} = 0,$$

for $f \in G_c$. Analogously the second limit. $\square$

Theorem 1. *If $g \in G_c$, then for any set $S \subset \mathbb{N}$ such that $S \in D_c$, i.e., it has the weighted asymptotic density, the set $g(S) \in D_c$ and $d_c\big(g(S)\big) = d_c(S)$.*

P r o o f. Let us denote $F_N^+(g) = \big\{g(j); j \le N \wedge g(j) > N\big\}$. For any set $S \subset \mathbb{N}$ and any $g \in Aut(\mathbb{N})$ we have

$$\big\{g(j); j \le N \wedge g(j) \in S\big\} \subseteq F_N^+(g) \cup \big(S \cap \{1, 2, \dots, N\}\big).$$

Now let $g \in G_c$ and $S \subset \mathbb{N}$, $S \in D_c$. Then the upper weighted asymptotic density

$$\begin{aligned}
\overline{d}_c\big(g^{-1}(S)\big) &= \limsup_{N\to\infty} \frac{1}{S(N)} \sum_{\substack{j\le N\\ j\in g^{-1}(S)}} c_j \\
&= \limsup_{N\to\infty} \frac{1}{S(N)} \sum_{\substack{j\le N\\ g(j)\in S}} c_j \\
&\le \limsup_{N\to\infty} \frac{1}{S(N)} \left(\sum_{\substack{j\le N\\ g(j)>N}} c_j + \sum_{\substack{i\le N,\, i\in S\\ i=g(j)}} c_j \right) \\
&= \limsup_{N\to\infty} \frac{1}{S(N)} \left(\sum_{\substack{j\le N\\ g(j)>N}} c_j + \sum_{\substack{i\le N,\, i\in S\\ i=g(j)}} c_{g^{-1}(i)} \right) \\
&= \limsup_{N\to\infty} \frac{1}{S(N)} \sum_{\substack{i\le N\\ i\in S}} c_{g^{-1}(i)},
\end{aligned}$$

for $g \in G_c$. If the sum

$$\sum_{\substack{N=1\\ i\le N,\, i\in S}}^{\infty} c_{g^{-1}(i)} = \infty,$$

then because $\lim_{i\to\infty} \frac{c_i}{c_{g^{-1}(i)}} = 1$, using the Stoltz theorem we have

$$\lim_{N\to\infty} \frac{\sum\limits_{i\le N,\, i\in S} c_i}{\sum\limits_{i\le N,\, i\in S} c_{g^{-1}(i)}} = 1,$$

and so it is equal to

$$\limsup_{N\to\infty} \frac{1}{S(N)} \sum_{i\le N,\, i\in S} c_i = \overline{d}_c(S).$$

Thus

$$\overline{d}_c\big(g^{-1}(S)\big) \le \overline{d}_c(S).$$

Using the fact that G_c is a group, we have $g, g^{-1} \in G_c$, so we obtain the reverse inequality

$$\overline{d}_c(S) = \overline{d}_c\Big(g\big(g^{-1}(S)\big)\Big) \leq \overline{d}_c\big(g^{-1}(S)\big),$$

and so

$$\overline{d}_c\big(g^{-1}(S)\big) = \overline{d}_c(S).$$

Similarly, for the lower weighted asymptotic density we have $\underline{d}_c\big(g^{-1}(S)\big) = \underline{d}_c(S)$. Thus if the set $S \in D_c$, then the set $g(S) \in D_c$, and

$$\overline{d}_c\big(g^{-1}(S)\big) = \overline{d}_c(S) = \underline{d}_c(S) = \underline{d}_c\big(g^{-1}(S)\big), \quad \text{i.e.,} \ \ d_c(S) = d_c\big(g(S)\big).$$

If the sum $\sum_{N=1, i\leq N, i \in S}^{\infty} c_{g^{-1}(i)} < \infty$, then $d_c\big(g^{-1}(S)\big) = 0$. In this case the sum

$$\sum_{\substack{N=1,\, i \leq N \\ i \in \mathbb{N}-S}}^{\infty} c_{g^{-1}(i)} = \infty,$$

Using the same way for the set $\mathbb{N} - S$, we obtain

$$d_c(\mathbb{N} - S) = d_c\big(g^{-1}(\mathbb{N} - S)\big) = 1 - d_c\big(g^{-1}(S)\big) = 1 - 0 = 1$$

and because $d_c(\mathbb{N} - S) = 1 - d_c(S)$ we have

$$d_c(S) = 0 = d_c\big(g^{-1}(S)\big).$$

□

Remark 1. We can construct the weights such that every permutation will preserve the weighted asymptotic density for any set $S \in D_c$.

Lemma 2. *Let the weights $\{c_n\}$ be such that*

$$\lim_{N\to\infty} \frac{c_N}{\sum\limits_{j=1}^{N} c_j} = 1.$$

Then for any $S \subset \mathbb{N}$ it holds that $S \in D_c$ if and only if S is finite or $\mathbb{N} - S$ is finite, and for any permutation $g \in Aut(\mathbb{N})$, if $S \in D_c$, then

$$g(S) \in D_c, \quad \textit{and} \quad d_c(S) = d_c\big(g(S)\big).$$

P r o o f. Let us suppose that S is infinite and also $\mathbb{N} - S$ is infinite. The sequence

$$\left\{ \frac{\sum\limits_{\substack{j \leq N \\ j \in S}} c_j}{c_N} \right\}_{N=1}^{\infty}$$

will tend for $N \in S$ to 1 and for $N \in \mathbb{N} - S$ to 0. Because the sets $S, \mathbb{N} - S$ are infinite, the 0 and 1 are the cluster points, and so

$$\underline{d}_c(S) = \liminf_{N\to\infty} \frac{1}{S(N)} \sum_{\substack{j\leq N \\ j\in S}} c_j = \liminf_{N\to\infty} \frac{\sum_{\substack{j\leq N \\ j\in S}} c_j}{c_N} = 0,$$

and

$$\overline{d}_c(S) = \limsup_{N\to\infty} \frac{1}{S(N)} \sum_{\substack{j\leq N \\ j\in S}} c_j = \limsup_{N\to\infty} \frac{\sum_{\substack{j\leq N \\ j\in S}} c_j}{c_N} = 1,$$

and the set S is not measurable. In the case S is finite, then

$$\underline{d}_c(S) = \liminf_{N\to\infty} \frac{1}{S(N)} \sum_{\substack{j\leq N \\ j\in S}} c_j = 0 = \limsup_{N\to\infty} \frac{1}{S(N)} \sum_{\substack{j\leq N \\ j\in S}} c_j = \overline{d}_c(S),$$

and $d_c(\mathbb{N} - S) = 1 - d_c(S) = 1$. Analogously for $\mathbb{N} - S$ it is finite. If $g \in Aut(\mathbb{N})$, then the image of finite set S is finite set $g(S)$ and so $d_c(S) = d_c\big(g(S)\big) = 0$, analogously for the case $\mathbb{N} - S$ it is finite. $\square$

Lemma 3. *Let the weights be not increasing and satisfy*

$$\lim_{N\to\infty} \frac{Nc_N}{S(N)} = 0.$$

Then if for permutation $g \in Aut(\mathbb{N})$ is $\lim_{j\to\infty} \frac{c_{g(j)}}{c_j} = 1$, then

$$\lim_{N\to\infty} \frac{1}{\sum_{j\leq N} c_j} \sum_{\substack{j\leq N \\ g(j)>N}} c_j = 0 = \lim_{N\to\infty} \frac{1}{\sum_{j\leq N} c_j} \sum_{\substack{j>N \\ g(j)\leq N}} c_{g(j)}.$$

P r o o f. Let the weights be not increasing and satisfy

$$\lim_{N\to\infty} \frac{Nc_N}{S(N)} = 0.$$

Then if there exist the constants k_1, k_2 such that $0 < k_1 \leq \frac{c_n}{c_{g(n)}} \leq k_2 < \infty$, then we have $c_n \leq k_2 c_{g(n)}$, and so

$$\sum_{\substack{j\leq N \\ g(j)>N}} c_j \leq k_2 \sum_{\substack{j\leq N \\ g(j)>N}} c_{g(j)} \leq k_2 N c_N,$$

because the weights are not increasing. So the

$$\lim_{N\to\infty} \frac{1}{S(N)} \sum_{\substack{j\leq N \\ g(j)>N}} c_j \leq \lim_{N\to\infty} \frac{1}{S(N)} k_2 N c_N = 0.$$

Analogously for the second limit we have

$$0 < \frac{1}{k_2} \le \frac{c_{g(j)}}{c_j} \le \frac{1}{k_1} < \infty,$$

so

$$c_{g(j)} \le \frac{1}{k_1} c_j,$$

$$\sum_{\substack{j>N \\ g(j)\le N}} c_{g(j)} \le \frac{1}{k_1} \sum_{\substack{j>N \\ g(j)\le N}} c_j \le \frac{1}{k_1} N c_N,$$

and

$$\lim_{N\to\infty} \frac{1}{S(N)} \sum_{\substack{j>N \\ g(j)\le N}} c_{g(j)} \le \lim_{N\to\infty} \frac{1}{S(N)} \frac{1}{k_1} N c_N = 0.$$

If the $\lim_{j\to\infty} \frac{c_{g(j)}}{c_j} = 1$, then the existence of the constants k_1, k_2 is guaranteed, so the proof is completed. □

A special case of weighted asymptotic density is the logarithmic density, where the weights are $c_j = \frac{1}{j}$, and the $\lim_{N\to\infty} \frac{N c_N}{S(N)} = \lim_{N\to\infty} \frac{N \frac{1}{N}}{S(N)} = 0$, so the condition of previous lemma is satisfied. The next corollary follows immediately from the lemma.

Corollary 1. *Let $d_c(A)$ be the logarithmic density and let $g \in Aut(\mathbb{N})$ be a permutation. If $\lim_{j\to\infty} \frac{c_{g(j)}}{c_j} = 1$, then g preserves the logarithmic density.*

Now we prove a special condition for a specific type of weights. Suppose that the sequence of weights $\{c_n\}$ has moreover two following properties:
For two sequences of positive integers $\{n\}, \{k(n)\}$ such that $k(n) \to \infty, n \to \infty$ we have

$$\lim_{n\to\infty} \frac{n}{k(n)} = 1 \Longrightarrow \lim_{n\to\infty} \frac{c_n}{c_{k(n)}} = 1. \tag{4}$$

There exists a positive real valued function $\omega(\delta)$ such that $\lim_{\delta\to 1} \omega(\delta) = 1$, and

$$\lim_{N\to\infty} \frac{S(\delta N)}{S(N)} = \omega(\delta). \tag{5}$$

Theorem 2. *Let the weights $\{c_n\}$ satisfy conditions (4) and (5). Then for every permutation $g : \mathbb{N} \to \mathbb{N}$ such that*

$$\lim_{n\to\infty} \frac{g(n)}{n} = 1, \tag{6}$$

and for every set $A \subset \mathbb{N}$ there holds: If A has the weighted density, then also its image $g(A)$ has the weighted density and $d_c(A) = d_c\big(g(A)\big)$.

P r o o f. Let the set $A \subset \mathbb{N}$ have the weighted density. If A is a finite set, then $g(A)$ is also finite and the assertion holds. Suppose that A is an infinite set. Let us choose an arbitrary $\varepsilon > 0$. Then the condition (6) implies that

$$(1-\varepsilon)n \le g(n) \le (1+\varepsilon)n, \qquad n \ge n_0 \tag{7}$$

for suitable $n_0 \in \mathbb{N}$. The set A can be decomposed into two subsets

$$A = A_0 \cup A_1, \quad \text{where} \quad A_0 = A \cap [0, n_0).$$

Clearly it holds $d_c(A) = d_c(A_1), d_c\big(g(A_0)\big) = 0$. Analogously we have

$$\overline{d}_c\big(g(A_1)\big) = \overline{d}_c\big(g(A)\big) \quad \text{and} \quad \underline{d}_c\big(g(A_1)\big) = \underline{d}_c\big(g(A)\big).$$

There it holds

$$S\big(g(A_1), N\big) = \sum_{\substack{g(a)\le N\\ a\in A_1}} c_{g(a)}.$$

From the inequalities (7) we obtain

$$\sum_{\substack{a\le \frac{N}{1+\varepsilon}\\ a\in A_1}} c_{g(a)} \le S\big(g(A_1), N\big) \le \sum_{\substack{a\le \frac{N}{1-\varepsilon}\\ a\in A_1}} c_{g(a)}. \tag{8}$$

The conditions (4) and (6) imply that $\frac{c_{g(a)}}{c_a} \to 1$ as $a \to \infty$. Thus if $\sum_{a\in A} c_a < \infty$, then also $\sum_{g(a)\in g(A)} c_{g(a)} < \infty$ and $d_c(A) = d_c\big(g(A)\big) = 0$. Suppose now $\sum_{a\in A} c_a = \infty$. From Stolz Theorem we obtain for arbitrary $\delta > 0$

$$\sum_{\substack{a\le \delta N\\ a\in A_1}} c_{g(a)} \sim S(A_1, \delta N), \qquad N \to \infty. \tag{9}$$

Now from (9) and (5) we have for $\delta > 0$

$$\frac{1}{S(N)} \sum_{\substack{a\le \delta N\\ a\in A_1}} c_{g(a)} \to \omega(\delta) d(A), \qquad N \to \infty. \tag{10}$$

And so from (8) we obtain

$$\omega\left(\frac{1}{1+\varepsilon}\right) d_c(A) \le \underline{d}_c\big(g(A)\big) \le \overline{d}_c\big(g(A)\big) \le \omega\left(\frac{1}{1-\varepsilon}\right) d_c(A).$$

Thus for $\varepsilon \to 0^+$ the condition (5) yields $d_c(A) = d_c\big(g(A)\big)$. $\square$

Nathanson and Parikh proved in [NP] following result:

Theorem A. *Let $f : \mathbb{N} \to \mathbb{N}$ be a one-to-one function such that if the set A of positive integers has asymptotic density, then the set $f(A)$ also has asymptotic density. Let*

$$\lambda = d\big(f(\mathbb{N})\big).$$

Then for every $A \in \mathcal{D}$ we have

$$d\big(f(A)\big) = \lambda d(A).$$

Using the ideas from theirs proof the following "formal" improvement can be made:

Theorem B. *Let $\mathcal{A}$ be a q-algebra of sets of positive integers and ν a finitely additive probability measure on $\mathcal{A}$ which has the Darboux property on $\mathcal{A}$. Let $g : \mathbb{N} \to \mathbb{N}$ be such injective mapping that $g(A) \in \mathcal{A}$ for $A \in \mathcal{A}$ and for $A, B \in \mathcal{A}$ it holds*

$$\nu(A) = \nu(B) \Rightarrow \nu\big(g(A)\big) = \nu\big(g(B)\big). \tag{i}$$

Then for every $A \in \mathcal{A}$ we have $\nu\big(g(A)\big) = \lambda\nu(A)$, where $\lambda = \nu\big(g(\mathbb{N})\big)$.

Proof. Let $\frac{p}{q} \in \langle 0, 1\rangle, p, q \in \mathbb{N}$. Suppose that

$$A \in \mathcal{A} \quad \text{and} \quad \nu(A) = \frac{p}{q}.$$

Using the Darboux property we get that there is a decomposition

$$\mathbb{N} = B_1 \cup \dots \cup B_q, \quad B_j \in \mathcal{A}, \quad \nu(B_j) = \frac{1}{q}, \quad j = 1, \dots, q,$$

thus

$$g(\mathbb{N}) = g(B_1) \cup \dots \cup g(B_q), \quad g(B_j) \in \mathcal{A}.$$

From (i) we obtain $\nu\big(g(B_j)\big) = \frac{\lambda}{q}, j = 1, \dots, q$. The set A can be decomposed into

$$A = A_1 \cup \dots \cup A_p, \quad A_i \in \mathcal{A}, \quad \nu(A_i) = \frac{1}{q}, \quad i = 1, \dots, p.$$

Thus we have a decomposition $g(A) = g(A_1) \cup \dots \cup g(A_p)$, $g(A_i) \in \mathcal{A}$. From (i) we get $\nu\big(g(A_i)\big) = \frac{\lambda}{q}, i = 1, \dots, p$, and so $\nu\big(g(A)\big) = \lambda\frac{p}{q}$.

Now we shall consider a function $\overline{g} : \langle 0, 1\rangle \to \langle 0, \lambda\rangle$ defined as follows: if $x \in \langle 0, 1\rangle$, then there exists $A \in \mathcal{A}$ that $\nu(A) = x$. Put $\overline{g}(x) = \nu\big(g(A)\big)$. The condition (i) provides that this definition is correct, and from the Dardoux property of ν on $\mathcal{A}$ we get that $\overline{g}$ is non decreasing on $\langle 0, 1\rangle$. We proved that $\overline{g}(x) = \lambda x$ for x rational and so $\overline{g}(x) = \lambda x$ for every $x \in \langle 0, 1\rangle$. $\square$

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